# BOHR'S ATOMIC MODEL FROM THE STANDPOINT OF THE

# GENERAL THEORY OF RELATIVITY AND OF THE

# CALCULUS OF PERTURBATIONS

An Investigation Submitted to the

Faculty of Massachusetts Institute of Tech-

nology as a Thesis in Partial Fulfilment of

the Requirements towards the Degree of

Doctor of Science

Cambridge, Mass., U.S.A.
May, 1924.

Certification for the Department of Physics

Dedicated to the Memory of my Mother

# Table of Contents

Part III.  Appendix

General Introduction to the theory
    of Perturbations.

## Acknowledgment

The author takes great pleasure in expressing his deep gratitude to Professor Paul Heymans, who has made the writing of this thesis possible, and to Professors Henry Bayard Phillips, Joseph Lipka and Norbert Wiener, to whom he owes his initiation into Mathematical Physics.

# Bohr's Atomic Model from the Standpoint of the General Theory of Relativity and of the Calculus of Perturbations

## Abstract

I. In the first part of this thesis, Bohr's atomic model is studied in detail from the standpoint of the general theory of relativity. A)Lenard's arguments: 1) that the theory of relativity stands in no causal relationahip to Sommerfeld's theory of fine structure, and 2) that it is not clear why the restricted theory of relativity is sufficient to account for the motion of the electron in the atom, whereas the general theory is required to explain the anomaly of the perihelion of Mercury, are examined at length. The former objection naturally leads to an examination of the more general question: Does the metric of space-time stand in a causal relationship to the advance of the perihelion of Mercury? It is shown that

$\alpha$) The Newtonian equations of dynamics are obtained from the geodesic equations in a four-dimensional continuum by cancellations which, so far as our analysis can show, are purely arbitrary.

$\beta$) Complementing Le Roux's and Painlevé's objec-

tions to the arbitrariness of Schwarzschild's so-
lution of Einstein's gravitational equations $G_{\mu\nu} = 0$
in a static sphero-symmetrical gravitational field,
it is proved that Schwarzschild's result is arrived
at by a series of suitable restrictive arbitrary assum-
ptions.   There exist in fact an infinite number of so-
lutions of the gravitational equation consistent with
the definition of static, spherical and symmetrical
field and the choice of the Schwarzschildian form
must be regarded as purely arbitrary.  It follows that
Einstein's equation is insufficient to fix uniquely
the relativistic Keplerian orbit.

   Y ) Von Gleich's and Zaremba's analysis are dis-
cussed in detail and found to hold.  The argument may
be summed up as follows:  For motions in a central
field the dynamic equations of classical mechanics must
be supplemented by a "relativistic" equation, but this
last equation is arbitrary because several forms of
this equation lead to the same formula for the peri-
helion advance.  Hence it cannot be maintained that the
metric of space-time stands in a necessary causal re-
lationship to the anomaly of the perihelion.  The gen-
eral conclusion is that Lenard's first objection holds.

   2)  Using Nordström's fundamental form in a static
sphero-symmetrical material field and Weyl-Eddington's
equation of motion of the electron in this field it is

shown: $\alpha$ )that the motion is plane, $\beta$ ) that the cur-
vature of the space-time in the region around the
nucleus almost vanishes (Laue's condition), $\gamma$ ) that
the field of the nucleus is very near static, $\delta$ )that
it is justified to treat hydrogen or hydrogenic atoms
as a relativistic one-body problem, $\varepsilon$ ) that our more
general equation of the orbit reduces to Sommerfeld's
equation if conclusions ($\alpha$) to ( $\delta$) are taken into ac-
count, $\theta$ ) that Einstein's equation for planetary mo-
tion follows from our equation of the orbit if the
terms depending on electric charges vanish, $\eta$ ) that
condition ($\beta$) is not satisfied for planetary motion but
condition ($\gamma$) is. $\zeta$ ) that the general relativistic
Keplerian orbit can be obtained by the familiar method
of the calculus of perturbations and yields results in
accordance with rigorous analysis. In closing, the weak-
ness of the Weyl-Eddington equation of motion is dis-
cussed.

B. An attempt is made to bring the quantum theory
into relativity. Following on the work of Mecke, it is
shown that the two fundamental postulates of the quantum
theory: minimal principle, and continuity principle can
be brought in harmony with relativity and lead to the
following results: $\alpha$ ) an explanation of the existence
of unmechanical orbits in the atom, $\beta$ ) a unified inter-
pretation of Bohr's second postulate (monochromatic radia-

tion) and his principle of correspondence. Mecke's "integral principle" (atomistic conception of action) can also be brought into the body of the theory of relativity, but, if it is to have any usefulness, privileged coördinate systems must be introduced. On the author's view, this is due entirely to the artificiality of our <u>present</u> quantum conditions and shows no weakness of the main argument. The fundamental meaning of the relativistic two-body problem, in which probably the periodic solutions are the natural quantized states, irrespective of any quantum rules, is brought out. It is shown next that the electron problem is not yet sufficiently developed to warrant an attack on the quantum problem from that side. In closing, Mohorovicic's objection that "different geometries hold at the same spot of the universe" (ex. atom in gravitational field) is examined and found to be untenable.

II. A. In the second part of this investigation, the applicability of the calculus of perturbations to the Bohr atom with more than one electron is studied in detail. The methods of quantization of Epstein, Bohr and Born and Pauli are examined. It is shown that it follows directly from the form of the integrals of the equations of dynamics, as determined by the classical theorems of Poincaré and Bruns, that neither of these methods can converge. In Epstein's case nothing is known

as to the remainder, and he disregards altogether a
possible libration of the momenta of the intermediate
orbit, a case which cannot be handled by Delaunay's
method, which he uses. The contention of Born and
Pauli, that the semi-convergency of the Newcomb-Lind-
stedt or Bohlin expansion of the perturbation function
is sufficient for practical atomic problems, is shown
to be untenable. Stress is laid on the fundamental
significance of commensurabilities in periodic times in
those methods that utilize the Newcomb-Lindstedt ex-
pansion and the impossibility of introducing angle var-
iables in certain cases. For continuity reasons, it
is concluded that neither the solution of Epstein nor
that of Born and Pauli are compatible with the adiabatic
hypothesis or the correspondence principle. The decisive
advantages of periodic solutions and the appropriateness
of Bohr's method as a first approximation, in those
cases to which it applies, is brought out clearly. The
general conclusions: For purely analytic reasons, the
calculus of perturbations is not in general applicable
to the atomic system with more than one electron.

B. Following Fermi, Poincaré's theorem is discussed
from a geometrical-statistical standpoint. We establish
the following theorem: Except for the dynamical system
of one degree of freedom, no continuous analytic hyper-
surface exists in phase-space, besides the energy-surface,

which can wholly contain a dynamical trajectory issued from one of its points. This holds under conditions which are explicitly stated.

C. An attempt is made to extend the adiabatic theorem to non-conditioned-periodic systems. The method of attack is geometrical-statistical and the result is in the main negative. We establish the following theorems: $\alpha$ ) Any dynamical system admitting a single integral, uniform and independent of time, admits an adiabatic transformation leaving this characteristic adiabatically invariant. As a particular corollary we have the well-known theorem that the energy of a quasi-ergodic system is adiabatically invariant, and the adiabatic invariant of harmonic motion. $\beta$ ) If a dynamical system admits more than one integral uniform and independent of time, none of these integrals is in general an adiabatic invariant. An exception is, for instance, the phase-integrals of a conditioned periodic system.

The results of Krutkow and Fock and Kneser are analyzed and discussed. In closing, the difficulties standing in the way of a generalization of the concept of adiabatic invariance to systems of higher degree of freedom are discussed, following Ehrenfest. It is pointed out that the theorem of the conservation of dimensions for the transformation from W, I-phase-space to

p,q-phase-space involves only differentiability conditions, which for physical reasons are satisfied in all likelihood, but no rigorous proof is given. Ehrenfest and Breit's example of the rotating dipole is explained from the standpoint of our general discussion.

D. A discussion is given of the separation of variables in Hamilton-Jacobi's equation, embodying the condition of separability of Levi-Civita, the latter's theorem on the separability of the variables in the kinetic energy and Dall'Acqua's proof of the separable forms of the potential.

E. For purposes of ready reference, Cauchy's existence proof and its generalization by Poincaré, Poincaré's periodic solution, the convergency of the expansion of the perturbation function and the theorems of Poincaré and Bruns are given at the end.

III. An appendix is added collecting certain classical results of celestial dynamics which are made use of in the text.

Part I. The Bohr-Sommerfeld Atomic Model from the

Standpoint of the General Theory of Relativity

1. General

The mathematical meaning of the restricted theory

of relativity is contained in the Lorentz transforma-

tion. That more is implied in the Lorentz group than

was originally supposed in 1905 by Einstein is some-

thing which can be safely said.[1] By applying the re-

stricted theory of relativity to Bohr's model of the

hydrogen atom, Sommerfeld was able to give a complete

satisfactory theory of the fine structure of hydrogen

lines and, besides, important extensions to the theory

of the general systematization of Röntgen spectra.[2]

But soon Lenard[3] cast two grave doubts on Sommerfeld's

interpretation of the fine structure: Why is it that

restricted relativity is sufficient to account for the

advance of the perihelion of an electronic orbit in an

atom, whereas general relativity is necessary to explain

this advance in the case of the planetary system? Does

the theory of relativity stand in a causal relationship

to the theory of fine structure? The answer to the first

of these questions was quite apparent to the writer

early in 1923, while studying De Donder's "La gravifique

einsteinienne" (Paris, 1921). That there are some very

deep-seated reasons why the restricted theory is sufficient to account for electronic motions, whereas the general theory is necessary to calculate planetary motions, will be shown in the sequence.

On the other hand, the correctness of Lenard's contention that there exists no necessary causal relationship between relativity and perihelion advance must be recognized. Aside from arguments of a very general kind which will be brought forth presently, it should be quite carefully emphasized at the very start that Paschen's experimental verification of Sommerfeld's theory of fine structure (1a) cannot be taken to be a proof of the restricted theory of relativity, but of any theory which leads to a law of variation of electronic mass $m = m_0 / \left( 1 - v^2/c^2 \right)^{1/2}$, as clearly shown by (2a) Lenard. Further, Paschen's figure for the fundamental hydrogen doublet $\Delta\nu_0 = 0.365$ cm$^{-1}$ has been questioned very recently by Gehrcke and Lau (3a), who find $\Delta\nu_0 = 0.292$ cm$^{-1}$, in accordance with Abraham's mass formula for a rigid electron (4a). For the present, the question rests altogether in the hands of the experimental physicist.

## A few Remarks on Relativistic Keplerian Orbits.

One of the best known results of Einstein's gravitational theory is the explanation of the advance of the perihelion of planetary orbits. In this paper we are largely concerned with the theory of electronic orbits in hydrogenic atoms from the standpoint of general relativity. The discussion of specific phases of the problem is left for later sections, but at the very outset of the investigation a question of fundamental importance presents itself for consideration: Is it a fact that the metric properties of space-time stand in a causal relationship to the advance of the perihelion in a relativistic Keplerian orbit? On account of the obvious fact that all future deductions are based upon the answer to this query, it might be properly taken up here.

The problem has been discussed from different angles by a number of writers, notably by Zaremba.[4] After a profound examination of the postulates of the theory of relativity he arrives at the important conclusion that "dans tous les cas où les relativistes ont crû avoir demontre qu'une proposition, confirmée par l'observation, est une conséquence de la théorie de la rélativité, ils ont appuyé leur thèse par des considerations basées non seulement sur les hypothèses de la relativité, mais en-

core sur quelque affirmation gratuite où sur des hypothèses logiquement incompatibles avec celles de la relativité". Further, he states that the postulates of the theory of relativity are insufficient to establish a correspondence between measurements and the numerical values of the symbols entering into the formulae of the theory: "Avant de parler de l'accord ou du desaccord de la théorie de la relativité avec les faits observés, il faudrait la completer au moyen d'hypotheses additionnelles". However, he is careful to add that "les résultats précedents ne constituent cependant nullement une réfutation de la théorie de la relativité" because it has not been shown that it is impossible so to complete the postulates of the theory of relativity as to transform it into a theory in which we may <u>actually</u> speak of a confirmation of theory by experiment. All this seems to point out that a re-examination of the problem is not only opportune, but necessary and unavoidable.

We are not concerned at all here with the numerous attacks which, for metaphysical or other reasons, [5] have been made from time to time against the theory of relativity. We approach the question, with due reverence to the formidable creative work of Einstein and Weyl, from the point of view of the mathematical physicist only.

The theory of the geodesics, which had already yield-

ed such beautiful results in classical dynamics, led

Einstein[6] to the following problem: What is the rela-

tion between the equations of the geodesics

$$\frac{d^2 x_k}{ds^2} + \left\{ \begin{matrix} i & j \\ & k \end{matrix} \right\} \frac{dx_i}{ds} \frac{dx_j}{ds} = 0 \quad (i,j,k = 1,2,3,4) \quad (1)$$

and the equations of motion of classical mechanics

$$\frac{d^2 x_k}{dt^2} - \frac{\partial U}{\partial x_k} = 0 \qquad (k = 1,2,3) \qquad (2)$$

In this last equation $U$ is the gravitational potential.

It may also be written in the well-known form:

$$\frac{d^2 x_k}{dt^2} + \frac{\kappa^2 x_k}{r^3} = 0 \qquad (2a)$$

Here $r^2 = \sum_{k=1}^{3} x_k^2$ , $\kappa$ is Gauss' constant, $k^2 = n^2 a^3 / 1+m$,

if $n = 2\pi/T$ is the mean angular velocity, T the

periodic time, a the semi-major axis in astronomical

units, m the ratio of the mass of the planet to the mass

of the sun. Since m is very small compared to 1, the

above relation is usually written $k^2 = n^2 a^3$.

In the relativity theory, $x_4 = ct$, where t is the

relativistic time, which of course is _not_ independent of

space coördinates. The above equation (2a) is therefore

written in terms of the variable $x_4$:

$$\frac{d^2 x_k}{dx_4^2} + \frac{k^2}{c^2} \frac{x_k}{r^3} = 0 \qquad k = 1,2,3 \ldots \quad (3)$$

If by t we mean the independent variable of the problem,

i.e. the old "absolute" time, then in classical mechanics

we obviously have $\ddot{x}_4 = 0$. Not so in relativity, for

the relativistic time $x_4$ is no more the independent
variable, because it is connected to the space coördi-
nates by the postulate that the square of the linear
element in four-dimensional space-time is an invar-
iant with respect to affine transformations, i.e. by
the condition equation $ds^2 = g_{\mu\nu} dx^\mu dx^\nu =$ invariant....(4)
From this it follows immediately that (1) and (4) can
never be rigorously identifiable, but only with greater
or less approximation.  The necessity of a metric based
on a quadratic fundamental form has been discussed by
Weyl and proved by Cartan.[7]

In order to identify (2a) with (1), Einstein sets
as a first approximation  $ds = dt = dx_4$[8]     Since both
$ds$ and $dx_4$ contain the factor $c$, he neglects $dx_k/ds$ for
$k = 1,2,3,$ in equation (1).  Otherwise this is justi-
fied because the planetary velocity is negligible com-
pared with the velocity of light.  From (1) Einstein gets

$$\frac{d^2 x_k}{dt^2} + \begin{Bmatrix} 44 \\ k \end{Bmatrix} = 0 \qquad k = 1,2,3.$$

and assuming that $g_{44}$, $g_{11}$, $g_{22}$, $g_{23}$   and the remaining
$g_{\mu\nu}$'s differ from 1, -1, 0, respectively, by small quan-
tities of the first order, we have, from the definition
of the Christoffel symbols of the first and second kind

$$g^{\mu\mu}\begin{bmatrix} \lambda\nu \\ \mu \end{bmatrix} = \begin{Bmatrix} \lambda\nu \\ \mu \end{Bmatrix}$$

of the 10 different components of the covariant funda-

mental tensor $g_{\mu\nu}$ only <u>four</u> remain in a first approx-
imation, one for each of the four equations (1). We
then get $d^2x_4 / dt^2 \doteq 0$, as in classical mechanics
and the remaining equations (1) give

$$\frac{d^2x_k}{dt^2} = \begin{bmatrix} 44 \\ k \end{bmatrix} \qquad k \doteq 1,2,3.$$

Here the $g_{k4}$ which involve space <u>and</u> time are rather
unwieldy, so Einstein sets them simply equal to zero.
The writer is unable to see that there is any other
reason for this except the desire to obtain equation (2)
and must therefore regard this as purely arbitrary. But
the advantage gained is great, for now we have simply

$$\begin{bmatrix} 44 \\ k \end{bmatrix} \doteq - \frac{1}{2} \frac{\partial g_{44}}{\partial x_k} \qquad k \doteq 1,2,3.$$

and we need only plate $- \partial g_{44} / \partial x_k \doteq 2\, \partial U / \partial x_k$ to ob-
tain the desired identification. But we cannot con-
clude from this arbitrary procedure that the relativis-
tic equations of motion ( i.e. the geodesic equations)
"reduce" to the Newtonian equations as a first approxi-
mation. For as shown above, setting $2U \doteq 1 - g_{44}$ is
the result of purely arbitrary cancellations, which, as
far as one can see, are exclusively born of the wish to

reduce $\begin{Bmatrix} 44 \\ k \end{Bmatrix}$ , to $- \frac{1}{2} \frac{\partial g_{44}}{\partial x_k}$ . If our standpoint is

correct, the statement of Einstein "Das Merkwürdige
an diesem Resultat ist, dass nur die Komponente $g_{44}$ des

Fundamentaltensors allein in erster Näherung die Be-
wegung des materiellen Punktes bestimmt" is difficult
to understand.

Let us now examine the advance of the perihelion
of a planet more closely. In a recent article, G. von
Gleich [9] has shown that _any_ law of attraction gives
results for the advance of the perihelion of a planet
(Mercury in his example) which differ from the general
relativity formula _only_ _by_ _a_ _numerical_ factor. Thus,
the latter gives the well-known expression [10]

$$\Delta\phi = \frac{24\,\pi^3\,a^2}{T^2 c^2 (1-e^2)} \qquad (e = \text{eccentricity}) \quad (43'' \text{ per century})$$

while restricted relativity gives [11]

$$\Delta\phi = \frac{4\,\pi^3\,a^2}{T^2 c^2 (1-e^2)} \qquad\qquad (7'' \text{ per century})$$

Weber's theory which assumes a law of attraction [12]

$$F = -\frac{k^2}{r^2}\left[1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{2r}{c^2}\frac{d^2 r}{dt^2}\right]$$

and Neumann's, which sets [13]

$$F = \frac{k^2}{r^2}\left(1 - \frac{1}{c^2}\frac{dr}{dt}\right)$$

give $\quad \Delta\phi = \dfrac{8\,\pi^3\,a^2}{T^2 c^2 (1 - e^2)} \qquad (14'' \text{ per century})$

Riemann's law with a potential [13a]

$$k^2\left(\frac{1}{r} - \frac{1}{c^2}\ f\left(x,y,z,\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)\right.$$

gives $\quad \Delta \phi = \dfrac{16\,\pi^3\,a^2}{T^2 c^2 (1 - e^2)}$ $\qquad$ (28" per century)

and the same result is obtained from Gauss's law which

sets

$$F = -\frac{k^2}{r^2}\left[1 + \frac{1}{c^2}\left(2u^2 - 3\left(\frac{dr}{dt}\right)^2\right)\right] \qquad (14)$$

Finally, the much-discussed paper of P. Gerber

which assumes a potential

$$\frac{k^2}{r}\left(1 + \frac{1}{c^2}\,\frac{dr}{dt}\right)^2 \qquad (15)$$

leads exactly the Einstein formula. However, it has

been shown by Laue that Gerber's justification of the

expression for his potential is untenable. $^{(15a)}$ Besides,

it is evident a priori that all these expressions for

the gravitational force or potential are purely em-

pirical. Other empirical expressions, as for example

Seeliger's $\quad -\dfrac{k^2}{r}\,e^{-\beta\,r}\quad$ also give the required pre-

cession, if $\beta$ is suitably chosen. In general for all

such empirical laws we may write,

$$\Delta \phi = \eta\,\frac{4\,\pi^3\,a^2}{T^2 c^2 (1 - e^2)}$$

$\eta$ being an integer, having the following values

$$\begin{aligned}
&\text{Einstein and Gerber:} \quad \eta = 6\\
&\text{Riemann and Gauss} \;: \quad \eta = 4\\
&\text{Weber and Neumann} \;\;: \quad \eta = 2\\
&\text{Lenard and Lorentz} \;: \quad \eta = 1
\end{aligned}$$

Similar considerations have led Gehrcke [15a] to the conclusion that this expression for the advance of the perihelion is a dimensional property, i.e. it must occur in <u>all</u> gravitational theories and Mohorovicic has extended the reasoning to show that it also plays a fundamental part in all other consequences of gravitational theory, as for instance, displacement of spectral lines, deviation of a light-ray, radius of curvature of the universe, etc. But in view of the well-known objections which can be made to so-called "dimensional" theory, too much weight cannot be attached to such conclusions. [16]

We may sum up the above evidence as follows: For motions in a central field, the dynamic equations of classical mechanics $\dfrac{d^2 x_k}{dt^2} = \dfrac{\partial U}{\partial x_k}$ (k = 1,2,3) with their well-known integrals $r^2 \dot{\phi} =$ const. (integral of angular momentum) $\dot{r}^2 + r^2 \dot{\phi}^2 = 2U + c_0 = k^2 \left( \dfrac{2}{r} - \dfrac{1}{a} \right)$ (Jacobi's integral) are only a first approximation to the facts. They must be supplemented by a "relativistic" condition equation $F(x, \dot{x}, t, c)$, but the form of this equation is as yet arbitrary in the sense that <u>several</u> forms of this equation will lead to the <u>same</u> expression for the perihelion advance. In other words, it cannot be maintained that the metric of space-time stands in a <u>necessary</u> causal relationahip to the motion of the perihelion.

The argument may still be presented from another angle.

As is well-known, the relativistic Keplerian orbits are the geodesics in a static-symmetrical spherical gravitational field. The solution of the Einstein gravitational equation $G_{\mu\nu} = 0$ for this case was given by Schwarzschild,[17]. By reasons of symmetry, he chooses polar coördinates, -- _choosing_ the coördinate system strictly amounts to begging the problem, as already brought out clearly by Eddington[18] and Le Roux[19] but specially by Painleve[20] -- and is led to assuming for the first fundamental form an expression $-ds^2 = R dr^2 + R' d\sigma^2 - R'' dt^2$; $R, R', R''$ being functions of $r$ only, $d\sigma^2$ the line-element on a sphere of unit radius and $t$ the relativistic "time". It might be pointed out here that there exists as yet no compelling reason for identifying this parameter t with the ordinary time, as pointed out with extreme clearness by Zaremba and Le Roux[21]. Expressed in symmetrical coördinates

$$d\sigma^2 = -\frac{4 dx\, dy}{(x+y)^2}$$

and we may write, making an obvious change of notation, $-ds^2 = g_{11} dx_1^2 + g_{22} dx_2^2 + 2 g_{34} dx_3 dx_4$ in which $g_{11}$, $g_{22}$ are functions of $x_1$ only and $g_{34}$ is of the form $-\frac{2\phi(x_1)}{(x_3+x_4)^2}$. If $\phi(x_1)$ is not a constant we may always change the variable $x_1$ so as to bring back $g_{34}$ to the form $g_{34} = -\frac{2x_1^2}{(x_3+x_4)^2}$. If now we form the ten Einstein equations $G_{\mu\nu} = 0$, we find that seven of them vanish identically and the remaining three give:

$$G_{11} = \tfrac{1}{4}\left(\frac{1}{g_{11}}\frac{\partial}{\partial x_1}g_{22}\right)^2 + \tfrac{1}{2}\frac{\partial^2}{\partial x_1^2}\log g_{22} - \frac{1}{g_{11}}\frac{\partial}{\partial x_1}$$

$$- \frac{1}{2g_{11}}\frac{\partial}{\partial x_1}g_{11}\left(\frac{1}{2g_{11}}\frac{\partial}{\partial x_1}g_{22} + \frac{2}{x_1}\right) = 0$$

$$G_{22} = \tfrac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{1}{g_{11}}\frac{\partial}{\partial x_1}g_{22}\right) - \tfrac{1}{2}\left(\frac{1}{g_{11}}\frac{\partial}{\partial x_1}g_{22}\right)\left(\frac{1}{g_{22}}\frac{\partial}{\partial x_1}g_{22}\right) +$$

$$+ \tfrac{1}{2}\left(\frac{1}{g_{11}}\frac{\partial}{\partial x_1}g_{22}\right)\left(\frac{1}{2g_{11}}\frac{\partial}{\partial x_1}g_{11} + \frac{1}{2g_{22}}\frac{\partial}{\partial x_1}g_{22} + \frac{2}{x_1}\right) = 0$$

$$G_{34} = \tfrac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{1}{g_{11}}\frac{\partial}{\partial x_1}g_{34}\right) - \tfrac{1}{2}\left(\frac{1}{g_{11}}\frac{\partial}{\partial x_1}g_{34}\right)\left(\frac{1}{g_{34}}\frac{\partial}{\partial x_1}g_{34}\right) + \frac{\partial^2}{\partial x_3 \partial x_4}\log g_{34} +$$

$$+ \frac{1}{2g_{11}}\frac{\partial}{\partial x_1}g_{34}\left(\frac{1}{2g_{11}}\frac{\partial}{\partial x_1}g_{11} + \frac{1}{2g_{22}}\frac{\partial}{\partial x_1}g_{22} + \frac{2}{x_1}\right) = 0$$

$G_{22} = 0$ can be integrated at once, giving

$$\frac{dg_{22}}{dx_1} = \frac{c}{x_1^2}\sqrt{g_{11}g_{22}}$$

c being an integration constant. $G_{11} = 0$ then gives
after a somewhat laborious computation

$$\frac{d}{dx_1}\log g_{11} + \frac{d}{dx_1}\log g_{22} = 0$$

whence $g_{11}\,g_{22} = $ const.

By virtue of the assumptions made above this con-
stant is equal to 1,
and we have $\dfrac{dg_{22}}{dx_1} = \dfrac{c}{x_1^2}$

from which $g_{22} = c' - \dfrac{c}{x}$ and $g_{11} = \dfrac{1}{c' - \dfrac{c}{x}}$

Introducing these in $G_{14} = 0$ gives $c' = 1$ and we have

obtained Schwarzschild's solution

$$ds^2 = \frac{dr^2}{1-\frac{c}{r}} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) - (1 - \frac{c}{r}) \, dt^2$$

But, it should be carefully noted <u>that we have arrived

at this result by a series of suitable restrictive as-

sumptions.</u> In fact there is an infinite number of so-

lutions [22], and hence the choice of Schwarzschild's

form to define the motion of a gravitating particle is

<u>wholly</u> <u>arbitrary</u>. For instance, Painlevé (l.c.) shows

that the form

$$ds^2 = (1 - \frac{c}{r}) \, dt^2 + 2drdt\sqrt{\frac{a}{r}} + dr^2 + r^2(d\phi^2 + \sin^2\theta \, d\phi')$$

gives the <u>same</u> perihelion <u>advance</u> as the Schwarzschild form.

The general conclusion is: Einstein's equations of the

gravitational field are <u>insufficient</u> to fix <u>uniquely</u> the

relativistic Keplerian orbits. As we shall see later

the same objections apply to Nordström's solution for the

field of an electrically charged heavy particle, which

lies at the basis of our further work.

# BIBLIOGRAPHY.

1) Le Roux, Journal de mathématiques pures et appliquées, Vol.1, p.205,1922; also Mohorovicic, Annalen der Physik, Vol.67,p320,1922.

1a) Paschen, Annalen der Physik, Vol.50,p.901,1916; Sommerfeld, Atombau und Spektrallinien,p.588, Braunschweig,1922.

2) Sommerfeld, l.c. Ref.(1a), Chapter VIII.

2a) Lenard, Über Äther und Uräther, pp.35/42,Leipzig,1921.

3) Lenard's remarks to Gerber's memoir in Annalen der Physik, Vol.65,p.593,1921; particularly p.599, footnote 2.

3a) Gehrcke und Lau, Annalen der Physik, Vol.67,p.388, 1922.

4a) Sommerfeld, l.c. Ref.(1a), p.646.

4) Zaremba, Journal de mathématiques pures et appliquées, Vol.1,p.105,1922.

5) Mohorovicic "Die Einsteinsche Relativitätstheorie und ihr mathematischer,physikalischer und philosophischer Charakter" p.53, footnote 73a, Leipzig,1923.

6) Einstein "Die Grundlage der allgemeinen Relativitätstheorie", p.58, Leipzig,1916.

7) E,Cartan, Journal de mathématiques pures et appliquées, Vol.2,p.167, 1923.

8) Einstein, l.c. Ref.(6)

9) G.von Gleich, Annalen der Physik,Vol.72,p.228,1923.

10) Eddington "Report on the Relativity Theory of Gravitation" p.51, London,1920.

11) Sommerfeld, l.c. Ref.(1a) p.738.

12) Tisserand, Mécanique céleste,Vol.IV, p.562.

13) See Enzyklopädie der mathematischen Wissenschaften, VI-2,pp.154 ff.

14) Tisserand, Comptes-Rendus de l'Académie des Sciences de Paris, Vol.110,p.313.

15) Gerber, Annalen der Physik, Vol.52,p.415,1917.

16) See for ex. N.Campbell, Phil.Mag., Vol.47,p.481,1924.

17) Schwarzschild, Sitzungsberichte der preussischen Akademie der Wissenschaften, p.189,1916.

18) Eddington, l.c. Ref.(10) p.43 or Mathematical Theory
    of Relativity, p.82, Cambridge,1923.

19) Le Roux, Comptes-Rendus de l'Académie de Sciences de
    Paris, Vol.172,p.1467, 1921, also l.c.Ref.(1)

20) Painlevé, Comptes-Rendus, Vol.173,p.679,1921.

21) Le Roux, Journal de mathématiques pures et appliquées.
    Vol.1, p.205, 1922.

22) Painlevé, l.c. Ref.(20); cf.also a remark of Weyl,
    "Raum, Zeit,Materie" p.256, Berlin, 1923.

(13a) Tisserand, l.c. Ref. (12)

(15a) Laue, Annalen der Physik, Vol. 66, p.284, 1921.

## The Relativistic Keplerian Orbit in the Hydrogen Atom

We consider from the standpoint of general re-
lativity the motion of a "small" particle (electron)
of mass m and charge -e in the field of a "large"
particle (nucleus) of mass M and charge E, having
spherical symmetry about its midpoint O. If certain
relations, to be discussed later, exist between e, m
and E, M, so that the field of the Large particle is
not appreciably disturbed by the field of the small
particle, the material field of the system, assuming
no other electro-magnetic action in its space-time, can
be also considered to have spherical symmetry about the
center of the large particle. Following more or less
arbitrary considerations of the nature outlined in
dealing with Schwarzschild's solution of the static sym-
metrical gravitational field and choosing the center of the
large particle as the origin of a system of spatial polar
coördinates r, $\theta$, $\phi$, Nordström[1], Jeffery[2] and others[3]
have found the metric fundamental from of the material
field to be:

$$ds^2 = c^2 \left(1- \frac{\gamma}{r} + \frac{\varepsilon}{r^2}\right) dt^2 - \frac{dr^2}{1-\gamma/r+\varepsilon/r^2} -r^2(d\theta^2 + \sin^2\theta \, d\phi^2)...(1)$$

where $\gamma = 2\kappa M$, $\varepsilon = \frac{\kappa E^2}{c^2}$ and $\kappa$ is the gravitational
constant divided by the square of the velocity of light,
$\kappa = 7.42 \times 10^{-29}$ in ordinary c.g.s. units. By the gen-
eral theory of relativity, the equations of motion, are,

in the case considered, (4)

$$\mu\left(\frac{du^k}{ds} - \tfrac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x_k} u^\mu u^\nu\right) = -\frac{\rho}{c} F_{kj} u^j \quad (k = 1,2,3,4) \quad (2)$$

$\mu$ and $\rho$ being the mass and charge density, respectively, of the electron, $u^k = \frac{(dx)^k}{ds}$ the generalized velocity (Eddington) or, more properly perhaps, the world-direction K (Weyl); $F_{kj} = \frac{\partial \phi_k}{\partial x_j} - \frac{\partial \phi_j}{\partial x_k}$ is the electromagnetic tensor and $\phi_k$ the components of the electromagnetic potential. In our case $\phi_1 = \phi_2 = \phi_3 = 0, \phi_4 = \frac{E}{r}$. We define, following Weyl, the mass and charge of the electron by the relations:

$$m\,ds = \mu\,dx \qquad -e\,ds = \rho\,dx \qquad (dx = dx_1\,dx_2\,dx_3\,dx_4)$$

and multiplying (1) by $dx/ds$ obtain Eddington's equation:

$$m\frac{du^k}{ds} = \frac{m}{2}\frac{\partial g_{\mu\nu}}{\partial x_k} u^\mu u^\nu + \frac{e}{c} F_{kj} u^j \quad (k=1,2,3,4) \quad (3)$$

In the first place, we note that, as a consequence of our assumptions, the motion takes place in a plane. This can be seen just as in the case of planetary motion in a static sphero-symmetrical gravitational field as follows: Consider Nordström's fundamental form (1) above, which determines the material field of the nucleus. By our assumptions, the trajectory of the moving electron in $3+1$ space-time is determined by equation (3) above. In our case the only $F_{ik}$'s which do not vanish are, covariantly,

$$F_{14} = -\frac{\partial \phi_4}{\partial x_1} \qquad , \quad F_{41} = \frac{\partial \phi_4}{\partial x_1}$$

hence the equation corresponding to $x_3 = \theta$ is <u>in-dependent</u> of the electric field, because the corresponding components of the electromagnetic tensor vanish. Therefore it depends only on the gravitational field and by following exactly the same process given in detail in Eddington [5], [6], is found to be

$$\frac{d^2\theta}{ds^2} + \frac{2}{r} \frac{dr}{ds} \frac{d\theta}{ds} - \sin\theta \cos\theta \left(\frac{d\phi}{ds}\right)^2 = 0,$$

which is identically satisfied for $\theta = \frac{\pi}{2}$. Hence if $\theta = \pi/2$ initially, $\theta = \pi/2$ always, and $\theta = \pi/2$ may be taken as the orbital plane. Since on it $d\theta = 0$, the line-element (1) reduces to

$$ds^2 = c^2 R^2 dt^2 - R^{-2} dr^2 - r^2 d\phi^2 \ldots \quad (4)$$

where $R^2 = 1 - \gamma/r + \epsilon/r^2$.

The equations (3) corresponding to the index $k = 4$ (variable $t$) and $k = 2$ (variable $\phi$) are

$$\frac{d^2t}{ds^2} + \frac{dt}{ds} \frac{dt}{ds} \frac{\partial}{\partial r} \log R^2 = \frac{eE}{mcR^2 r^2} \frac{dr}{ds}$$

$$mc\frac{d^2\phi}{ds^2} + \frac{2mc}{r} \frac{dr}{ds} \frac{d\phi}{ds} = 0$$

which integrated yield the energy-integral and the integral of angular momentum, respectively:

$$R^2 \frac{dt}{ds} = C + \frac{eE}{mcr} = W + mc^2 + \frac{eE}{mcr} \quad (5)$$

$$mcr^2 \frac{d\phi}{ds} = p \quad \textbf{(6)}$$

C and p are integration constants. We set for C the well-known energy expression of relativistic dynamics.

Although the equation (3) corresponding to the index k = 1, (variable r) is not needed in what follows, it has considerably physical interest, as first pointed out (publically) by Jaffé[7]. It is:

$$\frac{d^2 r}{ds^2} - \frac{1}{2}\left(\frac{dr}{ds}\right)^2 \frac{\partial}{\partial r}\log R^2 - rR^2\left[\left(\frac{d\theta}{ds}\right)^2 - \sin^2\left(\frac{d\phi}{ds}\right)^2\right] +$$

$$+ \frac{R^4 c^2}{2}\left(\frac{dt}{ds}\right)^2 \frac{\partial}{\partial r}\log R^2 = \frac{eEc}{mr^2}\frac{dt}{ds} \qquad (7)$$

From (4) we have:

$$\frac{dt}{ds} = \frac{1}{\left[c^2 R^2 - \left(\frac{d\sigma}{dt}\right)^2\right]^{1/2}} = \frac{1}{\left[c^2 R^2 - \left[R^{-2}\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right]\right]^{1/2}} \qquad (8)$$

where $d\sigma^2$ is the positive definite quadratic form of the space variables r, $\phi$. This is a useful equation which permits changing from derivatives with respect to s (proper time) to derivatives with respect to t (universal time). Using (8) in (6) and (7), we may write:

$$\frac{d}{dt}\left\{\frac{mcR^{-2}}{\left[c^2 R^2 - \left(\frac{d\sigma}{dt}\right)^2\right]^{1/2}}\frac{dr}{dt}\right\} = -\frac{\kappa c^2 Mmc}{r^2\left[c^2 R^2 - \left(\frac{d\sigma}{dt}\right)^2\right]^{1/2}}\left[1 + \frac{1}{c^2 R^4}\left(\frac{dr}{dt}\right)^2\right] +$$

$$+ \frac{\kappa E^2 mc}{r^3\left[c^2 R^2 - \left(\frac{d\sigma}{dt}\right)^2\right]^{1/2}}\left[1 + \frac{1}{c^2 R^4}\left(\frac{dr}{dt}\right)^2\right] + \frac{mcr\left(\frac{d\phi}{dt}\right)^2}{\left[c^2 R^2 - \left(\frac{d\sigma}{dt}\right)^2\right]^{1/2}} - \frac{eE}{r^2} \qquad (9)$$

$$\frac{d}{dt}\left(\frac{mc}{\left[c^2 R^2 - \left(\frac{d\sigma}{dt}\right)^2\right]^{1/2}}r^2\frac{d\phi}{dt}\right) = 0 \qquad (10)$$

the left-hand member of the former equation is essentially a rate of change of momentun -- i.e. a force -- which, since it contains $R^{-2}$, the coefficient of $dr^2$ in the fundamental form (4), is a <u>radial</u> force. Likewise,

the latter equation which contains $r^2$, in the coefficient of $d\phi^2$, is a _tangential force_. If we now consider the right-hand member of (9), we see that the radial force is made up of the following components:

1) The Newtonian attractive force, $-\dfrac{K\,Mm}{r^2}$, which, corrected for finite velocity of propagation, gives the first term on the right-hand member of (9)

2) The repulsive gravitational force between the electric charge E and the mass m, which, except for terms in c, varies as the cube of the distance and gives the second term.

3) The centrifugal force $mr\dot\phi^2$ (third term)

4) The attractive Coulombian force. (fourth term)

We come back now to the equation of the orbit. The fundamental form (4) above may obviously be written

$$c^2 R^2 \left(\frac{dt}{ds}\right)^2 - \frac{1}{R^2}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\phi}{ds}\right)^2 - 1 = 0$$

Introducing in this equation the energy-integral (5) and the integral of angular momentum (6), we have, writing as usual $u = 1/r$, the equation of the orbit:

$$\left(\frac{du}{d\phi}\right)^2 = -\frac{W}{c^2 p^2} - \frac{2mW}{p^2} + \frac{2meE}{p^2}\left(1 + \frac{W}{mc^2} + \frac{Kc^2 mM}{eE}\right) u -$$

$$- \left(1 + k\frac{m^2 E^2}{p^2} - \frac{e^2 E^2}{c^2 p^2}\right) u^2 + 2kM u^3 - \frac{K E^2}{c^2} u^4 \qquad (11)$$

whose integral is $\phi = \int \dfrac{du}{\sqrt{f(u)}}$ . Since $f(u)$ is

an entirely polynomial of the <u>fourth</u> degree in $u$,
the integration can be carried out by means of ellip-
tic functions.[8] It will be noticed that the orbit of

an electron in a static spherical symmetrical material
field depends on a polynomial of the <u>fourth</u> degree,
whereas the orbit of a material particle in a similar
gravitational field depends on a polynomial of the
<u>third</u> degree.[9] A complete discussion of the orbits in
the latter case has been given by De Jans.[10] A brief
discussion of the corresponding problem in the former
case has been given by Weyl[11]. The complete study
of these orbits is, as shown below, of scant interest
and can be left for another occasion.

The problem now before us is to show that, if we
consider the numerical relations between electronic and
nuclear charge and mass respectively in the hydrogen
atom, the above equation of the orbit reduces to the one
obtained by Sommerfeld through the dynamics of restrict-
ed relativity, i.e.

$$\frac{d^2 u}{d\phi^2} = \frac{m\,eE}{p^2}\left(1\ \frac{W}{mc^2}\right) - \left(1 - \frac{e^2 E^2}{p^2 c^2}\right)u \ \ldots \quad (12)$$

Now, differentiating equation (11), we obtain

$$\frac{d^2 u}{d\phi^2} = \frac{meE}{p^2}\left(1 + \frac{W}{mc^2} + \frac{K c^2 mM}{eE}\right) - \left(1 + K\frac{m^2 E^2}{p^2} - \frac{e^2 E^2}{c^2 p^2}\right)u +$$

$$+ 3 \chi M u^2 - \frac{2 \chi E^2}{c^2} u^3 \quad \ldots\ldots\ldots \quad (13)$$

An important remark might be made here. Sommerfeld's equation was obtained by neglecting the departure of the metric properties of space around the nucleus from those of Euclidean space, i.e. by neglecting the Riemann curvature due to the gravitational and electric action of the nucleus. Hence, it was pointed out long ago by Laue [13], in answering Lenard's objection already quoted, that all that is necessary in order to prove that the motion of the electron around the nucleus in the hydrogen atom is governed by the dynamics of special relativity is to show that the quantities $\gamma$ and $\xi$ which determine the departure of the line-element from the Minkowski (Euclidean) form, are small quantities. More precisely, it is required to show that the two quantities

$$\gamma_1 = \frac{2 \chi M}{r} \quad , \quad \xi_1 = \frac{\chi E^2}{c^2 r^2} \quad \ldots \ldots \quad (14)$$

are very small compared to 1. Laue's remark, however, is open to the very serious objection that, if we do so, we neglect altogether the consideration of the ratios m/M and e/E, which, as pointed out above, are essential for our method of reasoning. Laue's remark would clear up Lenard's objection only if it were known a priori that the ratios m/M, e/E are negligible, which is not by any

means the case. It might be argued that our own procedure is based upon just such an assumption, but it must be remarked that our method <u>gives the necessary information as to just how much the orbit depends upon the gravitational and the electrical magnitudes which fix the field</u> as well as the factors introduced by restricted relativity. To be sure, it is only an approximation, but an approximation which reasonably can tell how far it is from the facts.

In this connection, another important remark must also be made here. We have taken no account of the modification of the field due to the moving particle. A measure of this neglected influence is given by the ratio of the quantities $\gamma_1$, $\varepsilon_1$, which fix the curvature of the space-time, to the quotient $\left(\frac{d\sigma}{dt}\right)^2 \bigg/ c^2 R^2$ which measures the rate at which the field singularity due to the electron itself moves about in the pure nuclear field. We require that

$$\gamma_2 = \frac{\gamma_1 c^2 R^2}{\left(\frac{d\sigma}{dt}\right)^2} << 1$$

$$\text{and} \qquad \varepsilon_2 = \frac{\varepsilon_1 c^2 R^2}{\left(\frac{d\sigma}{dt}\right)^2} << 1$$

$$(15)$$

In order to prove these relations and show that our equation (13) reduces to Sommerfeld's equation (12), we must now consider the numerical relations in the hy-

drogen atom. Assume that the electron is in the innermost path $(n_\phi = 1, n_r = 0)$. We have[14]

$e = 4.774 \times 10^{-10}$    statcoulombs (Millikan's value)

$E = 4.774 \times 10^{-10}$    "

$m = 8.99 \times 10^{-28}$    grams

$M = 1.649 \times 10^{-24}$    "

$\left(\dfrac{d\sigma}{dt}\right)^2 \Big/ c^2 = 7.29 \times 10^{-3}$    (Sommerfeld's constant)

$r = 0.532 \times 10^{-8}$    cm.

$$\gamma_1 = \frac{2 \times 7.42 \times 10^{-29} \times 1.649 \times 10^{-24}}{0.532 \times 10^{-8}} = 4.61 \times 10^{-46}$$

$$\varepsilon_1 = \frac{7.42 \times 10^{-29} \times \overline{4.774}^2 \times 10^{-20}}{9 \times 10^{20} \times \overline{0.532}^2 \times 10^{-16}} = 6.62 \times 10^{-53}$$

It follows from this that Laue's conditions (14) are satisfied with extraordinary accuracy and to all intents and purposes the space in which the moving electron moves is Minkowskian. Let us see now how our conditions (15), which measure the departure of our one-body problem from the true two-body problem, are satisfied.

By above $R^2 = 1$, hence, $\gamma_2 = \dfrac{4.61 \times 10^{-46}}{7.29 \times 10^{-3}} = 0.634 \times 10^{-43}$

and $\varepsilon_2 = \dfrac{6.62 \times 10^{-53}}{7.29 \times 10^{-3}} = 0.923 \times 10^{-50}$

and to all intents and purposes, the hydrogen atom can be treated as a one-body problem. Further à plus forte raison, the material field of the hydrogen atom is spherical, symmetrical and static: The material field

of the nucleus is almost non-affected by the moving

electron. These conclusions hold, for still stronger

obvious reasons, in all hydrogenic ions of the type

$He^+$ , $Li^{++}$ , $Be^{+++}$ , etc; but they do not hold at all for

other ionic or atomic types on account of the inter -

penetration of the different electronic orbits. It is

possible then that different orbits may come close to-

gether over certain intervals (15) and as a result the

material field of atoms and ions of non-hydrogenic type

is qualitatively shown to be neither static (16) nor

sphero-symmetrical. This conclusion is of paramount im-

portance for what follows.

We conclude further, that all the terms depending

on the curvature of space-time -- i.e. all the terms

containing $\chi$ -- may be dropped from our equation of the

orbit (13). The latter then obviously reduces to Sommer-

feld's equation: Lenard's objection is now completely

answered.

With regard to electronic orbits of any azimuthal

and radial quantum number, $n_\phi$ and $n_r$, Sommerfeld's

discussion (17) shows that the greatest departure of the

relativistic from the classical Keplerian orbit -- great-

est precession of the perihelion -- occurs for large ec-

centricities, i.e. when $n_\phi = 1$ and $n_r$ is a large number.

But even for the highest values of $n_r$ which can occur

practically in atomic theory, our fundamental quantities $\gamma_1$ , $\gamma_2$ , $\varepsilon_1$ , $\varepsilon_2$ , stay of the same order of magnitude and all our conclusions hold, as an elementary computation readily shows.

The planetary orbits in a pure gravitational field are a particular case of our equation (13) and are obtained from it by placing $e = E = 0$, that is by dropping the curvature terms which depend on electric charges in the fundamental form (1). Simultaneously, of course, the electromagnetic tensor vanishes in (3). But in this case our conditions corresponding to (15) are satisfied for all planets in the solar system, while the Laue conditions (14) are not. <u>We have here the reason why restricted relativity is unable to account for planetary motions.</u> In fact, if we place $e = 0$, $E = 0$, we obtain, from (11)

$$\left(\frac{du}{d\phi}\right)^2 = -\frac{W}{p^2 c^2} - \frac{2mW}{p^2} + \frac{2Kc^2 m^2 M}{p^2}\,u - u^2 + 2KMu^3 \qquad (16)$$

or

$$\frac{d^2 u}{d\phi^2} = \frac{2Kc^2 m^2 M}{p^2} - u + 3KMu^2 \qquad (16a)$$

which are the well-known equations of a planetary orbit in the theory of Relativity (18). The last term causes the departure of the Einstein orbit from the Newtonian orbit. Suppose now we treat this departure as a small orbital perturbation would be accounted for in classical mechanics and write the solution of (16) in the usual form:

$$u = u_0 + u_1 \, \mathbb{K} M + u_2 (\mathbb{K} M)^2 + \ldots \ldots \qquad (17)$$

If now we substitute (17) in (16) and equate coeffi-
cients of M, we obtain a system of linear differential
equations with constant coefficients. The first of these
equations

$$\frac{d^2 u_0}{d\phi^2} = \frac{2 \mathbb{K} c^2 m^2 M}{p^2} - u_0$$

$$(19)$$

gives the Newtonian orbits

$$u_0 = \frac{2 \mathbb{K} c^2 m^2 M}{p^2} (1 + \varepsilon_0 \cos \phi) \qquad (18)$$

where, ~~as in~~ ^analogy to Newtonian dynamics, (c is introduced here only by the units we are using)

$$\varepsilon_0^2 = 1 + \frac{W(1 + 2mc^2)}{c^4} \left( \frac{1}{2 \mathbb{K} M m^2} \right)$$

and we have assumed that $\phi = 0$ is the azimuth of the
perihelion.

The term of the first order in $\mathbb{K}$ gives,

$$\frac{d^2 u_1}{d\phi^2} + u_1 = 3 \left( \frac{\mathbb{K} c^2 m^2 M}{p^2} \right)^2 (1 + \varepsilon_0 \cos \phi)^2$$

whose integral is

$$u_1 = c_1 \cos \phi + c_2 \sin \phi + 3 \left( \frac{c^2 m^2 M}{p^2} \right)^2 \left[ 1 + \varepsilon_0 \cos \phi + \varepsilon_0 \phi \sin \phi + \right.$$

$$\left. + \frac{\varepsilon_0^2}{6} (3 - \cos 2\phi) \right]$$

The integration constants are determined by the condi-
tions that $du_1/d\phi$ vanishes for $\phi = 0$ and that
$u = u_0 + \mathbb{K} M u_1$ gives the correct value of the term which

does not contain  u  in (16).  We get

$$u = u_o + \varkappa M u_1$$

$$= \frac{\varkappa c^2 m^2 M}{p}\left[1 + 3\varkappa MR\left(1 + \frac{\xi_o^2}{2}\right)\right]\left[1 + \xi_o\left[1 + \varkappa MR\left(\frac{1}{\xi_o^2} - \frac{3\xi_o^2}{2}\right)\cos\phi\right] + \right.$$

$$+ 3\varkappa M\xi_o R^2 \phi \sin\phi - \frac{\varkappa}{2} M\xi_o^2 R^2 \cos 2\phi .$$

What might be called the "Einstein orbit of the
first order" differs therefore from the Newtonian orbit:
1) By the value of the constant which determines the
semi-latus rectum of the Newtonian orbit, (1st term)
2) By the precession of the perihelion. It can be easily
shown that the second term $(3\varkappa M\xi_o R^2 \phi \sin\phi)$ gives ex-
actly the Einstein formula, already quoted before,
3) by a departure from the elliptic form.  This is in
agreement with the rigorous analysis given by de Jans.[18a]
The maximum lengthening of the radius vector is, at the
aphelion,

$$\Delta r_{max} = \frac{\varkappa M\xi_o^2}{2(1 - \xi_o)^2}$$

and even for Mercury remains far beyond experimental ob-
servation.

From the fact that, as pointed out before, the in-
tegration of the differential equation of the orbit can
be effected by elliptic (doubly-periodic) functions, we
see at once that all orbits in the hydrogen atom are very
nearly strictly periodic.  The qualification arises here

from the fact that the material field is not <u>rigorously</u>
static nor sphero-symmetrical.  But the electronic
orbits in other atomic systems need not be periodic be-
cause the field is neither static nor sphero-symmetrical.
It may be that this corresponds to the radiation of
energy by gravitational and electromagnetic waves, as
suggested by Jeffery.  But if we admit that the atom is
quantized in the Bohr sense, it must follow that the
quantum postulates govern not only electromagnetic but
also gravitational radiation and the possibility sub-
sists that satisfactory <u>approximations</u> in the stationary
If not exact solutions,
states, may be obtained by periodic solutions of the
types discussed elsewhere in this thesis, as already
(14)
pointed out by the writer.  Jeffery has made the very
interesting suggestion that the solution of the relati-
vistic two-body problem <u>may</u> under certain conditions be-
come periodic, <u>these</u> periodic orbits then would be the
quantized paths.  The fundamental importance of the two-
body problem for the quantum theory thus becomes evident.
We shall now attempt to develop as far as our resources
permit, a different relativistic interpretation of the
quantum theory of stationary orbits, which at least brings
to light in an extraordinarily clear manner the why of
some of its obscure points.  It must be admitted that our
quantization rules which, as known, are essentially an
<u>addition</u> to classical dynamics, are highly artificial:

the real solution of the problem of the quantized
orbits must be independent of any such arbitrary rules.
The paramount importance of Jeffery's suggestion should
not, therefore, be overlooked.

A closing remark which concerns our method of de-
termining the equation of the electronic path must be
made here. Our fundamental equation of motion (2) is
based on empirical facts without any theoretical in-
terpretation and the path of the electron is not a geo-
desic in the space-time described by Nordström's funda-
mental form. This is because the electric force de-
scribes a world-condition which is totally different from
mechanic force. All that we have done is to write the
tensor equation

$$m\left[\frac{d^2x}{ds^2} + \left\{\begin{matrix}\alpha\ \beta\\\sigma\end{matrix}\right\} u^\alpha u^\beta\right] = \frac{e}{c} F_{\mu\sigma} u^\mu$$

which corresponds to the elementary equation of electro-
statics

$$m \frac{d^2x}{dt^2} = X e.$$

$\frac{e}{mc} F_{\mu\sigma} u^\mu$ would then represent the "deviation" of
the actual path from the geodesic path. The problem is
then to explain why the electron deviates from the geo-
desic path. Whether such an interpretation can be worked
out from the generalizations of the Einstein theory given

by Weyl [20] and Eddington [21] is, so far as the writer

is aware, unknown. The most satisfactory interpreta-

tion with which the writer is acquainted is De Donder's,

who, on the basis of his "Théorème du tenseur asymétrique" [22]

is able to give what appears to be the most satisfactory

extant unification of electromagnetic and gravitational

theories.

Another difficulty presents itself here which may

also be carefully noted. We have assumed that the only

effects to be dealt with in the atomic system are those

due to the nucleus and the electron. How is our reason-

ing to be altered if the atom is immersed in another field,

say the gravitational field of the earth? The most

obvious answer is that the atom relies only on the pro-

perties "im Kleinen" of the surrounding space. But im

Kleinen the gravitational field of the earth, for instance,

is Euclidean and we are led to the difficulty, pointed out

already by Mohorovicic [23], of having to assume different

geometries for different regions of the world. Thus in

our case, if we have a nucleus in the earth's field, im

space is
Kleinen Euclidean with respect to the earth and non-Euclid-

ean with respect to the nucleus. This would obviously

be a contradiction. But he has overlooked that we are

dealing with different degrees of smallness. To speak

in very rough terms, take the following example: the

earth, a mountain, a square foot on that mountain.

Within a curved surface (earth) we have a space of

different curvature (mountain, our atom) and in this

mountain (i.e. in this atom) the <u>im Kleinem</u> Euclidean

space. Mohorovicic's objection therefore does not seem to be

valid.

## BIBLIOGRAPHY.

1) G.Nordström, Verslag Koninklijke Akademie van
   Wetenschappen te Amsterdam, Vol.26,p.1204,1918.

2) G.B.Jeffery, Proceedings of the Royal Society of
   London, Vol.99,p.123,1921.

3) See for ex. C.Longo, Nuovo Cimento,p.191,1918, De
   Donder "La gravifique einsteinienne" p.95 and
   p.179, where a reference is given to the work of
   Vanderlinden, Paris, 1921; Weyl,"Raum,Zeit,Materie"
   p.257, Berlin,1923.

4) Weyl, l.c. Ref.(3). Note that Weyl sets c equal to 1.
   Also Eddington "Mathematical Theory of Relativity",
   p.190, Cambridge,1923.

5) Eddington, l.c., Ref.(4) p.85; also Laue "Die Rela-
   tivitätstheorie" Bd.2.,S.216, Braunschweig,1921.

6) Jeffery,l.c. Ref.(2) p.128.

7) Jaffe, Annalen der Physik, Vol.67,p.215,1922.

8) Whittaker and Watson "Modern Analysis" Par.22.7,
   Cambridge, 1916.

9) See for ex. Eddington, l.c. Ref.(4) p.86; De Donder,
   l.c.Ref.(3) p.98;  Weyl, l.c. Ref.(3), p.267.

10) De Jans. "Sur le mouvement d'une particule materielle
    dans un champ de gravitation a symetrie spherique",
    Paris-Bruxelles, 1923 (Publication de l'Académie
    Royale de Belgique)

11) Weyl, Annalen der Physik, Vol.54,p.134,1917.

12) Sommerfeld,"Atombau und Spektrallinien", p.737,
    Braunschweig, 1922.

13) Laue, l.c. Ref.(5), p.238 or Annalen der Physik, Vol.
    66, p.284,1921.

14) Sommerfeld, l.c. Ref.(12) p.9 and p.262.

15) Cf. M.Born Die Naturwissenschaften, Vol.11,p.542,1923.

16) N.Bohr, l.c. Ref.(15) p.615.

17) Sommerfeld, l.c. Ref.(12) p.570.

18) Cf.Eddington, l.c.(4) p.86, De Donder, l.c. Ref.(3)
    p.88, Weyl, l.c. Ref.(3),p.261.

18a) De Jans, l.c. Ref. 10, p.91.

18b) Vallarta, Journal of Mathematics and Physics, Vol.3,

p.181,1924.

19) Jeffery, l.c. Ref.(2) p.130.

20) Weyl, l.c. Ref.(3), p.298.

21) Eddington, l.c. Ref.(4),p. 213.

22) De Donder, l.c. Ref.(3), p.25, 48, 81.

23) Mohorovicic " Die Einsteinsche Relativitätstheorie und ihr mathematischer, physikalischer und philosophischer Charakter" p.70, Berlin,1923.

# The Relativistic Interpretation of the Quantum Theory

Bohr's first postulate is that <u>closed</u> electronic orbits are possible without the simultaneous occurrence of irreversible radiation.  That non-radiating <u>open</u> orbits are indeed possible in classical electro-dynamics was pointed out as early as 1911 by Born,[1] then discussed by S. R. Milner and G. A. Schott,[2] who showed that the only class of motions unaffected by irreversible radiation is that in which the path of the electron is a hyperbola described with a constant velocity component parallel to the conjugate axis of the orbit.  Mie,[3]  in his remarkable article on the field of an electron rotating around a mass centre, has shown that Bohr's postulate is not only by any means the foundation of the modern quantum theory of the atom, but rather the essential hypothesis of <u>any</u> electric theory of matter which is built up on the assumption of the existence of electrons.  He has further shown that a rigorous reasoning starting from the principle of the relativity of gravitation (<u>not</u> of the general principle of relativity) leads to the startling conclusion that an inertial electron rotating around an uncharged gravitational center does <u>not</u> radiate.  The hope of finding a modified Maxwellian theory which is in accordance with the existence of non-radiating orbits

need not, therefore, be given up as yet. Much light
may be thrown on this point if we attempt a relativis-
tic (interpretation of the quantum) theory. One possible answer, depending on the
solution of the relativistic two-body problem, has
already been mentioned. It has also been suggested
that the puzzles on the quantum theory are to be
traced to an unsatisfactory conception of the nature
and properties of the electron. It is apparent that
such an alternative offers possibilities which cannot
be disregarded a priori. It thus becomes necessary to
consider the problem of the electron somewhat more
closely.

Consider again Nordström's fundamental form in the
material field of the electron

$$ds^2 = - \frac{dr^2}{1 - \frac{\gamma}{r} + \frac{\varepsilon}{r^2}} - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) + (1 - \frac{\gamma}{r} + \frac{\varepsilon}{r^2})dt^2$$

$$(4)$$

As pointed out by Eddington $\frac{\gamma}{2}$ can be identified
with the gravitational mass of the electron and $\varepsilon/_{4\pi}$ with
its electric charge. As we approach the electron from
infinity $R^2 = 1 - \frac{\gamma}{r} + \frac{\varepsilon}{r^2}$ decreases continuously
to a minimum at $r = \frac{2\varepsilon^2}{\gamma} = \frac{e^2}{4\pi m}$, if m is the elec-
tronic mass, then increases and reaches the value plus
infinity at $r = 0$. Thus, <u>there is no singularity in the</u>

<u>material field of the electron except at r = 0.</u>  In
the relativity theory it is therefore possible to have
an electron which is rigorously a <u>point-singularity</u>
but has finite mass and finite charge.  In the old
theories of electromagnetic mass, we could not have
a point-singularity, because the mass and charge would
become infinite.$^{(5)}$  The difference between the nature
of the singularity in the case of the material field
and the gravitational field might also be emphasized;
at the same time, the notion of <u>radius</u> of the electron
loses all precise meaning.

The electron, which in the older electromagnetic
theories is looked upon as a substantial aggregate in
a non-substantial field, appears in the relativity
theory as a region which is not at all sharply bounded
with respect to its field.  In his epoch-making "Grund-
lage einer Theorie der Materie"$^{(6)}$ , Mie has investigated
how such "energy knots" move about in space-time just as
a wave moves over the surface of the ocean; this con-
ception, it is seen, means that there is no unicity of
the substance of the electron.  His theory, which has
been further investigated by Weyl$^{(7)}$  is based upon the
postulate, which indeed pervades the whole theory of re-
lativity and is one of its most beautiful consequences,
that not only the field is a consequence of matter, but
conversely matter is generated by the field.  The most

important consequence of this view is <u>that the protons</u>
(8)
<u>and electrons are not the last constituents of matter.</u>
The problem of matter is thus to explain why the field
admits just such aggregates of "energy-knots" as the
protons and electrons and no others. In other words
the problem of matter is solved when we find the ans-
wer to the query: Why do the field laws admit just the
solutions that correspond to the proton and the electron.
Whether the answer to this question involves also the
explanation of the existence of non-radiating orbits,
we do not essentially know, but it must be admitted that
we are yet far from this answer.

An attempt from the converse standpoint is that of
(9)
Tetrode . He looks upon the observable electrons and
protons as the primary entities, assumes that the ac-
celeration of any electron is conditioned by other elec-
trons and looks upon the electromagnetic field as a pure
mathematical entity, which serves a purpose only so long
as the quantum phenomena can be neglected. The essential
new conception of his theory is that the instantaneous
state of an atom is conditioned not only by its past but
by the "arithmetische Mitte der nächsten Vergangenheit
und Zukunft". The absorption of energy is predetermined
by its emission. His speculation is interesting in that
it allows an explanation of the mechanism of radiation,

but has not been worked through sufficiently so far.

All that we have done so far is to find a substitute for the main difficulty of the older electronic theories: why is it that the electron does not explode following the Coulombian repulsion between neighboring parts? An unsatisfactory answer, now discarded, was for example Abraham's: the electron is a rigid sphere and the charge is rigidly bound to its surface. This view is of course untenable in Relativity.

Another well-known attempt is that of Poincaré.[10] His electron is deformable but is prevented from exploding by <u>applying</u> a suitable pressure, of hypothetical origin, on the electronic surface. The Poincaré electron has been re-examined from the standpoint of gravitational theory by van den Berg, de Donder and Vanderlinden[11], but the arbitrary introduction of the applied Poincaré tensor brings into their theory an element completely foreign to the spirit of Relativity. We may therefore sum up the present status of the question in the statement: The relativistic interpretation of the electron problem is itself not yet sufficiently developed to enable the quantum postulates to be referred to the problem of the electron's constitution.

Of course, more or less bold speculations are allowed in the electron problem, provided not too much is claimed for them. One of the most interesting is that

of Mohorovicic,[12] with Lenard[13] the foremost expo-
nent of the modern reversion to the theory of a sub-
stantial ether. Starting from the hypothesis of a
discontinuous gaseous ether, "die wir als eine etwas
Kompressible Flüssigkeit auffassen werden", Mohorovicic
shows, by combining it with his theory of gravitation,[14]
that the pressure and density in this ether decrease
with decreasing distance from matter. Changes in either
of these propagate with finite velocity, and, carrying
out the analogy with a gas still further, finds that
the "temperature" of the ether is proportional to the
gravitational potential, a speculation which originally
is due to Fricke.[15] He is further led to consider two
kinds of mass--baric and antibaric -- a conception due
to O. Wiener.[16] It should be remarked in this connection
that, as already noted by Herglotz, Wiener's equations
are insufficient to determine his "Grundgesetz der Er-
haltung der Äthergeschwindigkeit", for the simple reason
that there are more unknowns than equations. Mohorovicic,
as all his predecessors who adhered to the ether theory,
finds entirely inadmissible values for the ether pres-
sure and density at the proton and electron
$(p_0 = 8.28 \times 10^{43}$ g/cm$^2$, $\rho_0 = 9.2 \times 10^{22}$ g/cm$^3$ for the proton;
$p_0 = 7.16 \times 10^{30}$ g/cm$^2$ , $\rho_0 = 7.96 \times 10^{9}$ g/cm$^3$ for the elec-
tron) so concludes that the electron and proton are not
the last constituents of matter. He thus revives the old

nineteenth century conception of the atom as an aggregate of static tension centers and confronted with the same old difficulties:why do such centers exist? How did they arise? He promises to answer these questions at another occasion. Aside from this, his generalization of the fundamental equations of electrodynamics is quite objectionable, but this point is outside the scope of this paper. Let us now look at the problem from another entirely different standpoint.

Let us postulate, as in the general relativity theory, that all natural phenomena are manifestations of the four-dimensional metric properties of space. The atomistic concept, which is the essential idea in the quantum theory, thus far refused to find a place in relativity, but, as far as the writer is aware, it has not been shown that any contradiction exists between the postulate of relativity and the hypothesis of the quantum theory. To be sure, certain consequences of the restricted theory, especially the inertia of energy, is difficult to interpret from the atomistic standpoint. (17) We now consider the following problems: What are the necessary theoretical principles of any atomistic theory and can these principles be brought into the body of the relativity theory? (18)

In the Mie-Weyl theory, mass-density is a tensor mass itself a vector; since neither a tensor nor a vector

can conceivably have an atomistic structure, mass-density and mass cannot be interpreted atomistically. This, it must be carefully emphasized, is perfectly in accordance with the field theory of matter, as Reichenbächer has shown,[19] and also in accordance with the facts, for obviously no other interpretation except the statistical one is possible for the mass-effects which are interpreted in the relativity theory. The only physical entities which conserve their scalar character in the general theory of relativity are entropy, electric charge and action. The last is just the one to which the quantum theory ascribes an atomistic structure. The conception of an atomistic action in a three-dimensional continuum leads, we believe, to the well-known difficulties of the quantum theory, which finds its worst expression in the concept of the emission of light quanta.

From a relativistic standpoint, the quantum theory of the atom is based upon three principles: the minimal principle, the integral principle and the continuity principle. Let a and b be two universal constants, K the Gaussian curvature of the metric fundamental form $ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} -- K = g^{\mu\nu} R_{\mu\nu}$, if $R_{\mu\nu}$ is the contracted Riemann-Christoffel tensor—the world-function of the field, which in general is a function of the covariant fundamental tensors and their derivatives and of the variables defining the material field. Then the most

g

general form of the Hamilton minimal principle is
written (De Donder)

$$\delta S = \delta \int_R (a + b K + \Lambda) \, d\omega = 0$$

where $d\omega$ is the element of volume, $d\omega = \sqrt{-g}\, dx_1\, dx_2\, dx_3\, dx_4$

and the variation is taken with respect to the fundamen-
tal tensors and their derivatives, within the region R
of space-time considered.  On the boundaries of R, all
the variations vanish. [20]  We postulate that $\delta S = 0$
is satisfied over every finite region of space-time.

For our purposes, a more restricted formulation of
the Hamilton principle is sufficient.  We formulate it,
following Weyl [21]  as follows:  The total action is the
sum of the substance-action of mass

$$(S_m) = \int dm \int \sqrt{-g_{\mu\nu}\, dx^\mu\, dx^\nu}$$

plus the substance-action of electricity

$$(S_e) = \int de \int \phi_\kappa \, dx_\kappa$$

and the field-action of electromagnetic energy

$$(S_f) = \frac{1}{4} \int F_{ik} F^{ik} \, d\omega .$$

In the first two equations, the outer integration is to
be taken over all substance elements, the inner over
that part of the world-line of an arbitrary substance
element which lies inside the world-region considered.
In the last expression, the integration is to be taken

over the world-region considered. dm is the mass element, de the element of charge, $\phi_\kappa$ , as above, the electrodynamic potential, $F_{ik}$ , the electromagnetic tensor. We shall enunciate the <u>integral principle</u> as follows: The line integral of the total action S over any closed world-line is an integral multiple of Planck's element of action h, i.e. $\oint S ds = nh$. In this form, the integral principle was first stated, but not used, by Mecke.[22] The conservation principle will be stated and taken up later.

We return now to the minimal principle and some of its interesting consequences. In the case of an electron, i.e. a single point-singularity, the outher integrations can be carried out at once, giving,

$$S = \int (m_0 c \sqrt{g_{\mu\nu} dx^r dx^\nu} + e \phi_\kappa u^k) ds = \int L(x_k, u^k) ds \qquad (1)$$

and Hamilton's principle gives at once the Lagrangian equations,

$$\frac{d}{ds} \left( \frac{\partial L}{\partial u^k} \right) - \frac{\partial L}{\partial x_k} = 0 \qquad (2)$$

Now, we have, since $\sqrt{g_{\mu\nu} u^r u^\nu} = 1$, identically,

$$p_i = \frac{\partial L}{\partial u_i} = - m_0 c \, g_{ik} \, u^k + e \, \phi_i \qquad (23) \qquad (3)$$

$p_i$ being the impulse vector. Further

$$L(x_k, u^k) = - m_0 c \, g_{ik} \, u^k \, u^i + e \, \phi_i u^i = p_i u^i \qquad (4)$$

and, as shown in De Donder, l.c.,

$$g^{ik} (p_i - e \phi_i) (p_k - e \phi_\kappa) = - m_0^2 c^2 \qquad (5)$$

where, however, it is to be noticed that our m is
written cm in De Donder. From (3)

$$m_0 c \, g_{ik} \, u^k = p_i - e \, \phi_i$$

whence by multiplying by $g^{ik}$, and contracting,

$$m_0 c \, u^k = - g^{ik} (p_i - e \, \phi_i) \tag{6}$$

from which we get the "Hamiltonian" function

$$H(x_k, p_k) = p_i \, \frac{g^{ik}}{m_0 c} \, (p_k - e \, \phi_k) \tag{7}$$

and, from (4) and (7)

$$u^i = \frac{dx_i}{ds} = \frac{\partial L}{\partial p_i} = \frac{\partial H}{\partial p_i} + \frac{p^i}{m_0 c} \tag{8}$$

We now establish the fundamental result that <u>only
for static fields</u> do the above equations in space-time
reduce to the Hamiltonian canonic equations in space <u>and</u>
time. By this last expression: "space <u>and</u> time", we
mean that we <u>privilege</u> the coordinate $x_4$ relatively to
the other three coordinates $x_1, x_2, x_3$. In the case of
a static field $\phi_1 = \phi_2 = \phi_3 = 0$ , so

$$P_4 = - m_0 c \, g_{4k} \, u^k + e \, \phi_4 = - H \tag{9}$$

where $- m_0 c \, g_{4k} \, u^k = - m_0 c^2 + \frac{mv^2}{2}$ can be interpreted

as the Kinetic and $e \, \phi_4$ as the potential energy.

Now, from (5)

$$\frac{\partial p_4}{\partial p_i} = - \frac{g^{ik}(p_4 - e \, \phi_k)}{g^{4k}(p_k - e \, \phi_k)} \tag{10}$$

therefore, one of the two triplets of Hamiltonian e-
quations is

$$\frac{dx_k}{dx_4} = -\frac{\partial p_4}{\partial p_k} \tag{11}$$

which, it will be noted, follows directly from the de-
finition of the impulse vector (3) and hence does not
involve the minimal principle. On the other hand, the
remaining Hamilton triplet, follows directly from the
minimal principle, through equations (2), but can be
reduced to the Hamilton form only if the field is static
(independent of time). We have

$$\frac{dp_k}{ds} = \frac{\partial p_k}{\partial x_j} u^k + \frac{\partial p_4}{\partial x_j} u^4 \tag{12}$$

but by (5):

$$\frac{\partial p_4}{\partial x_j} = \left(\frac{\partial p_4}{\partial x_j}\right)_{p_j} - \frac{\partial p_k}{\partial x_j} \frac{u^k}{u^4} \tag{13}$$

hence the Hamiltonian system is now

$$\left. \begin{array}{l} \dfrac{dp_k}{dx_4} = \left(\dfrac{\partial p_4}{\partial x_k}\right)_{p_j} \\[3mm] \dfrac{dp_4}{dx_4} = \left(\dfrac{\partial p_4}{\partial x_4}\right)_{p_j} \end{array} \right\} \tag{14}$$

Now, if the field is static, $p_4$ does not contain $x_4$ and
the second of (14) vanishes identically. Under this
restriction we may place

$$x_4 = ct, \quad cp_4 = m_0 c^2 + W,$$

and (11) and (14) give, in classical notation,

$$\dot{p}_k = - \frac{\partial W}{\partial x_k} \qquad \dot{x}_k = \frac{\partial W}{\partial p_k}$$

It is therefore seen that from a relativistic stand-
point, the Hamiltonian equations hold only if the
field is static. It has been pointed out before that,
if Bohr's views of interpenetrating orbits are cor-
rect, the field of the electron in an atomic system
of more than one electron cannot in general be static,
hence the motion of the electrons, particularly, let it
be emphasized, the optical electrons, need not be gov-
erned by the canonic equations. We have here a theo-
retical reason for the existence of unmechanical orbits
in the atom $(25)$     i.e. orbits not governed by the canonic
equations and not obtainable through the Hamilton-
Jacobi equation -- which have been so often emphasized
by Bohr and others.

The minimal Hamilton principle, to which in the last
analysis all problems in physics can be reduced, yields
thus four Lagrangian equations (2) in space-time, whereas their
integrals contain in general eight integration constants.
Of these eight parameters, four are fixed by the initial
conditions, and four remain arbitrary. Here is un-
doubtedly a weakness of our present reasoning, to be
traced back to the fact that, in order to bring the problem

within the limits of our ability, we have considered the field of a _single_ electron only.  These arbitrary constants must now be determined quantum-theoretically, and hence the introduction of the _integral_ _principle_. But just as the presence of our arbitrary constants is very likely a result of the limitations in our reasoning, so is the introduction of the integral principle _purely arbitrary_.  There is no doubt whatever in the writer's mind that, as brought out before, the true solution of the _quantum problem lies in the relativistic solution of the two-body problem_.  In other words, it is the opinion of the writer that the periodic solutions of the two-body problem contain _no arbitrary constants_.  These then are the natural quantized orbits, which, on account of the limitations in our reasoning, we have now to fix by introducing arbitrary additional conditions.

To cover the weakness of our previous treatment and fix, then, the remaining four arbitrary constants, we _assume_ with Mecke that the integral principle $\oint L ds = nh$, can be split up by separation of variables into four conditions

$$I_k = \oint p_k \, dq_k = n_k h.$$

For $k = 1, 2, 3$ (space coördinates), these are the well-known Sommerfeld-Wilson phase-integrals, while the rela-

tion corresponding to k $\leq$ 4 (time coördinates) contains something new, which we shall discuss later. Since now we are using a <u>specific</u> coördinate system (separable system), we have given up the general relativistic invariance of $\oint L ds$ and introduced a restriction in the choice of variables which is wholly foreign to the principle of relativity; merely another reflection of the weakness of our reasoning. The arbitrariness and purely heuristic value of the Sommerfeld-Wilson conditions, is, we believe, thrown thereby into an extremely clear light. On the other hand, if we believe that a <u>privileged</u> system of coordinates is implied in the quantum theory, then it cannot be expected that quantization in a system of coordinates chosen at random will lead to correct values for the energy terms, as already pointed out by the writer.[26]

From the orbital equations, the frequency of the emitted radiation is determined in Bohr's theory by the principle of correspondence. Under Mecke's separability assumption, it is of course always possible, as shown exhaustively in Part II, to find a contact-transformation from separable variables to angle variables, which is a fundamental step in the application of the correspondence principle. We now consider the following problem: How is the correspondence principle to be interpreted in

the relativistic four-dimensional metric space? Let us introduce now the continuity principle, which we shall enunciate again somewhat artificially, as follows: The total action remains unchanged for any change of the partial actions $I_k$, i.e. $\Delta S = S_2 - S_1 = 0$. $s_1$ is a function of _four_ quantum numbers $n_k$ and $s_2$ the _same_ function of four different quantum numbers $n_k'$. Assuming, as always, that an introduction of angle variables is possible, we may write, for each of the coördinates $x_k$

$$f(x_k) = \frac{1}{n_k n_k' h^2} \sum_{n_k, n_k' = -\infty}^{\infty} A_{n_k n_k'} \; e^{ \frac{2\pi}{h}(s_2^k - s_1^k)}$$

where, as usual, the coefficients in the Fourier expansion are given by

$$A_{n_k n_k'} = \iint f(x_k, x_k') \; p_k \; p_k' \; e^{j \frac{2\pi}{h}(s_2^k - s_1^k)} dx_k \; dx_k'$$

and, by the continuity principle,

$$s_2^4 - s_1^4 = \sum s_1^k - \sum s_2^k ,$$

whence $f(x_1 x_2 x_3) = \sum_{n_k n_k'} \sum_{k=1}^{3} B_{n_k n_k'}^k \; e^{j \frac{2\pi}{h}(s_2^4 - s_1^4)}$

where the coefficients B are products or sums of products of the coefficients A. This however, is nothing else than Bohr's frequency postulate, for, again under the assumption of a static field

$$s_2^4 - s_1^4 = (p_4' - p_4) x_4$$

and the frequencies of the harmonics in the above
equation are Bohr's frequencies,

$$\nu = \frac{1}{h} (p_4' - p_4) = \frac{1}{h} (E' - E)$$

and the correspondence principle <u>appears</u> therefore
<u>as the natural generalization of the frequency postu-</u>
<u>late</u>. The meaning of the fourth phase integral and the
corresponding quantum $n_4$ is thus seen at once.

We see, therefore, in conclusion, that at least
in a semi-formal way, the quantum postulates can be
brought into a four-dimensional metric without too
much forcing, even though with a good deal of artifici-
ality. Whether now the atomicity of action entails the
atomicity of electricity and, if so, how one is to be
brought into the realm of the other are questions which
for the present we cannot answer.

In its general formulation, the integral principle
states merely that the world-line of any quantized
system must lie <u>within</u> a <u>closed</u> region of space-time.
A coördinate transformation which, be it again noted,
leaves the above formulation invariant, can be interpret-
ed then, in the sense of Einstein's equivalence principle,
as the <u>effect of an external field</u>. Thus, the Zeeman
effect introduces the Coriolis-centrifugal field, which,
from a relativistic standpoint can be identified with a
(27)
rotating coördinate system;   the Stark effect trans-

forms cartesian to parabolic coördinates, etc.  No such
interpretation is apparent in Mecke's restricted form.

BIBLIOGRAPHY.

1) S.R.Milner, Philosophical Magazine,Vol.41,p.405,1921.

2) G.A.Schott,          "                    "  ,Vol.45,p.769,1923.

3) G.Mie, Annalen der Physik, Vol.70,p.489,1916.

4) Eddington,"Mathematical Theory of Relativity" p.186,
   Cambridge, 1923.

5) See for ex. Max Abraham's "Theorie der Elektrizität",
   Vol.2,p.22,Leipzig,1920.

6) Mie, Annalen der Physik, Vols.37,38,39,1912/13.

7) Weyl, Raum,Zeit,Materie,p.210,Berlin,1923; also his
   "Mathematische Analyse des Raumproblems" Berlin,1923.

8) See also Reichenbächer, Zeitschrift für Physik, Vol.
   15,p.276,1923.

9) H.Tetrode, Zeitschrift für Physik,Vol.10,p.317,1922.

10) H.Poincaré, Rendiconti del Circolo Mate,atico di
    Palermo, Vol.21,p.129,1906.

11) De Donder, "La gravifique einsteinienne",p.96,
    Paris,1921.

12) S.Mohorovicic, Zeitschrift für Physik,Vol.18,p.47,
    1923.

13) P.Lenard, "Über Relativitätstheorie, Äther, Gravita-
    tion" Leipzig,1920,;"Über Äther und Uräther",Leipzig,
    1921; then Tomaschek, Annalen der Physik,Vol.73,p.105
    1924 with explanatory remarks by Lenard in the same
    Vol. of the Annalen, p.89.

14) Mohorovicic, Naturwissenschaften,Vol.21,p.145,1922.

15) Fricke "Eine neue und einfache Deutung der Schwer-
    kraft" p.14, Wolfenbüttel, 1919.

16) O.Wiener, Abhandlungen der mathematischen-physischen
    Klasse der sächsischen Akademie der Wissenschaften,
    Vol.38, No.4, 1921,p27.

17) Mohorovicic "Die Einsteinsche Relativitätstheorie und
    ihr mathematischer, physikalischer und philosophischer
    Charakter",p.56, Berlin,1923.

18) Weyl, Raum,Zeit,Materie,p.210, Berlin,1923.

19) Reichenbächer, Annalen der Physik,Vol.63,p.115,1920.

20) De Donder, l.c. Ref.(11),p. 22.

21) Weyl, l.c. Ref.(18) p.195.

22) R.Mecke, Zeitschrift für Physik, Vol.21,p.26,1924.

23) De Donder,l.c.(11) p.85,eq.359.

24)    ibid. p.47.

25) Vallarta, Journal of Mathematics and Physics,Vol.3,
    p.180,1924.

26)    ibid. p.178.

27) Eddington,l.c.Ref.(4) p.112 or De Donder Chapter X.
    l.c. Ref.(11)

Part II. The Application of the Calculus of Per-
turbations to Bohr's Atomic Model.

## General

In Part I of this investigation we have come
to the conclusion that <u>in general</u> the equations of
classical dynamics do not govern the motions of the
electrons in an atomic system of more than one elec-
tron. We now investigate the following problems:
Consider an atomic system with more than one elec-
tron and assume that the motions of the electrons in
the stationary states <u>can</u> be described by the equa-
tions of classical dynamics, is it possible to de-
termine these motions by the methods of the calculus
of perturbations? Are there any limitations and, if
so, how far can they be so determined? Can such a
system be quantized by our present quantum rules? If
not, how are our present conditions for quantization
to be altered?

# The Methods of Quantization of Bohr, Epstein and Born and Pauli

Methods for the application of the calculus of perturbations to the quantum-theory of statinnary orbits in non-conditioned-periodic atomic systems have been proposed by a number of writers. Perhaps the first attempt in this direction was made by Burgers in 1917 and Bohr's fundamental investigation a year later has become almost a classic.[1] However, the two most complete attempts in this direction are those of Epstein[2] and of Born and Pauli[3], which we proceed to examine.

The earliest investigations to find the general integrals of the three-body problem gave the result, already pointed out elsewhere,[*] that terms -- so called "secular" terms -- occur in the eccentricities and inclinations of the planets which increase linearly with time. The "stability theorem" of Laplace and Lagrange showed that these terms are the constant terms of a trigommetric series and so the possibility of finding purely trigonometric expressions for the coördinates of the moving planets was opened up. The importance of this trigonometric expansion is this: Suppose that any coördinate $q_k$ can be expressed as a trigonometric series

$$q_k = \sum_{m_1 \ldots m_s = -\infty}^{\infty} A_{m_1 \ldots m_s} e^{j(m_1 w_1 + \ldots + m_s w_s)}$$

[*] see Appendix.

where the w's are linear functions  of time.  If

$$\sum A_{m_1 \ldots m_n}$$ is finite, then the upper and lower bounds

of the corresponding coördinates can be readily found,

whereby the stability question is solved once for all;

next, the above series, if it exists and is absolutely and

uniformly convergent, gives the value of the correspond-

ing coördinate for _all_ values of the time.  If on the

other hand, the convergence is not absolute and uniform,

no such conclusions can be drawn, and  it will not be

legitimate to apply this expansion outside of the domain

of convergency.  The _form_ of the expansion will be always

the same, but the coefficients will not be the same over

each separate region.

The first to study wuch trigonometric expressions

systematically and thoroughly was Gylden, who, unfortuna-

tely for science, died before he could complete his in-

vestigations.  Long before him, Delaunay had given a

particular instance where the coördinates can be expressed

in this form without giving any attention to the conver-

gency of his method:  In his celebrated "Théorie du Mouve-

ment de la lune" (1860) [4]  he established the Fourier ex-

pansion of the coördinates of the moon and showed that its

motion can be thus satisfactorily described over given

finite intervals.  In this particular case, the perturbing

body (sun) is far removed from the perturbed (moon) and

therefore it is questionable a priori whether Delaunay's

method can be applied to more general cases; on the other hand, it points the way to formal processes leading to pure trigonometric integrals of the three or n-body problem. This was first explicitly carried out by Tisserand[5]. Let us illustrate the process involved by a simple example.

Suppose it is required to find an integral of the canonic equations of a system of two degrees of freedom.

$$\dot{q}_1 = \frac{\partial F}{\partial p_1} \ , \quad \dot{q}_2 = \frac{\partial F}{\partial p_2} \ , \quad \dot{p}_1 = -\frac{\partial F}{\partial q_1} \ , \quad \dot{p}_2 = -\frac{\partial F}{\partial q_2} \ \ldots (1)$$

of the form $F = \sum A_{ij} \cos(i \, p_1 + j \, p_2) + \bar{\Phi}$

where $A_{ij}$ and $\bar{\Phi}$ are assumed to be analytic functions of $q_1$ and $q_2$. From the perturbing function we select any one term $A_{i \, j} \cos(i_1 p_1 + j_1 p_2)$ and write

$$F_1 = \bar{\Phi} + A_{i_1 j_1} \cos(i_1 p_1 + j_1 p_2)$$

If now we consider the canonic system

$$\dot{q}_1 = \frac{\partial F_1}{\partial p_1} \ , \quad \dot{q}_2 = \frac{\partial F_1}{\partial p_2} \ , \quad \dot{p}_1 = -\frac{\partial F}{\partial q_1} \ , \quad \dot{p}_2 = -\frac{\partial F}{\partial q_2} \ \ldots \ldots (2)$$

we may look upon the coördinates thereby determined as the coördinates of an _intermediate_ orbit $O_1$ and by suitable variation of constants of $O_1$ , the equations of the actual path $O$, which is determined by (1), can be satisfied. This introduction of intermediate paths is the characteristic trait of Delaunay's method.

More in general, the process is conceived as follows:

The Hamiltonian function of the dynamical system
under consideration is split into two parts

$$F = F_1 + R_1$$

of which $F_1$ describes a system which is conditioned per-
iodic, and is so chosen that $R_1$ -- the perturbing function --
is "as small as possible"; the motion $O_1$ given by the
function $F_1$ is then called the "first intermediate orbit".
This first intermediate motion $O_1$, since it is conditioned
periodic, can be described by means of angle variables
$w_k = \nu_k t + \ell_k$ and their canonically conjugate momenta $I_k$
(Epstein, l.c., shows that these are canonic variables
not only with respect to $F_1$, but also with respect to $F$).
Next it is _assumed_ that the perturbing function is per-
iodic and regular everywhere within the domain considered.
If so, $R_1$ can be expanded in an absolutely and uniformly
convergent Fourier series

$$R_1 = \sum_{m_1 \ldots m_n = -\infty}^{\infty} A_{m_1 \ldots m_n} e^{j \sum_{k=1}^{n} m_k w_k}$$

the A's being functions of the momenta $I$ . Suppose now
that A is the numerically largest coefficient of this
series. We consider the system

$$F_2 = F_1(I) + A(I) e^{j \sum m_k w_k}$$

It is easy to see that $F_2$ again describes a conditioned
periodic motion which we shall call the "second inter-

mediate motion". We write then,

$$F = F_2( I, w) + R_2( I, w)$$

$R_2$ standing for that part of the Fourier series which does not contain the term A. If we now introduce new angle variables $W'_k$ to describe the motion determined by $F_2$, $F_2$ becomes a function of the canonically conjugate momenta $I'$ and the process may be repeated on the next largest coefficient of the Fourier expansion until formally the desired approximation is obtained.

Epstein does not say a word with regard to how far the process can be legitimately carried out; in fact it would seem that he is of the opinion that nothing is known as to whether it is or not convergent. But, as was first pointed out by Smekal[6], it follows directly from the theorems of Poincare and Bruns that the method cannot in general converge. He further points out that examples can be given in which those particular solutions of dynamic problems, which are conditioned periodic, have at most n-s independent periods, so that Epstein's expression for $F_1$ need not be convergent in any finite region of a continuous n-dimensional manifold. Even assuming that the expansion were convergent, it does not give an approximation to the motion, but only to the energy of this motion, which of course is all that is wanted in the quantization.

Two aspects of the question, both of fundamental importance, are left entirely untouched in Epstein's paper: First, the relation of his solution to the adiabatic hypothesis and second, the connection between his method and the essential features of the principle of correspondence. Let us again examine this question from the standpoint of the quantum-theory of periodic systems.

Periodic properties of the motion of an atomic system, play, as known, a fundamental part in all the investigations of Bohr and his collaborators. He takes the view that, the solution of the canonic equations of a system with n-degrees of freedom being, in general, too complicated "scarcely offers sufficient basis for fixing and describing discrete stationary states of the system", therefore that only such periodic properties need be taken into account. (7)   He further points out that in those cases where it has been possible to fix the stationary states, the general solutions of the canonic equations is of simply or multiply periodic character, but is careful to explain that in more complicated cases "we must be prepared for the fact that the (canonic) equations will not prove sufficient for the description of the motion in the stationary states". Let us review briefly Bohr's method for the determination of the stationary states in periodic systems: An s-periodic non-degenerate system is fixed by

s-conditions $I_k = n_k h$ (k = 1,2...s), where the I's are canonic variables conjugate to the angle variables w, of the system (8) and for a separable system coincide with the phase-integrals $\oint p_k dq_k$ (9). Since the energy of the system is a function of the I's only, it is clear that Bohr's conditions are necessary and sufficient for the quantization of the system. During the motion of the system, the I's remain constant and only the w's increase linearly with time.

Next consider the case where a "perturbation", i.e. an external force which is small compared with the internal forces of the system, is acting on the system. According to the usual procedure of celestial mechanics, described at length elsewhere in this paper, this perturbation is described by considering at each instant the osculating motion of the system, i.e. the motion which would take place if the perturbation at that instant vanished. Let $\lambda \Omega$ ($\lambda$ = perturbing parameter) be the potential of the applied forces and assume, of course, the unperturbed motion to be periodic. Then the osculating motion is also periodic, the rate of change of the canonic variables describing this osculating motion is

$$\dot{I}_k^0 = -\lambda \frac{\partial \Omega}{\partial w_k} \quad , \quad \dot{w}_k^0 = \nu_k^0 + \lambda \frac{\partial \Omega}{\partial I_k} \quad k = 1,2....s$$

$$\dot{\alpha}_\kappa^0 = -\lambda \frac{\partial \Omega}{\partial \beta_\kappa} \qquad \dot{\beta}_\kappa^0 = \lambda \frac{\partial \Omega}{\partial \alpha_\kappa}$$

$$\Omega = \Omega_0(I,\alpha,\beta) + \sum \Omega_{m_1...m_s}(I,\alpha,\beta) \cos 2\pi(m_1 w_1 + ... + m_s w_s + \delta_1 + ... + \delta_s)$$

and the nature of the perturbation is <u>essentially</u> different according as to whether $\Omega_0$ depends on the action variables $I_1 \ldots I_s$ or contains in addition the variables $\alpha$ and $\beta$ which will be so in general of $s < n$. If $s = n$ the perturbations are always of multiply periodic character and the quantum conditions are sufficient to specify the stationary states. For <u>this</u> case a contact-transformation,

$$I'_k = I^0_k + \lambda \frac{\partial S}{\partial w_k} \qquad \alpha'_\kappa = \alpha^0_\kappa + \lambda \frac{\partial S}{\partial \beta_\kappa} \qquad k = 1,2 \ldots n$$

$$w'_k = w^0_k - \lambda \frac{\partial S}{\partial I_k} \qquad \beta'_\kappa = \beta^0_\kappa \qquad - \frac{\partial S}{\partial \alpha_\kappa}$$

where

$$S = \frac{1}{2\pi} \sum \frac{\Omega_{m_1 \ldots m_n}}{m_1 w_1 + \ldots + m_n w_n} \sin 2\pi \, (m_1 w_1 + \ldots + m_s w_s)$$

and the energy of the perturbed motion is given by

$$E_1 = E \, (I^1_1 \ldots I^1_s) + \lambda \, (I^1_1 \ldots I^1_s)$$

but if $\Omega_0$ depends on the $\alpha's$ and $\beta's$, the above process breaks down. In the latter case, a contact-transformation such as the one above does indeed remove all coordinates from $E$, but not from $\Omega$. We find in fact

$$E_1 = E(I^1_1 \ldots I^1_s) + \lambda \Omega_0 (I^1_1 \ldots I^1_s; \alpha^1_1 \ldots \alpha^1_{s-u}, \beta^1_1 \ldots \beta^1_{s-u})$$

$\alpha$ and $\beta$ being functions of t. The secular perturbations of the first order are given by

$$\dot{\alpha}^1_\kappa = -\lambda \frac{\partial \Omega_0}{\partial \beta_\kappa} \qquad \dot{\beta}^1_\kappa = \lambda \frac{\partial \Omega}{\partial \alpha_\kappa} \qquad k = 1,2 \ldots s-u.$$

Assume now that the solutions of these equations are again multiple periodic of perodicity  u.   Then the system is quantized, from Bohr's standpoint, by introducing besides the  s  conditions $I_k = n_k h$ ,  u  <u>new</u> conditions  $I_j^i = n_j h$.  That these conditions are adiabatically invariant follows at once from Burgers proof.

The logical development of Bohr's method is the investigation of M. Born and W. Pauli, Jr. [10]   . It is seen, from our summary presentation of Bohr's method, that his quantization is limited to the perturbation of the <u>first</u> order, which evidently cannot be always sufficient.

We have already presented the methods used in Celestial Mechanics to obtain <u>formal</u> solutions of the dynamical equations as a power expansion in the perturbing parameter.   Born and Pauli seem to be the only writers working on the problem of the quantization of general atomic systems who have páid any attention to the fundamental question of the <u>convergency</u> of the processes they use. They also draw attention to the distinction that must be established in the treatment according as to whether the unperturbed system is or not degenerate.  Just as in Bohr's method, given the Hamilton function  $H_o$  of the unperturbed system in terms of the action variables  $I_k^o$ , the problem is to find functions $I_k$, $w_k$ of $I_k^o$ and $w_k^o$ , which are canonically conjugate and have the same properties

with respect to the perturbed system as the variables $I_k^o$ , $w_o^k$ with respect to the unperturbed system. We shall not go into the detail of Born-Pauli's reasoning, for in the non-degenerate case, it is identical with the Newcomb-Lindstedt method that is common in Celestial Mechanics and has already been expounded at length elsewhere. With respect to the general character of the perturbed orbits, it is to be noted that only <u>periodic</u> changes can take place in the constant orbital elements of the unperturbed motion on account of the fact that these, by hypothesis, can be described by angle variables. In other words, <u>secular perturbations are altogether excluded</u>, if the system is non-degenerate. If the system is degenerate, the method must be essentially modified, and this is the most important contribution of Born and Pauli.

If the unperturbed system is degenerate, Born and Pauli have shown that the problem depends on a solution of a partial differential equation of Hamilton-Jacobi type: if this equation can be integrated by separation of variables, no further difficulties stand in the way of the solution. It is then readily shown that the secular terms can be expanded in Fourier series, but while in the case of periodic terms, the <u>amplitudes</u> are proportional to the perturbing force and the frequencies are finite and <u>constant</u>, in the case of secular terms, the

amplitudes are finite and the frequencies proportional
to the perturbing force. But if the Hamilton- Jacobi
equation is not integrable by separation of variables,
the method breaks down. Bohr then has suggested that
in this case we are not really dealing with insufficient
analytical resources, but that inherently it is no long-
er possible to assign mechanically a discrete set of
stationary orbits; in other words, that "sharp" quanti-
zation is no longer possible and that "sharp" spectral
lines must give way to "diffuse" lines [11]. Whether
this is or is not the case cannot be decided on the basis
of present experimental evidence.

Let us consider the point a little closer. We are
again dealing here with the so-called "higher" and "lower"
commensurabilities, which play, as known, a preponderant
part in Celestial Mechanics. If the mean motions of two
electrons in an atom is approximately commensurable, so
that,

$$n : n_1 = p : q \quad \underline{nearly}$$

(p,q prime integers), we have already seen that large
terms may arise in the perturbation function. Commensur-
abilities are closely connected with degeneracy and cause
librations or oscillations of the orbital elements and,
most important of all, the coördinates are no longer
single-valued functions of the integration constants.
Uncer such conditions "sharp" quantization is obviously

impossible, for the rules $I_k = n_k h$ admit more than
one interpretation. Indeed, it is conceivable that
uniform continuous manifolds do exist, for sufficiently
close "packing" of the singularities of these functions
where no single finite set of rules of the type chosen
by Sommerfeld and Wilson would yield a discrete set of
stationary orbits. The outlook seems slightly better
in the case of "higher" commensurabilities, for then
large terms can occur in the perturbing function and the
general character of the motion remains essentially un-
altered. The following points should however be noted:
1) A term of the form $A_1 \cos (m_1 w_1 + m_2 w_2 + \cdots + m_n w_n)$ in
the perturbing function can only cause small periodic
variations which in the vicinity of the commensurability
is of the order of magnitude of $\sqrt{G}$. 2) The angle $\sum m_k w_k$
increases indefinitely with time, but if the Jacobi co-
ordinates of the perturbed point lie at any time within
a small continuous region, nothing but libration about
fixed angles can take place. 3) If the Jacobi coordi-
nates do not lie within any such region, then the mean
anomaly can undergo only continuous changes. From Bohr's
standpoint, therefore, lower commensurabilities must give
rise to diffuse lines, while higher commensurabilities can
still be more or less adequately handled by his method of
periodic solutions.

In any case, whatever be the nature of these commen-

surabilities, it follows immediately from the theorems
of Bruns and Poincare that it is <u>in general</u> impossible
to describe the motion of a perturbed motion by means
of multiple Fourier series whose degree of periodicity
is equal to the number of degrees of freedom of the
system (general solutions)$^{(12)}$.    Hence it is <u>in general</u>
impossible to describe an atomic system of more than
one electron by angle variables and therefore to intro-
duce constant-in-time action variables $I_k$ which can be
used to quantize the system.  In other words, an atomic
system of more than one electron is in general non-quan-
tizable by our available quantum rules, <u>unless</u> the motion
can be described by particular periodic solutions of the
types we have considered at length elsewhere.

From an entirely different standpoint, the same con-
clusion is arrived at in a recent paper of Smekal,$^{(13)}$
which was recently discussed by the writer.  Smekal points
out that, according to an unpublished theorem of Herglotz,
if a continuous 2n-dimensional region of initial values
of coördinates and momenta exist where n-fold Fourier
series converge everywhere uniformly, then a contact-trans-
formation can be always found which permits obtaining the
general solutions directly; but since then all the coördi-
nates are cyclical, the system is separable and con-
ditioned-periodic.  Hence the conclusion drawn by the

writer. No general solution of a non-conditioned periodic system can be expressed as a uniformly convergent Fourier series over any finite region of p,q. This is of course closely allied with the fact, known since the days of Poincare, that the "stable" [14] solution of the n-body problem do not generate any continuous 2n-dimensional manifold. One important point should not be overlooked here: If the correspondence principle has at all anything but a purely formal meaning, the Hamiltonian function must be a <u>continuous</u> function of the action variables $I_1 \ldots I_n$. Hence the conclusion: The demands of the correspondence principle are in sharp contradiction with the properties of the general solutions of the n-body problem. Otherwise expressed, <u>even after</u> we neglect the radiation reaction, the requirements of dynamics and electrodynamics cannot be <u>simultaneously</u> satisfied.

Again, if the quantized stationary states are to be looked for among the general solutions, then the <u>stability</u> of the atom becomes shrouded in darkness, for we cannot bring any proof that the distances of the electrons from each other and the nucleus stay alsays within finite bounds. This seems to be a not inconsiderable argument for the adoption of periodic solutions, in the absence of anything better.

The writer is as yet unable to state whether such particular periodic solutions will yield satisfactory values

for the energy-terms, but at the same time is not in a
position to agree with the remark of Bohr,[15] that the
class of periodic solutions that he has discussed in his
paper is the same as the class first proposed by Smekal
and discussed by the writer. But the full discussion of
this point must be left until later.

In spite of the fact that the method of Born-Pauli
is not in general convergent, -- except semi-convergent
over limited intervals of time, as shown by Poincare[16] --
these two authors believe that it is sufficient for the
calculation of energy levels, because, they say " Man
kann die doch ziemlich spärlichen Glieder mit allzu
kleinen Nennern einfach fortstreichen" which, on their
view, is justified because the "diffuse" character of
the lines, caused by commensurable relation, is negli-
gible compared to other agencies, like Doppler effect etc.
That, aside from any of the purely theoretical arguments
which we have brought up, this point of view is actually
untenable, is best shown by Born and Heisenberg's[17] at-
tempt to apply the method in question to the quantization
of the excited helium atom. In this remarkable paper, the
first systematic investigation of all possible orbits in
the excited helium atom is carried out, with a view to
obtaining the values of the energy terms. The result is
negative: no values which are at all comparable to ex-
periment are obtained. How much the theoretical values

differ from the experimental ones is best shown by
the following table:

### Table I

| | Azimuthal quantum number of outer electron | | | | | |
| | 2 | | 3 | | 4 | |
| | Calc. | Obs. | Calc. | Obs. | Calc. | Obs. |
| Orthohelium | -0.0625 | -0.067 | -0.0289 | -0.0036 | -0.0166 | -0.00103 |
| Parhelium | 0.0137 | 0.0103 | 0.040 | -0.0025 | 0.0017 | -0.0014 |

From the point of view which has been developed in
the preceding pages, no other result could indeed be ex-
pected. It is, granted of course that Born and Heisenberg's
proof that the calculus of perturbations can be applied
to the case on hand is correct, but again it follows direct-
ly from the theorem of Bruns that, aside from the solu-
tions found by Born and Heisenberg, infinitely many other
solutions of non-conditioned-periodic type exist in the
immediate neighborhood of their solutions, which may
be stable in a dynamical sense, and which cannot be obtained
by their method. This, let it be noted, introduces con-
siderable difficulties from the point of view of the adia-
batic principle and the correspondence principle. If we
require, according to the former, that any one stationary
orbit may be adiabatically transformable into a correspond-
ing orbit, we cannot carry out the transformation for the
simple reason that the initial orbit is not continuously
connected with any other orbit: In other words, the first

requirement of the adiabatic principle -- "infinitely

slow change of the parameters" has no sense whatever

here, because any infinitesimal change of the orbital

parameters will lead at once through an unallowable dis-
(18)
continuity      to an entirely different orbit.

The difficulties are no less when we consider the

matter from the standpoint of the principle of corres-

pondence.  Suppose we require that only such changes

between quantized orbits are allowable which can be con-

nected by a continuous manifold of conditioned periodic

solutions.  By Bruns' theorem, no such manifold exists.

Next, assume that the solutions are continuous not in

an analytical sense but in the sense of the correspond-

ence principle.  Born and Heisenberg then find that other-

wise allowable orbits become unstable.  We must conse-

quently alter our whole conception of stability in order

to keep this second hypothesis.

Epstein's method can mean anything only if the De-

launay transformation from which the successive approxi-

mations are obtained, actually does result in decreasing

the perturbing function.  This can only be the case if

there exists any definite reason for eliminating a given

Fourier term.  The Delaunay method assumes in fact that
(19)
none of the intermediate orbits has a libration in the p's.

If this is not the case, we must take into account the

possibility that by a variation of the elements of such

an intermediate orbit we are led alternately within and

without the orbit in question. How the trigonometric

integrals are to be derived in this case is totally un-

known, so far as the writer is aware. Such a fadt is

entirely disregarded in Epstein's method. It is just

for this reason that his quantization of the crossed

electric and magnetic fields cannot be looked upon as

free from objection, for it is clear that, if nothing

is known as to how large the perturbing function may

become after the approximating process has been consid-

ered ended, the possibility is always extant that that

part which has been neglected may be actually of the

order of magnitude, or greater, than the part which has

been taken into account.

Before closing this paragraph, a general fundamental

difficulty in the application of the methods of the

calculus of perturbations to quantum-theoretical problems,

which has already been emphasized several times before,

might properly be discussed here: We refer to the pres-

ence of commensurabilities in the motions of electrons in

quantized atomic systems. The difficulty has already been

discussed by Born and Heisenberg [20], but neither ex-

hausted nor cleared up, in the writer's opinion.

The difficulty in question may be concisely expressed

as follows: In the applications of the calaulus of per-

turbations in celestial mechanics, the case where commen-

surable periodic times exist among the planets can be re-

garded as infinitely improbable, i.e. it is reasonably assumed that no <u>exact</u> commensurabilities can occur. It is on this fact that Poisson's stability proof is based. [21] On the other hand, such commensurabilities <u>must</u> be present in the motion of electrons in atomic systems, for that is just what the quantum conditions require of the unperturbed motion, as already pointed out by Born and Heisenberg themselves. [22]  It has been shown exhaustively in this paper that when such commensurable ratios do exist, the Newcomb-Lindstedt method, which lies at the basis of the developments of Born and Pauli, breaks down at the outset. Hence the important conclusion: If our quantum conditions are correct, the Newcomb-Lindstedt perturbation method is <u>not applicable in general</u> to atomic systems. In other words, if our quantum conditions are correct <u>all</u> atomic systems are <u>inherently</u> degenerate, which may serve as another argument for the adoption of periodic solutions of lower periodicity than the number of degrees of freedom.

It is true that in such a case of degeneration we may still evade the difficulty by utilizing the Bohlin expansion (23) in powers of $\mu^{\frac{1}{2}}$ instead of the Newcomb-Lindstedt expansion in powers of $\mu$ . But again here the expansion is <u>not</u> convergent (24). Hence the following conclusion: For purely analytical reasons, the methods of the calculus of perturbations are not in general applicable to atomic systems with more than one electron.

# BIBLIOGRAPHY.

1) Bohr. Quantentheorie der Linienspektren, pp.58/88, Braunschweig, 1923.

2) Epstein, Zeitschrift fur Physik, Vol.8,p.211,1921.

3) Born and Pauli, Zeitschrift fur Physik, Vol.10,p.137, 1922.

4) Poincaré, "Les méthodes nouvelles de la mécanique céleste" Vol.2,p.315, Paris 1893.

5) Tisserand, Annales de l'observatoire de Paris, Vol. 18, 1885.

6) In a reference to Epstein's paper, Note (2), in Physikalische Berichte, Vol.3,p.1109, 1922.

7) Bohr, Zeitschrift fur Physik, Vol.13,p.117,1922; also Annalen der Physik, Vol.71,p.228,1923.

8) Bohr, l.c. Note (1), p.39, footnote.

9) Burgers, Annalen der Physik, Vol. 52,p.202, footnote, 1916.

10) Born and Pauli, l.c. Note (3), p.141.

11) Bohr, Ref.(1) pp.134/140,especially footnote on p.140.

12) Vallarta, Journal of Mathematics and Physics, Vol.3, p.108,1924.

13) Smekal, Zeitschrift fur Physik, Vol.11,p.294,1922.

14) Poincaré, l.c. Ref.(4), Vol.3 Ch.XXVI,Paris,1899.

15) L.c.Ref.(7), p.134, footnote.

16) Poincaré, l.c. Ref.(4), Chap.VIII.

17) Born and Heisenberg, Zeitschrift fur Physik, Vol.16, p.229, 1923.

18) Cf. on this point the discussion given by Léon Brillouin, "La théorie des quanta et l'atome de Bohr", p.162, Paris, 1922.

19) Cf. Charlier, Mechanik des Himmels, Vol.2, p.446, Leipzig, 1907.

20) Born and Heisenberg, Zeitschrift fur Physik, Vol. 14, p.44, 1923.

21) Poincaré, l.c. Ref. (14) Vol. 3, p.141.

22) Born and Heisenberg, l.c. Ref.(20), p.46.

23) Poincaré, l.c. Ref.(21), Chapter XIX, XX.

# The Form of the Integrals of the Equations of Dynamics
## Discussion of Poincare's Theorem

In 1887 Bruns [1] showed that the three-body problem admits no algebraic integrals other than those already known. Two years later Poincaré [2] proved that not only are there no other algebraic integrals but even that no uniform transcendental integral over any finite bounded domain can exist. While Poincaré established his theorem only for sufficiently small masses, Bruns proved his for any system of masses.

We have been led elsewhere in this thesis to the study of conservative dynamical systems for which the energy F, besides depending on the coordinates and momenta, is an analytic function of a parameter $\mu$, which, for sufficiently small values of $\mu$, can be expanded in a power series in $\mu$ : $F = F_0 + \mu F_1 + \mu^2 F_2 + \ldots$. In the case of a periodic system, the configuration of the system -- and hence all the $F_k$'s except $F_0$ -- depend periodically upon the q's. $F_0$ is a function of momenta only (px). The canonic equations of such a system are called, as known, normal equations. All perturbation problems studied in celestial mechanics are, as we have shown elsewhere, reducible to normal canonical equations and Poincaré [3] proved that such systems do not admit any other

first integral, analytic everywhere and independent of time, besides the energy integral. Geometrically this means, as pointed out by Fermi [4] that in 2n-phase space $\Gamma$ it is impossible to find any family of hypersurfaces, other than that for which $F = $ const., such that the trajectory of the representative point in $\Gamma$ lies wholly on one of these hypersurfaces. We shall now proceed to discuss the geometrical significance of Poincaré's theorem and show that for $n > 2$ no continuous analytic hypersurface exists, besides the energy-surface, which can wholly contain a dynamical trajectory issued from one of its points. Analytically formulated: No continuous analytic function $\Phi (q, p, \mu)$ satisfying the normal canonic equations exists such that if $\Phi = 0$ initially, $\Phi = 0$ always, unless from $\Phi = 0$ follows identically $F = $ const.

Suppose the surface $S_\mu = \phi (q, p, \mu) = 0$ has the above property; it is required to show that in general $S_\mu$ coincides with one surface of the family $F = $ const. Expand $\Phi$ in a power series in $\mu$ and let $\Phi = \Phi_0 + \mu \Phi_1 + \mu^2 \Phi_2 + \cdots$. where $\Phi_\kappa$, except $\Phi_0$, are functions of p and q. We see at once that if $S_\mu$ is known for all values of $\mu$, the $\Phi_\kappa$'s are not completely determined, for $\Phi_0$ is assumed to vanish on $S_0$, but otherwise is arbitrary, $\Phi_1$. is determined on $S_0$ by the condition that for sufficiently small $\mu$ the two surfaces $S_\mu$ and $\Phi_0 + \mu \Phi_1 = 0$ differ by quantities of

the order of $\mu^2$, but otherwise is arbitrary outside of $S_o$ and so on for all other $\bar{\Phi}$'s. We proceed to show now that $\Phi_o$ can be chosen independently of p.[5] Analytically, the proof has already been given before. Geometrically it is necessary to show that all points having a coördinate p which belong to a point on $S_o$ are on $S_o$ and nowhere else. But if $P_o$ is a point on $S_o$, $S_o$ contains the entire trajectory of the phase-point of the unperturbed system which starts at $P_o$. If then $F_o$ does not involve the p's, the points of this trajectory are such that for them p = const., while the q's are linear functions of time, i.e. $q_k = \nu_k t + \delta_k$. In n-space (q) this trajectory is a straight line. Since F is periodic by hypothesis, we may transform all points on these lines to points within a hypercube of side $2\pi$ and vertex at the origin, which will be filled everywhere densely by the transform $T_o$, provided no commensurabilities exist among the $\nu_k$'s. In this case all the points within the unit cube are the transforms of points on $S_o$. If on the other hand relations of the form $\sum m_k \nu_k = 0$ ($m_k$ integers) exist on $S_o$ then we may always assume that such a relation does not hold identically <u>throughout space</u> and that $S_o = \Phi_o = \sum m_k \nu_k \gtreqless 0$ is the equation of $S_o$, for $\Phi_o$ is wholly arbitrary. Hence in every case, $\bar{\Phi}_o$ is independent of q, which was to be shown.

Analytically, the property that $S_\mu$ contains the tra-

jactories issued from one of its points is expressed by saying that $(F_1, \bar{\Phi}) = 0$ (6) is a consequence of $\bar{\Phi} = 0$. If then from this last equation we get for example $p_1$ as a function of $p_2 \ldots p_n$, $q_1 \ldots q_n$, $\mu$, substitute in $(F_1, \bar{\Phi}) = 0$ and expand in a power series in $\mu$, the coefficients of all powers of $\mu$ should vanish. Now, in this expansion the first term $(F_0, \Phi_0)$ vanishes identically throughout space, because $F_0$ and $\bar{\Phi}_0$ depend on $p$ only; the coefficient of $\mu$ is $(F_0, \Phi_1) + (F_1, \Phi_0)$ calculated on $S_0$ and we must have, on $S_0$, $(F_0, \Phi_1) + (F_1, \bar{\Phi}_0) = 0$.

Since $F_1$ and $\bar{\Phi}_1$ are periodic in the q's, we may write

$$F_1 = \sum_{m_1 \ldots m_n = -\infty}^{\infty} A_{m_1 \ldots m_n} \, e^{j(m_1 q_1 + \ldots + m_n q_n)}$$

$$\bar{\Phi}_1 = \sum_{m_1 \ldots m_n = -\infty}^{\infty} B_{m_1 \ldots m_n} \, e^{j(m_1 q_1 + \ldots + m_n q_n)}$$

where the coefficients A and B are functions of the p's only. We have, after an easy calculation $(F_0, \bar{\Phi}_1) + (F_1, \bar{\Phi}_0) =$

$$\sum_{m_1 \ldots m_n = -\infty}^{\infty} e^{j(m_1 q_1 + \ldots + m_n q_n)} \left( A_{m_1 \ldots m_n} \sum_j m_j \frac{\partial F_0}{\partial p_j} - B_{m_1 \ldots m_n} \sum_k m_k \frac{\partial \bar{\Phi}_0}{\partial p_k} \right) = 0$$

which must be identically satisfied if for any $p_j$ we write its value in terms of the remaining p's. On $S_0$ all the coefficients must vanish, i.e.

$$A_{m_1 \ldots m_n} \sum_j m_j \frac{\partial F_0}{\partial p_j} = B_{m_1 \ldots m_n} \sum_k m_k \frac{\partial \bar{\Phi}_0}{\partial p_k}$$

from which follows that if, exceptionally, on a point

on $S_o$ we have $\sum m_k w_k = 0$, we also have $\sum_k m_k \frac{\partial F_0}{\partial p_k} = 0$

because in general $B_{m_1 \ldots m_n} \neq 0$. In the system of two

degrees of freedom ($n = 2$) it may happen that $S_o$ has for

its equation $\nu_2 / \nu_1 = \alpha$, $\alpha$ being an irrational number. As

known, it is then impossible to find two integers $m_1$,

$m_2$ such that for a point on $S_o$ $m_1 \nu_1 + m_2 \nu_2 = 0$. In

this case we have

$$A_{m_1 \ldots m_n} = \frac{B_m}{m_1 \nu_1 + m_2 \nu_2} \sum_k m_k \frac{\partial \bar{\Phi}_0}{\partial p_k} \qquad (1)$$

and, formally at least, all the $\Phi$'s may be determined.

It remains to discuss the convergency of the series, but

in this case it is always possible so to choose $\alpha$ that

the series does converge $^{(7)}$. For the system with 2 de-

grees of freedom we thus have that besides the family of

hypersurfaces $F =$ const., an infinite number of families

of hypersurfaces $\bar{\Phi} =$ const. exist having the property

above.

On the other hand, if $n > 2$, matters are consider-

ably altered. Let for simplicity, but without loss of

generality $n = 3$. We may then have the following cases:

1) Of the three ratios $\nu_1 / \nu_2$ , $\nu_2 / \nu_3$ , $\nu_3 / \nu_1$ , one of them

is rational, but neigher of the other two are constant.

Then, on $S_o$ a dense set of points exists for all points

on which, say $\nu_1/\nu_2$ is rational. For any one of the points forming this set, two integers $m_1$, $m_2$ can be found such that $m_1\nu_1 + m_2\nu_2 = 0$. And then by the equation (1):

$$m_1\frac{\partial \Phi_0}{\partial p_1} + m_2\frac{\partial \Phi_0}{\partial p_2} = 0$$

and hence $\quad \nu_1/\nu_2 = \dfrac{\partial \Phi_0}{\partial p_1} \bigg/ \dfrac{\partial \Phi_0}{\partial p_2}$

and since the points for which this holds from a dense set on $S_0$, the latter must be identically satisfied. Likewise we may show that $\nu_3/\nu_1 = \dfrac{\partial \Phi_0}{\partial p_3} \bigg/ \dfrac{\partial \Phi_0}{\partial p_1}$

Hence on $S_0$ we have

$$\nu_1 : \nu_2 : \nu_3 = \frac{\partial \Phi_0}{\partial p_1} : \frac{\partial \Phi_0}{\partial p_2} : \frac{\partial \Phi}{\partial p_3}$$

and since $\nu_1 : \nu_2 : \nu_3 = \dfrac{\partial F_0}{\partial p_1} : \dfrac{\partial F_0}{\partial p_2} : \dfrac{\partial F_0}{\partial p_3}$ by the

canonic equations therefore the derivatives of $\Phi_0$ and $F$ with respect to correesponding p's are proportional. Therefore $\Phi_0 = 0$ coincides with one of $F_0 = $ const., for example, with $F_0 = -c_0$. On $S_0$ we have now $\Phi_0 = F_0 + c_0$

and since $\Phi_0$ is arbitrary outside of $S_0$, we may take $\Phi_0 = F_0 + c_0$ throughout space. $S_0$ being thus determined $\sum m_k \nu_k$ cannot vanish identically on $S_0$ unless all the m's vanish. Eliminating this case, we may divide through

both sides of (1) by $\sum m_k \nu_k = \sum m_k \dfrac{\partial \bar{\Phi}_o}{\partial p_k}$ and

find $A_{m_1 \ldots m_n} = B_{m_1 \ldots m_n}$ , except $A_{o \ldots o} \neq B_{o \ldots o}$ ,

leading to $\bar{\Phi}_1 = F_1 + f_1 (p_1 \ldots p_n)$. We thus have shown

$$\left.\begin{aligned} \bar{\Phi}_0 &= F_0 + c_0 \\ \bar{\Phi}_1 &= F_1 + f_1 (p_1 \ldots p_n) \end{aligned}\right\} \qquad (2)$$

Lastly suppose that for a certain r we have shown

$$\left.\begin{aligned} \bar{\Phi}_0 &= F_0 + c_0 & \bar{\Phi}_{r-1} &= F_{r-1} + c_{r-1} \\ \bar{\Phi}_1 &= F_1 + c_1 & \bar{\Phi}_r &= F_r + f_r(p_1 \ldots p_n) \end{aligned}\right\} \quad (3)$$

the c's being constants. We want to prove now that ana-
logous equations hold for $r+1$. For $r = 1$, (3) and (2)
are identical, so for any r $\bar{\Phi}_r - F_r =$ const. and hence
$S_\mu$ coincides with one of the surfaces of the family
$F =$ const. From (3) we see that the first $(r+1)$ terms
of the expansion of $\Phi = 0$ vanish identically throughout
space and that the coefficient of $\mu^{r+1}$ is

$$(F_0, \bar{\Phi}_{r+1}) + (F_1, f_r) + (F_{r+1}, F_0) = (F_0, \bar{\Phi}_{r+1} - F_{r+1}) +$$

$$+ (F_1, f_r) = F_r$$

taken on $S_0$. On $S_0$ we must therefore have

$$(F_0, \bar{\Phi}_{r+1} - F_{r+1}) + (F_1, f_r) = 0 \qquad (4)$$

and reasoning on this equation exactly as above, it is

found that on $S_o$ the partial derivatives of $F_o$ and $f_r$ with respect to all the p's are proportional and since on $S_o$, $F_o$ is a constant, $f_r$ must also be a constant, but since $f_r$ is arbitrary outside of $S_o$, we may put $f_r = $ const. $= c_r$ throughout space $\Gamma$ . (4) then reduces to $(F_o, \bar{\Phi}_{r+1} - F_{r+1}) = 0$ and letting

$$\bar{\Phi}_{r+1} - F_{r+1} = \sum_{m_1 \cdots m_n = -\infty}^{\infty} c_{m_1 \cdots m_n} e^{j(m_1 q_1 + \cdots + m_n q_n)}$$

(4) becomes,

$$\sum_{m_1 \cdots m_n = -\infty}^{\infty} e^{j(\sum_{k=1}^{n} m_k q_k)} c_{m_1 \cdots m_n} \sum_{k} m_k \nu_k = 0$$

Therefore on $S_o$, $c_{m_1 \cdots m_n} \sum m_k \nu_k$ must vanish and for all $C$'s except $C_{o \cdots o}$ must vanish on $S_o$. Let us write

$$C_{o \cdots o} = f_{r+1} (p_1 \cdots p_n) \qquad \text{on } S_o ;$$

since $\bar{\Phi}_{r+1}$ is arbitrary outside $S_o$, we may extend this equation throughout space. Our proof is now complete. The <u>sufficient,</u> but not necessary, conditions for the validity of this proof are:

1)  $n > 2$.

2)  No linear relations exist between the fundamental frequencies $\nu_k$ of the unperturbed system.

3)  None of the relations $\nu_i / \nu_j = $ const. follows from another similar relation $\nu_k / \nu_l = $ const.

4)  $\sum m_k \nu_k$ does not vanish on any of the surfaces $F_o = $ const. except for $m_k = 0$ $(k = 1, 2 \cdots n)$.

5) None of the coefficients $B_{m_1 \ldots m_n}$ vanish on $F_0 = \text{const.}$ for $\sum m_k \nu_k = 0.$

If commensurable relations exist between the periods, some of the terms of the perturbation function are no longer periodic and become secular. This case will not be examined here.[8]

# BIBLIOGRAPHY.

1) Bruns, Acta Mathematica, Vol.2, p.25, also Whit-
   taker, Analytical Dynamics, Chapter XIV, Cambr-
   idge,1917.

2) Whittaker, l.c., also Poincaré "Les methodes nou-
   velles de la mecanique celeste" Vol.1,p.253, Pa-
   ris, 1892.

3) Poincaré, l.c. Ch.V, Vol.1,p.233; also Whittaker,
   Analytical Dynamics, p.381.

4) E.Fermi.  Nuovo Cimento, Vol.26,p.106, 1923.

5) Poincaré. l.c. p.236, par. 82.

6) For the meaning of the Poisson symbol (F, Φ ), see
   this thesis, Appendix p.   or Poincare, l.c. Vol.
   1, p.14, also Whittaker, p.299.

7) Poincaré, l.c. Ch.III, p.44, also Charlier, Mecha-
   nik des Himmels, p.255,Vol.2, Leipzig, 1907.

8) Poincaré, lc. p.240, Vol.1.

## The Adiabatic Invariance of Quantized Orbits

It is well-known that the adiabatic theorem of Ehrenfest [1] states that if a dynamical system is at a certain instant in a quantized state and its "mechanism" -- in the larger sense given to this term by P. Hertz [2] -- is altered by an infinitely slow adiabatic process, this system stays quantized in the same state throughout and at the end of the adiabatic transformation.

In order that the adiabatic theorem may have a definite meaning it is required that the final configuration be independent of the way in which the transformation from the initial state is carried out, i.e. independent of any one of the intermediate states. That this is so for non-degenerate systems which admit a separation of variables, more generally, for systems which can be described by angle variables [2a], in the initial and all intermediate states, was shown by Burgers [3]. How important this last clause is, is pointed out very clearly in an example due to Fermi [4]. Degenerate systems have also been discussed by Burgers who has shown that the energy of such systems is adiabatically invariant.

The simpler cases of atomic theory which belong to the conditioned-periodic type having been studied with

greater or less thoroughness -- Stark and Zeeman effects, fine structure, etc. -- the problems which are now presenting themselves for consideration belong to the class which is non-separable so far as known, and does not admit of an expression in terms of angle variables, above all the helium atom, an example of a three-body problem, and the hydrogen molecule, an example of the (4a) "reduced" four-body problem. All attempts to reduce the treatment of such systems to that of conditioned-periodic systems may be now looked upon as failures. The object of the present part of this paper is then: How can the adiabatic theorem be extended to non-conditioned-periodic systems? It is clear that the answer to the question may lead to information as to what quantum conditions fix the stationary orbits in this more general type of dynamical system.

The system is assumed to be holonomic and conservative, of $n$ degrees of freedom. As usual, its phase is fixed by the coördinates of a point P -- the phase-point -- in 2n-phase-space $\Gamma$ . Through every point of $\Gamma$ we can pass a hypersurface E, characterized by the fact that for all points on it, the energy of the evolving system is a constant. If we set this energy equal to the energy-constant of our conservative system, its whole trajectory will then be upon this hypersurface. Quasi-ergodic systems, for example, are such that their trajectory on E passes infinitely

(5)

near every point on E . Our system may, in certain
cases, admit, besides the energy-integral, other inte-
grals which are independent of time, in which case the
trajectory in phase-space will fill up a manifold of lower
dimensionality. Let $\overline{\Phi}_K(p,q) = c_k$ (K = 1,2 ...s;    )
be s of these integrals, the c's being arbitrary constants.
Through every point of $\Gamma$ passes a manifold of 2n - s
dimensions G, which is the intersection of the s hypersur-
faces $\Phi_K = $ const, and the trajectory through a given point
must be all contained in G. In the same way as for quasi-
ergodic systems, the trajectory on G passes infinitely
near every point on G. Geometrically, the statistical
character of the trajectory is determined when the hyper-
surfaces $\Phi_K = $ const. are known. These hypersurfaces will
(6)
be called, following Fermi and others,    the <u>characteristics</u>
of the system. A quasi-ergodic system has therefore, but
a single characteristic.

A conservative system which admits a separation of
variables (conditioned-periodic system) has as many charac-
teristics as degrees of freedom, corresponding to the  n
constants $(\alpha)$ of the complete integral of the Hamilton-
(7)
Jacobi equation;    a greater number can only exist when
the system is degenerate, i.e. when linear relations with
integral coefficients exist between the fundamental fre-
quencies. In the classical example of the two-dimensional
anisotropic oscillator, the phase-space $\Gamma$ is four-dimen-

sional and the trajectory lies on a two-dimensional surface G: the system has therefore _two_ characteristics. If on the other hand, the binding is isotropic, the system is simple-periodic and G is one-dimensional, which corresponds to _three_ characteristics.

Let the coördinates of the point P in phase-space be $x_1 \ldots x_{2n}$, -- instead of the more conventional $q_1 \ldots q_n$, $p_1 \ldots p_n$. We investigate the following problem: What is the probability that, at a given instant, $x_1 \ldots x_{2n-s}$ have values between $x_1$ and $x_1 + dx_1 \ldots x_{2n-s}$ and $x_{2n-s} + dx_{2n-s}$, while the s remaining x's have values such that they correspond to points on G. Now it is shown in statistical mechanics that, in order that a distribution in $\Gamma$ be stationary, i.e. in statistical equilibrium, it is necessary that the statistical density be constant on each one of the hypersurfaces G.[8] Let j be the Jacobian

$$j = \frac{\partial(\Phi_1 \ldots \Phi_s)}{\partial(x_{2n-s+1} \ldots x_{2n})}$$

then the element of volume in $\Gamma$ is

$$dv = \frac{1}{j} \, dx_1 \ldots dx_{2n-s} \, d\Phi_1 \ldots d\Phi_s$$

which, since during the motion of the system $d\Phi_1, \ldots, d\Phi_s$ are constants, may be written:

$$d\sigma = \frac{1}{j} \, dx_1 \ldots dx_{2n-s}$$

whence the required probability is

$$P_1 = \frac{1}{j} \frac{d\sigma}{\int \frac{d\sigma}{j}}$$

the limits of integration being taken so as to include all values of $x_1 \ldots x_{2n-s}$ which belong to the points on G.

For quasi-ergodic systems the only characteristic is the energy W. The probability that the representative point of such a system in phase-space $\Gamma$ be within a space element of angle $d\omega$ is measured by the product of the hypervolume between W and W + dW and the element $d\omega$. This probability is, hence

$$P_2 = \frac{\dfrac{r^{2n-1} \, d\omega}{\dfrac{\partial H}{\partial r}}}{\displaystyle\int \frac{r^{2n-1} \, d\omega}{\dfrac{\partial H}{\partial r}}}$$

the integral being taken throughout the unit reference hypersphere.

The problem is now the following: Let the system depend on m parameters $\lambda_1 \ldots \lambda_m$: When are the final values of the characteristics independent of the way in which an adiabatic transformation of the system is carried out, through adiabatic variation of the parameters $\lambda_1 \ldots \lambda_m$? By above, this relation between the former and the latter is given by the system of differential e-quations

$$\frac{d\bar{\mathfrak{X}}_k}{d\lambda_i} = \frac{\displaystyle\int \frac{\partial \bar{\mathfrak{X}}_k}{\partial \lambda_i} \frac{d\sigma}{j}}{\displaystyle\int \frac{d\sigma}{j}} \qquad \begin{array}{l} k = 1, 2 \ldots s \\[1mm] i = 1, 2 \ldots m \end{array}$$

and if the values of $\bar{\mathfrak{X}}_k$ are known for $\lambda_1, \lambda_2 \ldots \lambda_m = 0$, the in-

tegral of this system gives the desired variation be-
tween the $\Phi$'s and the $\lambda$'s. In order that the final
values of the characteristics be independent of the way
in which these parameters are altered -- provided only
that the change is adiabatic -- the condition of <u>unlimited</u>
<u>integrability</u> of this system must be satisfied.

It can be easily shown that for a quasi-ergodic
system the condition of unlimited integrability is satis-
fied. For in this case the above system reduces to the
single equation, corresponding to the only characteristic W

$$dW = \frac{\int \frac{\frac{\partial W}{\partial \lambda_\kappa}}{\frac{\partial W}{\partial r}} r^{2n-1} d\omega}{\int \frac{1}{\frac{\partial W}{\partial r}} r^{2n-1} d\omega} \, d\lambda_\kappa \qquad \text{(added on } k = 1,2\ldots m)$$

Assuming $m = 2$, and denoting the coefficients of
$d\lambda_1 , d\lambda_2$ by $L_1$, $L_2$, the condition of unlimited integra-
bility is;

$$\frac{\partial L_1}{\partial \lambda_1} + L_2 \frac{\partial L_1}{\partial W} = \frac{\partial L_2}{\partial \lambda_2} + L_1 \frac{\partial L_2}{\partial W}$$

and we obtain after a somewhat long computation:

$$\frac{\partial L_1}{\partial W} = \frac{1}{\left(\int \frac{r^{2n-1} d\omega}{\frac{\partial W}{\partial r}}\right)^2} \left[ \int \frac{r^{2n-1} d\omega}{\frac{\partial W}{\partial r}} \left[ (2n-1) \int \frac{r^{2n-2} \frac{\partial W}{\partial \lambda_1} d\omega}{\left(\frac{\partial W}{\partial r}\right)^2} + \right. \right.$$

$$+ \int \frac{r^{2n-1} \frac{\partial^2 W}{\partial \lambda_1 \partial r} d\omega}{\left(\frac{\partial W}{\partial r}\right)^2} - \int \frac{r^{2n-1} \frac{\partial W}{\partial \lambda_1} \frac{\partial^2 W}{\partial r^2}}{\left(\frac{\partial W}{\partial r}\right)^3} d\omega \right] -$$

$$- \int \frac{r^{2n-1} \frac{\partial W}{\partial \lambda_1}}{\frac{\partial W}{\partial r}} d\omega \left[ (2n-1) \int \frac{r^{2n-2} d\omega}{\left(\frac{\partial W}{\partial r}\right)^2} \right.$$

$$\left. - \int \frac{r^{2n-1} \frac{\partial^2 W}{\partial r^2} d\omega}{\left(\frac{\partial W}{\partial r}\right)^3} \right]$$

$$\frac{\partial L}{\partial \lambda_2} = \frac{1}{\left( \int \frac{r^{2n-1} d\omega}{\frac{\partial W}{\partial r}} \right)^2} \left[ \int \frac{r^{2n-1} d\omega}{\frac{\partial W}{\partial r}} \left[ - (2n-1) \int \frac{r^{2n-2} \frac{\partial W}{\partial \lambda_1} \frac{\partial W}{\partial \lambda_2} d\omega}{\left(\frac{\partial W}{\partial r}\right)^2} \right. \right. +$$

$$+ \int \frac{r^{2n-1}}{\frac{\partial W}{\partial r}} d\omega \left( \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} - \frac{\frac{\partial^2 W}{\partial \lambda_1 \partial r} \frac{\partial W}{\partial \lambda_2}}{\frac{\partial W}{\partial r}} \right) -$$

$$- \int \frac{r^{2n-1} \frac{\partial W}{\partial \lambda_1}}{\left(\frac{\partial W}{\partial r}\right)^2} d\omega \left( \frac{\partial^2 W}{\partial \lambda_2 \partial r} - \frac{\frac{\partial W}{\partial \lambda_2} \frac{\partial^2 W}{\partial r^2}}{\frac{\partial W}{\partial r}} \right) \right] -$$

$$- \int \frac{r^{2n-1} \frac{\partial W}{\partial \lambda}}{\frac{\partial W}{\partial r}} d\omega \left[ (2n-1) \int \frac{r^{2n-2} \frac{\partial W}{\partial \mu} d\omega}{\left(\frac{\partial W}{\partial r}\right)^2} \right. +$$

$$+ \int \frac{r^{2n-1} d\omega}{\left(\frac{\partial W}{\partial r}\right)^2} \left( \frac{\partial^2 W}{\partial \lambda_2 \partial r} - \frac{\frac{\partial W}{\partial \lambda_1} \frac{\partial^2 W}{\partial r^2}}{\frac{\partial W}{\partial r}} \right) \right]$$

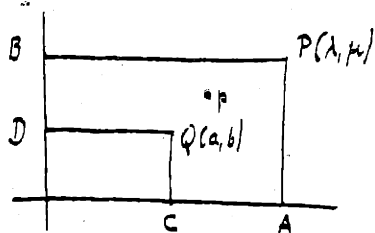and likewise for $\frac{\partial L_2}{\partial \lambda_1} + L_1 \frac{\partial L_2}{\partial W}$ . It is seen that $\lambda_1$ and $\lambda_2$ are symmetrical, so that the above integrability condition is satisfied. Hence the theorem: The

energy of a quasi-erogdic system is an adiabatic invar-
iant. Otherwise expressed: Any dynamical system having
a _single_ characteristic admits an adiabatic transformation
leaving this characteristic adiabatically invariant.

For dynamical systems with _more than one characteristic_
it is seen, by carrying through a similar computation,
which of necessity is more complicated, that the condition
of unlimited integrability is _not in general_ satisfied.
This can be best seen by the following physical example
due to Fermi [9] : Suppose that a point p is moving in-
side the polygon APBDQCA under the
action of no forces, but in such a
manner that a perfectly elastic im-
pact takes place every time it
strikes against one of the sides of the polygon. Under such
conditions it is obvious that the absolute values of the
components **u,v** of the velocity along the two axes stay
constant during the motion, so that the system has _two_
characteristics. Let (a,b) be the coördinates of the fixed
point Q, ($\lambda,\mu$) those of the variable point P. We thus
have a physical system having two characteristics and de-
pending on two parameters $\lambda,\mu$ . If now the position of
P varies adiabatically, u and v are given by the equations:

$$\frac{d}{d\lambda} \log u = \frac{2\mu}{ab - \lambda\mu} \quad , \quad \frac{d}{d\mu} \log v = \frac{2\lambda}{ab - \lambda\mu}$$

neither of which satisfies the conditions of integrability.

Hence the characteristics depend on the way the transformation of $P$ is carried out, and we have the theorem: If a dynamical system has more than one integral independent of time -- i.e. more than one characteristic -- none of these integrals is in general an adiabatic invariant.

Certain exceptions exist in which the system does admit adiabatic invariants. The most important is that studied by Burgers, characterized by the fact that the system can be described by angle variables. In this case there are n adiabatic invariants, i.e. the n phase integrals. Another particular case which is also important, first pointed out by Fermi (l.c.), is that in which the energy only depends explicitly on the parameters of the system. For the former is an adiabatic invariant, as already shown above, while the other characteristics are such that, by hypothesis,

$$\frac{\partial \bar{\varepsilon}_r}{\partial \lambda_i} = 0$$

We thus arrive at the following important conclusion: The adiabatic theorem is in general applicable to systems in which only one characteristic depends explicitly on the parameters of the system.

Krutkow and Fock (10) have proved the adiabatic invariance of the phase integral of a pendulum -- system of one degree of freedom -- for the particular case that the length of the string is a linear function of the time. Very recently H. Kneser (11) has proved the same theorem for any system of one degree of freedom, by methods which dif-

fer considerably from those employed here. The follow-
ing remarks may throw some light upon this question:
Ehrenfest and Burgers showed that the _average_ change of
the phase-integrals during a slow transformation for
all the phases of a multiple periodic motion vanishes,
provided the system does not become degenerate during the
transformation. Sommerfeld pointed out, however, that
this condition of _slow_ transformation is not enough; the
time variation of the parameter determining the mechanism
of the system -- in the general sense given to this ex-
pression above -- must be "_unsystematic_", that is, such
that the _time_ average can be asymptotically substituted by
the _phase_ average. **(12)** That this condition is not satis-
fied with sufficient accuracy for pendular motion was
pointed out by Krutkow and Fock who then proceeded to es-
tablish the adiabatic invariance of the phase integral by
assuming the linear law, as pointed out above.

Kneser, on the other hand, enumerates his theorem as
follows: Let the time variation of the parameter be
given by $\lambda = f(t)$ for $0 < t < T$, then the _total_ varia-
tion of the phase integral $I/_T - I/_0$ approaches the limit
0 as T increases indefinitely. His proof is carried out
quite rigorously. The fundamental geometrical-statistical
bearing of the question is not touched upon in his paper.
That this is essential is clearly brought out in the papers
of Ehrenfest **(12a)**. It seems therefore that the treatment
adopted in this paper is to be preferred. The ex-

treme generality of the methods involved is again em-

phasized.

Some other difficulties which occur in the genera-
lization of the concept of adiabatic invariance to sys-
tems of higher degree of freedom may be discussed here.
Consider again a system of n degrees of freedom specified
by coördinates $q_1 \ldots q_n$ and their conjugate momenta $p_1 \ldots p_n$
and the Hamiltonian function $H(p,q)$. Its phase-path in
2n-phase space comes in general as near as desired to any
point in an s-dimensional manifold $G_s$, which is contained
of course in the (2n-1) hypersurface $H(p,q) = E$. If the
motion is simple-periodic $s = 1$;  s has its maximum value
$s = 2n - 1$ if the motion is quasi-ergodic. Thus s can
have all integral values within 1 and 2n-1. Now, suppose
that the motion can be described by angle variables, that
is, that q and p $_\wedge$ can be expanded in an r-fold Fourier
$\overset{\text{and its derivatives}}{}$
series $(1 \leq r \leq 2n-1)$

$$p, q = \sum_{m_1 \ldots m_r = -\infty}^{\infty} A_{m_1 \ldots m_r} e^{J\left[2\pi (m_1 \nu_1 t \ldots m_r \nu_r)t + \delta_1 + \ldots + \delta_r\right]}$$

in which there are no "degenerate" terms, that is no re-
lations of the form $\sum m_k \nu_k = 0$. Now consider an
r-dimensional region $(\xi_1 \ldots \xi_r)$ and in it the lines $\xi_k = \nu_k t$
$(k = 1,2 \ldots r)$. By means of the Fourier series above, a
correspondence is established between points of $\xi_j \ldots \xi_r$ and
points on $G_s$ in phase-space $(p,q)$. Suppose we divide the

$\xi$ -space into unit "cubes"; then each line $\xi_\kappa$ pierces our first cube and an infinite number of other cubes; we take for the intercepted segments in the 2nd, 3rd ...... cube homologous segments within the first cube; on account of the absence of relations of the form $\sum m_k \nu_k = 0,$ (13) these segments form within the first cube a dense set; hence the $\xi$ -paths come as near as desired to every point (14) in an r-dimensional region, while the corresponding point in $G_s$ describes a certain phase path coming as near as desired to every point on $G_s$. Were the correspondence between $\xi$ -space and s-space unique (one-to-one) and continuous, then $r = s$, because this transformation is such as to leave the number of dimensions invariant (Brouwer's theorem (15) ). But it has not been shown that our correspondence is unique and continuous, hence we cannot yet maintain that a system of n degrees of freedom must have less than 2n-1 independent periods.

However, it should be noted that the unicity and continuousness of the transformation from $\xi$ to p,q involves differentiability conditions only, which in all probability are satisfied by our relations, at least within the domain considered, entirely for obvious physical reasons.

The significance of these considerations for the quantum theory will be readily appreciated, for the quantization rules (Sommerfeld-Wilson and Bohr) hold only for systems for which the degree of periodicity u is less or at most equal

to the number of degrees of freedom. In any case the possibility of constructing periodic systems with more than n independent periods is already disturbing enough. This difficulty is discussed and beautifully illustrated in a recent note of P. Ehrenfest [16]. As an example, consider the following: [17] A rigid electric dipole with moment of inertia I rotates freely in the xy-plane about its midpoint O, in such a way that as soon as the angle $\phi$ which its axis makes with the x-axis, exceeds the limits $-2\pi n \leq \phi \leq 2\pi n$, the dipole suffers a perfectly elastic reflection. n is a large irrational number. The dipole has then a "doubly"-periodic motion, one of discontinuous period $T = 8\pi n/\dot{\phi}$ , in which it rotates 2n times in one direction and 2n times in the opposite direction, the other harmonic of period $2\pi/\dot{\phi}$ due to its rotation about its axis. Ehrenfest and Breit point out that in this case the single quantum condition for a system of one degree of freedom $\oint pdq = nh$ leads to entirely absurd values of the energy, in fact to no quantization if n is chosen sufficiently great. This result is due altogether, in the light of the preceding discussion, to the fact that our quantum conditions cannot be applied to systems where the degree of periodicity is greater than the number of degrees of freedom. In this case we would have to begin by investigating the adiabatic invariants of the problem. It should also be carefully emphasized that we

are not dealing here with an example where $u > 2n - 1$, because one period is that of a harmonic function ("true period $2\pi/\dot\phi$), the other that of a discontinuity ("discontinuous" period $8n\pi/\dot\phi$ ).

Our general theory of adiabatic invariants leads to another important conclusion: Bohr's fundamental principle of the existence and permanence of quantum numbers demands the existence of adiabatic invariants and, since we have shown that such adiabatic invariants exist with certainty only for periodic systems, it must be concluded that atomic systems admit periodic solutions, of the types already discussed. This again seems to the writer to be a not inconsiderable argument for the adoption of particular periodic solutions, of the type he has already advocated,[18] in the dynamic description of atoms with more than one electron.

# Bibliography.

1) Ehrenfest, Annalen der Physik, Vol.51,p.327,1916.

2) P.Hertz. Annalen der Physik, Vol.33,p.235,1910.

2a) cf.Burgers' discussion at the end of Ref.(3) below.

3) Burgers, Annalen der Physik, Vol.52,p.195,1917; also Verslag Koninklijke Akademie van Wetenschappen te Amsterdam, 25 November 1916.

4) Fermi. Nuovo Cimento, Vol.25,p.173,1923.

5) P. and T. Ehrenfest, Enzyklopädie der mathematischen Wissenschaften, Vol.IV, Art.32,p.32,footnote 90, Leipzig,1907/14. See also Schaefer, Einführung in die theoretische Physik,Vol.2,p.440 footnote, Berlin, 1921. Regarding the existence of quasi-ergodic systems, see Ehrenfest, Zeitschrift für Physik Vol.19,p.243, footnote on Herglotz-Artin's example, 1923.

6) Fermi, Nuovo Cimento, Vol.25,p.267,1923.

7) Jacobi, Vorlesungen über Dynamik,p.163, Berlin,1864.

8) Gibbs "Elementary Principles in Statistical Mechanics" p.8, New York,1902; also for ex. Schaefer l.c. Ref.(5) Vol.2, p.434.

9) Fermi. Nuovo Cimento, Vol.25,p.171,1923.

10) Krutkow and Fock, Zeitschrift fur Physik, Vol.13, p.195,1923.

11) H.Kneser. Mathematische Annalen, Vol.91,p.155,1924.

12) Sommerfeld, Atombau und Spektrallinien,p.376 and p. 720, Braunschweig,1922.

12a) See for ex. Die Naturwissenschaften (Bohr Heft) Vol.11,p.543,1923, where numerous literature references are given. Cf.also L.Brillouin "La théorie des quanta et l'atome de Bohr" Chap. VII,Paris,1922.

13) Fraenkel "Mengenlehre", Berlin, 1923.

14) See l.c. Ref.(13), also Perron "Die Irrationalzählen" p.156, Leipzig,1921. This theorem is ascribed to Stäckel by Burgers: Annalen der Physik, Vol.52,p.200 footonote 3,1917; see also Kronecker, Gesammelte Werke, Vol.3,p.47.

15) Brouwer, Mathematische Annalen, Vol.70,p.161,1911;

Vol.71,p.305,1912; Vol.72,p.55,1912.

16) Ehrenfest, Zeitschrift für Physik, Vol.19,p.242,1923.

17) Ehrenfest and Breit, Zeitschrift für Physik, Vol. 9,p.207,1922.

18) Vallarta, Journal of Mathematics and Physics, Vol.3, p.108,1924.

4a) Nordheim. Zeitschrift für Physik, Vol.19, p.69, 1923.

# The Separation of Variables in Hamilton-Jacobi's Equation

It is well-known that for dynamical systems of n degrees of freedom, for which the motions corresponding to the several degrees of freedom are independent -- separable or conditioned-periodic systems -- the set of Sommerfeld-Wilson rules $I_k = \oint p_k dq_k = n_k h, \quad k = 1, 2 \ldots n.$ furnish the necessary and sufficient conditions for the quantization of the system. (1) The heuristic nature of these rules is sufficiently well-known to require any (2) further comment. It thus becomes of interest to discuss the following problem: Which types of the Hamilton-Jacobi equation can be integrated by separation of variables? This is known in the literature as Stäckel's problem, by whom it was first propounded in 1891 in his celebrated "Habilitationsschrift": "Über die Integration der Hamilton-Jacobischen Differentialgleichung mittels Separation der Variabeln" (Halle, 1891). Stäckel gave the solution of the orthogonal case and also the conditions of possibility (3). Later, in 1904, Levi-Civita (4) gave these conditions in explicit form and first pointed out clearly the part played by the potential function in the problem. Both Stäckel and Levi-Civita gave the complete solution of the problem, so far as the kinetic energy alone is concerned, for the system with two degrees of freedom, (5) and Dall'Acqua (6) for the system with three

degrees of freedom. In 1911, Burgatti [7] gave $n+1$
types of solutions for systems of n variables, without
showing that those are also possible, but his conclusions
are more based on a genial intuition than on any logi-
cal reasoning.

At the basis of the present discussion, lies the
following fundamental theorem established by Levi-
Civita (l.c.) and which forms the starting point of the
important investigations of Dall'Acqua [8] which we now
take up in considerably simplified form: If a conser-
vative holonomic dynamical system is integrable by se-
paration of variables, the equation defining the geo-
desics on the surface specified by the first fundamental
form $ds^2 = \sum a_{ij} dx_i dx_j$ , corresponding to the kinetic
energy T of the system, obtained from its Hamiltonian
(or Lagrangian) function by letting the potential vanish,
is also integrable by separation of variables. Such
dynamical systems are called geodesic. It is well-known
that the converse theorem is not in general true, i.e.
if the geodesic case is separable it does not always
follow that the dynamic problem is separable, except in
the case where the potential is essentially zero. [9] To
this type belong the cases studied by Levi-Civita.

Let $H(q_r \frac{\partial W}{\partial q}) = h_o$ be our Hamiltonian-Jacobi equa-
tion. The condition that it be integrable by separation

of variables is expressed by saying that $\dfrac{\partial W}{\partial q_k} = f_k(q_k)$

where $f_k$ depends on $q_k$ only. Writing, following Dall'Acqua,

$$\rho_\kappa = \frac{\dfrac{\partial H}{\partial q_k}}{\dfrac{\partial H}{\partial p_k}}$$

the condition of separability is expressed by $\dfrac{dp_k}{dq_k} = -\rho_\kappa$ or by the condition

$$\frac{\partial \rho_\kappa}{\partial q_j} - f_l \frac{\partial \rho_\kappa}{\partial p_j} = 0 \qquad k \neq j \qquad (1)$$

which must be satisfied for all values of k and j from 1 to n. As shown by Levi-Civita[10] these conditions are necessary an sufficient. Let $T = \frac{1}{2} a_{rs} \dot{q}_r \dot{q}_s$ (the

summation convention is adopted throughout this article) be the kinetic energy, U the potential of the forces acting on the system. If $a^{rs}$ are the contravariant fundamental tensors of the first fundamental form, then $K = \frac{1}{2} a^{rs} p_r p_s$, and the Hamiltonian function, is $H = K + U$. By the canonic equations

$$\dot{q}_r = \frac{\partial H}{\partial p_r} = \frac{\partial K}{\partial p_r} = \frac{1}{2} a^{rs} p_s$$

whence $\rho_r = \dfrac{\partial H}{\partial q_r} \Big/ \dot{q}_r$. We now note that

$$\dot{q}_r^1 \frac{\partial p_r}{\partial q_s} \ , \quad \dot{q}_r^1 \frac{\partial \rho_r}{\partial p_s} \ , \quad \dot{q}_r \rho_s \ \text{(not summed) are inte-}$$

gral rational (entire) functions in the p's or q's. There-

fore equation (1) multiplied by $\dot{q}_r \dot{q}_s$, i.e.

$$\dot{q}_s \left( q_r^2 \frac{\partial \rho_r}{\partial q_s} \right) - \left( \dot{q}_r \frac{\partial \rho_r}{\partial p_s} \right) (\dot{q}_s \rho_s) = 0 \qquad (2)$$

is an entire function, which must be satisfied for all values of p; we may therefore set each coefficient equal to zero. The terms not involving U are those of the fourth degree in p, while those of lower degree in p involve both K and U. If we consider only the geodesic problem ( $U = 0$ ) all these terms vanish. We thus have Levi-Civita's theorem given above.

In (2), the left-hand side is an entire function in p or q. The first term is obviously divisible by the $q$'s. The second may or may not. We thus have two possible cases in the indices s:

$1^O$ The function $\dot{q}_k \rho_k$ is divisible by $\dot{q}_k$ or zero.

$2^O$ The function $\dot{q}_k \rho_k$ is non-vanishing and non-divisible by $\dot{q}_k$.

Case 1. If $L_k$ is an entire function in p, we shall have $\rho_s = L_s$, or by above,

$$\frac{\partial U}{\partial q_s} = 0 , \qquad \frac{\partial K}{\partial q_s} = \dot{q}_s L_s \qquad (3)$$

and it becomes quite evident that $L_s$ is linear in the p's.

Case 2. If s belongs to this case ($\dot{q}_s \rho_s$ is not divisible by $\dot{q}_s$) then $q_r^2 \frac{\partial \rho_r}{\partial p_s}$ is divisible by the $q$'s. If we denote by $M_{rs}$ an entire function in p, we shall have:

$$\dot{q}_r^2 \frac{\partial p_r}{\partial p_s} = \dot{q}_s M_{rs} \qquad (r \neq s)$$

which, by above, separating the coefficients of the different powers, becomes:

$$a^{rs} \frac{\partial U}{\partial q_r} = 0 \tag{4}$$

$$\frac{\partial^2 K}{\partial q_r \partial p_s} q_r - \frac{\partial K}{\partial q_r} a^{rs} = \dot{q}_s M_{rs} \quad (r \neq s) \tag{4a}$$

and, from the last equation, the M's are linear homogeneous functions in p. Substituting in (2), we may separate the latter into the two conditions,

$$\frac{\partial^2 U}{\partial q_r \partial q_s} \dot{q}_r - \frac{\partial U}{\partial q_r} \frac{\partial^2 K}{\partial q_s \partial p_r} - \frac{\partial U}{\partial q_s} M_{rs} = 0 \tag{5}$$

$$\frac{\partial^2 K}{\partial q_r \partial q_s} \dot{q}_r - \frac{\partial K}{\partial q_r} \frac{\partial^2 K}{\partial q_s \partial p_r} - \frac{\partial K}{\partial q_s} M_{rs} = 0 \tag{5a}$$

Differentiating (4a) with respect to $q_s$, (5a) with respect to $p_s$ and subtracting, we get

$$\frac{\partial K}{\partial q_s} \frac{\partial M_{rs}}{\partial p_s} + 2 \frac{\partial^2 K}{\partial q_r \partial p_s} \frac{\partial^2 K}{\partial q_s \partial p_r} - 2 \frac{\partial^2 K}{\partial q_r \partial q_s} a^{rs} =$$

$$\dot{q}_s \frac{\partial M_{rs}}{\partial q_s} \tag{6}$$

The M's are easily found as follows: Differentiating (4a) with respect to $p_s$ we have:

$$\frac{\partial a^{ss}}{\partial q_r} \dot{q}_r = a^{ss} M_{rs} + \dot{q}_s \frac{\partial M_{rs}}{\partial p_s} \cdot \qquad (r \neq s)$$

Differentiating again,

$$\frac{\partial a^{ss}}{\partial q_r} a^{rs} = 2 a^{ss} \frac{\partial M_{rs}}{\partial p_s} \qquad (r \neq s) \qquad (7)$$

whence

$$M_{rs} = \frac{1}{2(a^{ss})^2} \frac{\partial a^{ss}}{\partial p_s} (2a^{ss} \dot{q}_r - a^{rs} \dot{q}_s) \quad (r \neq s) \quad (8)$$

The M's having been determined, the rest of the proof is only a matter of straight-forward calculation. In this way Dall'Acqua proves that the necessary and sufficient condition that a dynamical system which is separable in the kinetic energy be integrable by separation of variables, is that its potential <u>depend on a certain number of arbitrary functions each of a single variable</u> and this number characterizes the different cases.

To show this last part of the theorem, we note that the Hamilton-Jacobi equation may be written in the Jacobi expression:

$$a^{st} p_s p_t = 2(U + h_o)$$

If now we place all the q's equal to zero with the exception of the $r^{th}$ q, $p_r$ remains unaltered since by hypothesis it is a function of $q_r$ only. Suppose we pick out the terms in the summation which contain $p_r$. Indicating the <u>arbitrary</u> initial values of $p_r$ by $c_r$, the above equation becomes under the specified conditions

$$p_r^2 - 2p_r \, f_{(r)} + \phi_{(r)} - U_{(r)} - 2h_o = 0$$

where 
$$f_{(r)} = - a_{(r)}^{rs} \, c_s$$

$$\phi_{(r)} = a_{(r)}^{st} \, c_s \, c_t$$

the indices in parenthesis indicating that the corresponding term is taken with respect to $q_r$ only. The problem is now to specify the functions $f$ and $\phi$. Denoting by $f_{(r)}(\alpha)$ a linear form in the constants of the first group, which depends on the variable $q_r$ only, Dall'Acqua shows that a necessary and sufficient condition to be satisfied by the potential of systems of the first class is,

$$p_{(r)} = f_{(r)}(\alpha)$$

and for problems of the second class,

$$\phi_{(r)} = \phi'_{(r)}(\alpha) + \Phi'_{(r)}(\beta)$$

$\Phi'_r(\beta)$ being a linear form of $\beta = a^{ss}\left[ c_s^2 + 2a_o^{st} \, c_s \, c_t \right]$ with coefficients that depend on $q_r$ only. The energy-constant $h_o$ is then determined to be,

$$h_o = \tfrac{1}{2}\left[ \phi'(\alpha) + \Phi''(\beta) \right]$$

We thus have, for the potential U,

$$p_{(r)} = f_{(r)}(\alpha) \qquad \qquad \text{(r of 1st class)}$$

$$p_{(r)} = f_{(r)}(\alpha) \pm \sqrt{F_{(r)}(\alpha) + \phi_{(r)}(\beta) + U_r} \qquad \text{(r of 2nd class)}$$

$$2h_o = \phi'_o(\alpha) + \Phi''_o(\beta)$$

which solve the problem, provided the discriminants of

the two forms  f   and $\phi$  are both non-vanishing.

The elimination of the constants of the $\alpha$  and $\beta$ groups among these three equations gives a Hamilton-Jacobi equation which is integrable by separation of variables.  For details, reference must be made directly to Dall'Acqua's papers.

# BIBLIOGRAPHY.

1) Bohr. Quantentheorie der Linienspektren, pp.21 ff. Braunschweig, 1923.

2) Sommerfeld, Annalen der Physik, pp.5/10, Vol.51, 1916.

3) Charlier, Mechanik des Himmels, Vol.1,p.80, Leipzig,1902.

4) Levi-Civita, Mathematische Annalen, Vol.59,p.383, 1904.

5) Vallarta, Journal of Mathematics and Physics, Vol. 3, p.108 , 1924.

6) Dall'Acqua, Mathematische Annalen, Vol.66,p.398, 1908.

7) Burgatti, Rendiconti della Reale Accademia dei Lincei, Vol.20,p.108,1911.

8) Dall'Acqua, Rendiconti del Circolo Matematico di Palermo, Vol.33,p.341,1912.

9) Vallarta, Journal of Mathematics and Physics, Vol. 3,p.174,1924.

### Cauchy's existence proof and its generalization by Poincaré. Poincare's periodic solutions.

Consider a differential equation

$$\dot{x} = \phi(x,t) \qquad\qquad (1)$$

and let the function $\phi$ be such that it can be expanded in a power series in t in the vicinity of a point $t_o$, $x_o$. The integral of this equation can then be written

$$x = x_o + c_1(t - t_o) + c_2(t - t_o)^2 + \ldots \qquad (2)$$

which converges within the circle of convergence. The coefficients $c_1$, $c_2$ etc. are then unique and (2) is the only analytic integral which has the value $x = x_o$ for $t = t_o$. This is Cauchy's well-known classical "existence theorem". It was extended by Poincaré [11] who deduced from this extension certain theorems of the greatest importance in the general theory of orbits.

Consider the system of differential equations

$$\dot{x}_k = \phi_k(x_1 \ldots x_n,\ t, \mu) \qquad k = 1, 2 \ldots n \qquad (3)$$

and let

$$x_k = \theta_k(t, \mu) \qquad\qquad (4)$$

be those solutions of the above equations such that the values of $x_1, x_2 \ldots x_n$ vanish for $t = 0$. Assume now that for a certain system of values of $x_1$, $x_2 \ldots x_n$, $t, \mu$ one of the functions $\theta_k$ ceases to be holomorphic. We then say that this system of values corresponds to a

singular point of equations (3). Poincaré's extension
of Cauchy's theorem is then to the effect that:

For any non-singular point of the system of equations

$$\dot{x}_k = \phi_k(x_1 \ldots x_n, t, \mu) \qquad k = 1, 2 \ldots n,$$

$\theta_1(t, \mu) \ldots \theta_n(t, \mu)$ can be expanded in a power series
in $\mu$ (<u>not</u> in t <u>and</u> $\mu$), for all values of t between 0
and $t_o$, provided $|\mu|$ is sufficiently small, and $\phi_1 \cdots \phi_n$
can be expanded in positive powers of $x_1 - \theta_1(t, 0) \ldots$

$\qquad x_n - \theta_n(t, 0)$ for all values of t between zero
and T.

From this it follows that if the expansion of
in powers of $\phi_k$ and $x_k - \theta_k$ converges for all real
values of t, the expansion of the integral also converges
for any value of t, provided $\mu$ is sufficiently small.

We may therefore write

$$x_k = \theta_k(t, \beta_1 \ldots \beta_n, \mu) = \theta_k(t, 0) + \frac{\partial \theta_k}{\partial \beta_1}\beta_1 + \cdots + \frac{\partial \theta_k}{\partial \beta_n}\beta_n + \frac{\partial \theta_k}{\partial \mu} + \cdots \qquad (4)$$

If in particular this integral is a periodic function
of t, then it is convergent for all values of t, also for
$t = \infty$. In this case it is therefore enough to find in-
tegrals that converge in the interval $0 \to T$. We have
therefore, to determine the conditions under which the
motion defined by (3) is <u>periodic</u>. Poincaré proceeds as
follows:

We have assumed that the function $\phi$ depends on a para-
meter $\mu$ -- the so-called "perturbing" parameter . Let us

further suppose that the equations (3) have been integrated for $\mu = 0$ and that in <u>this</u> case certain <u>periodic</u> solutions have been found, under what conditions can we state that periodic solutions also exist for small $\mu$ ? The simplest case that we can think of is that for which the coordinates $x_k$ have the same values at $t = 0$ and $t = T$. The derivatives of the coordinates at these two epochs must then have the same values and the motion is then necessarily periodic. This, however, is not by any means a necessary condition for the existence of periodic solutions. For if we are dealing with the motion of mass-points, a periodic solution exists whenever the configurations of the mass-points, charges, etc. and their instantaneous changes are the same at $t = 0$ and $t = T$. We consider now the first case.

If the difference in the initial values $\beta_k = \theta_k(0,\mu) - \theta_k(0,0)$ is taken into account, the integrals of (1) may be written

$$x_k = \theta_k(t, \beta_1 \cdots \beta_n, \mu)$$

For $\mu = 0$, the motion is periodic of period T, by hypothesis, so that $\theta_k(T,0) = \theta_k(0,0)$ $(k = 1,2\ldots n)$ When does a periodic motion exist for $\mu \neq 0$? Evidently such is the case when $\theta_k(t,\mu) = \theta_k(T,\mu)$ or when by (4)

$$\psi_K = \sum_{k=1}^{h} \left[ \frac{\partial x_k}{\partial \beta_i}\bigg|_T - \frac{\partial x_k}{\partial \beta_i}\bigg|_0 \right] \beta_i + \sum_{K=1}^{n} \left( \frac{\partial x_k}{\partial \mu}\bigg|_T - \frac{\partial x_k}{\partial \mu}\bigg|_0 \right) = 0 \quad (5)$$

which by (4) may be written as

$$\psi_k = \sum_{k=1}^{n} \int_0^T \Theta_k \, dt$$

$$= \sum_{i,n=1}^{n} \beta_i \int_0^T \frac{\partial \Theta_k}{\partial \beta_i} \, dt + \sum_{k=1}^{n} \mu \int_0^T \frac{\partial \Theta_k}{\partial \mu} \, dt + \ldots \qquad (6)$$

which may be written

$$\psi_k = \sum_{i,k=1}^{n} A_{ik} \beta_i + \sum_{k=1}^{n} A_{ko} \mu + \ldots \text{higher power in } \beta \text{ and } \mu \quad (7)$$

The n equations $\psi_k = 0$ solved for $\beta_k$ give $\beta_1 \ldots \beta_n$

as a power series in $\mu$ : $\beta_k = p_k(\mu)$ $(k = 1,2 \ldots n)$

provided the determinant

$$D = \begin{vmatrix} A_{11} & \cdots\cdots & A_{1n} \\ \cdots\cdots\cdots\cdots \\ A_{n1} & \cdots\cdots & A_{nn} \end{vmatrix}$$

does not vanish.

In dynamics, equations $\psi_k = 0$ are not in general independent. Assuming that an integral $F(x_1 \cdots x_n, t) = c$ of equations (3) exist, which is periodic in t with period T, we have,

$$F\left[\Theta_k(T,\mu), T\right] = F\left[\Theta_k(0,\mu), 0\right] = F\left[\Theta_k(T,\mu), 0\right]$$

or, since $\Theta_k(T,\mu) = \Theta_k(0,\mu) + \psi_k$

$$F\left[\Theta_k(0,\mu) + \psi_k, 0\right] - F\left[\Theta_k(0,\mu), 0\right] = 0$$

The left-hand side can be expanded in powers of $\psi_1 \cdots \psi_n$ and vanishes when $\psi_k = 0$. Hence if n - 1 $\psi$'s vanish, the nth $\psi$ also vanishes. In particular,

if $\psi_k = 0$ $(k = 1 \ldots n-1)$, $\psi_n = 0$ if $\dfrac{\partial F}{\partial x_n} \neq 0$.

The second important case in which periodic solutions can exist, characterized by the fact that the mass-points have the same configuration at $t = 0$ and $t = T$, are investigated by Poincaré as follows $\overset{(12)}{.}$ Consider the canonic equations:

$$\dot{x}_k = \frac{\partial F}{\partial y_k} \qquad \dot{y}_k = -\frac{\partial F}{\partial x_k} \qquad k = 1,2 \ldots s \qquad (7)$$

where $F$ is a function of $x_1 \ldots x_n$, $y_1 \ldots y_n$ and $\mu$ having the following properties:

1) For all real values of $y_k$ $(k = 1 \ldots s)$, $F$ can be expanded in a power series of

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \ldots \ldots$$

2) $F$ depends only on $x, \ldots x_n$.

3) $F$ is periodic in $y_1 \ldots y_n$ with period 2

For $\mu = 0$, equations (7) can be easily integrated. We have in fact, for $\mu = 0$,

$$\dot{x}_k = 0 \qquad \dot{y}_k = -\frac{\partial F}{\partial x_k} = \nu_k \left( k = 1,2 \ldots s. \right)$$

which give $x_k = a_k \qquad y_k = \nu_k t + \delta_k$

where $a_k, \nu_k$ and $\delta_k$ are constants. If $\nu_k T$ is a multiple of $2\pi$, the motion is obviously periodic of period $T$. The problem is now: when do periodic solutions of the same period exist when $\mu \neq 0$?

Let, in this case, for $t = 0$:

$$x_k = a_k + \beta_k + \phi_\pi \quad , \qquad y_k = \delta_k + \gamma_k + \gamma_\pi + \psi_\pi$$

and instead of $x_k$, $y_k$ introduce new variables $\phi_\pi$ and $\psi_\pi$ defined by

$$x_k = a_k + \beta_k + \phi_k \quad , \qquad y_k = u_k t + \delta_k + \gamma_k + \psi_k$$

and obtain the set of equations:

$$\dot{\phi}_\pi = \frac{\partial F}{\partial \delta_k} \quad , \qquad \dot{\psi}_\pi = -\frac{\partial F}{\partial x_k} - \nu_k$$

If $\phi_k(0) = \phi_k(T)$ and $\psi_k(0) = \psi_k(T)$ the motion is again evidently periodic with period T. Since these equations are not independent (see above), $\psi_s(T) = 0$ follows from the other equations.

We have therefore $s - 1$ equations

$$\psi_k(T) = 0 \qquad k = 1,2 \ldots s - 1$$

and, letting one of the parameters $\beta_k$ be equal to zero ($\beta_s = 0$), we obtain

$$\psi_k(T) = \sum_{i,k=1}^{s-1} -\beta_k \int_0^T \frac{\partial^2 F_0}{\partial a_i \partial a_k} \, dt + \sum_{i=1}^{s-1} \mu \int_0^T \frac{\partial F_1}{\partial a_i} \, dt + \text{(higher}$$

$$\text{powers of } \beta_1 \ldots \ldots \beta_{s-1} \text{ and } \mu) = 0 \qquad (8)$$

Since $\frac{\partial^2 F_0}{\partial a_i \partial a_x} = \text{const}$, we have, dividing by T.

$$\sum_{i,k=1}^{s-1} \frac{\partial^2 F_0}{\partial a_i \partial a_k} \beta_k + \sum_{i=1}^{s-1} \frac{\mu}{T} \int_0^T \frac{\partial F_1}{\partial a_i} \, dt = 0$$

which determines the parameters $\beta_1, \beta_2 \ldots \beta_{s-1}$. In order

that one solution only may exist, it is necessary and
sufficient that the Hessian

$$
\begin{vmatrix}
\dfrac{\partial^2 F_0}{\partial a_1 \partial a_1} & \cdots\cdots & \dfrac{\partial^2 F_0}{\partial a_1 \partial a_{s-1}} \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\dfrac{\partial^2 F_0}{\partial a_{s-1} \partial a_1} & \cdots\cdots & \dfrac{\partial^2 F_0}{\partial a_{s-1} \partial a_{s-1}}
\end{vmatrix} \neq 0
$$

We have therefore the theorem: The necessary and
sufficient condition that s - 1 different $\beta's$ exist is
that the Hessian of $F_o$ with respect to $a_1\ a_2 \ldots\ a_{s-1}$
be non-vanishing.

Now, we may calculate $\phi_\kappa$ by the equation

$$
\phi_\kappa(T) = \sum_{\kappa=1}^{s} \gamma_\kappa \int_0^T \frac{\partial^2 F}{\partial y_i \partial y_k}\ dt + \mu \int_0^T \frac{\partial^2 F}{\partial y_i \partial \mu}\ dt + \ldots
$$

whence

$$
\frac{\phi_\kappa(T)}{\mu} = \sum_{\kappa=1}^{s} \gamma_\kappa \int_0^T \frac{\partial^2 F_1}{\partial y_i \partial y_k}\ dt + \int_0^T \frac{\partial F_1}{\partial y_i}\ d_t \qquad (9)
$$

and it is seen that in order that the periodic solution
we consider here may be the analytic continuation of the
solution previously obtained for $\mu = 0$, it is __necessary__
that the set of values $\gamma_1 \ldots \gamma_s$ which we obtain from (9)
shall vanish with $\mu$. But equation (9) contains one
term which is independent of $\mu$, hence this term must vanish.
If the solution is periodic, the s equations must be satis-

fied

$$\int_0^T \frac{\partial F_1}{\partial y_j} \, dt = 0 \qquad j = 1,2\ldots.s \qquad (10)$$

The investigation of this condition is carried out by Poincare as follows [13] : Introduce the average value $\left[F_1\right]$ of $F_1$ defined by

$$\left[F_1\right] = \frac{1}{T}\int_0^T F_1 dt$$

Equations (10) may then be written

$$\frac{\partial \left[F_1\right]}{\partial y_i} = \frac{\partial \left[F_1\right]}{\partial \delta_i} = 0 \qquad i = i,2,\ldots.s$$

whence $\left[F_1\right]$ must be a maximum or a minimum. Since $F_1$ is periodic, we have by Fourier's theorem:

$$F_1 = \sum_{m_1\ldots m_s = \infty}^{\infty} A_{m_1\ldots m_s} \cos(m_1 y_1 + \ldots. + m_s y_s + h)$$

$$= \sum A_{m_1\ldots m_s} \cos\left[(m_1 \nu_1 + \ldots + m_s \nu_s)t + m_1 \delta_1 \ldots + m_s \delta_s + h\right]$$

$$= \sum A \cos \omega$$

Further $\quad \dfrac{\partial F_1}{\partial \delta_j} = - \sum Am_1 \sin \omega$

and $\quad \left[F_1\right] = \displaystyle\int A \cos \omega$

where $\displaystyle\int$ denotes summation over such values of m for which $\sum m_1 \nu_i = 0.$ We have therefore the important theorem first established by Poincare:

Let the characteristic function F be expandable in a power series of a sufficinetly small parameter $\mu$ :

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \ldots\ldots$$

then, the necessary and sufficient conditions that the canonic equations admit a periodic solution are that

$$H(F_o) \neq 0$$

$$\frac{\partial [F_1]}{\partial \sigma_i} = 0 \qquad\qquad i = 1,2 \dots s$$

where $H(F_o)$ is the Hessian of $F_o$ with respect to the s-1 of the variables $x_1 \dots x_s$. Besides, if the solution is to have $s$ independent periods, the condition $H(F_1) \neq 0$ must be satisfied.

The case $H(F_o) = 0$ occurs frequently in astronomical applications. Its treatment, although easy, will not be taken up here [14] because it is of small importance in those applications with which we are mainly concerned.

## BIBLIOGRAPHY.

11) Poincaré, Les méthodes nouvelles de la mécanique céleste" Vol.1,p.58,Paris,1892;  also Charlier "Mechanik des Himmels" Vol.2,p.172,Leipzig,1907.

12) Poincaré,l.c., Ref.(11) p.110,Vol.1.

13) Poincaré,l.c., Ref.(11) p.113,Vol.1.

14) Poincaré,l.c., Ref.(11) p.117,Vol.1.

## The Convergency of the Expansion of the Perturbation Function.  Theorems of Poincaré and Bruns.

The question as to the convergency of the series which we have developed and made use of elsewhere in this paper is of paramount importance, not only as regards its intrinsic mathematical significance, but also to lend weight to the conclusions derived from an application of the method of the calculus of perturbations to specific problems.  The questions to be discussed in this section are:

1)  What is the region of convergency of the series used in the calculus of perturbations?

2)  What is the magnitude of the error made by neglecting terms beyond a certain predetermined term in a given series expansion?

Many investigations have been carried out with regard to the first problem, but many points in it are still awaiting a solution.  The most important result thus far available is that <u>no</u>  <u>solution</u> <u>of</u> <u>the</u> <u>three-body</u> <u>problem</u> <u>obtained</u> <u>by</u> <u>the</u> <u>method</u> <u>of</u> <u>perturbations</u> <u>holds</u> <u>over</u> <u>unlimited</u> <u>periods</u> <u>of</u> <u>time</u>.  Nor has it been possible to prove that any such solution does or does not exist.  The important question as to the limiting values of the coördinates in the three-body problem -- the so-called "stability problem" -- is

therefore still open. Perhaps the most promising way
of attacking this problem is through the method of
Hill's curves. In fact Hill was the first to give a
rigorous stability proof for a certain class of motion
of the three-body problem. The most general investiga-
tion of this type is that of Bohlin [1]. He showed that
no general conclusions on the maximum and minimum values
of the relative distances of the moving particles can be
drawn, it is only possible to conclude that **all** these
distances _cannot_ _increase_ indefinitely at the _same_ time.
In fact in the n-body problem the equation of Hill's limit-
ing curve is

$$\sum_{i,j=1}^{n} \frac{k^2 m_i m_j}{r_{ij}} + h = 0$$

and the above conclusion follows at once.

It will be recalled that it has been known since the
days of Laplace and Lagrange that no secular terms con-
taining the semi-major axis, the eccentricity and incli-
nation occur in the expression for the perturbation of the
first order. How the matter stand for the whole power
expansion is something which until to-day remains esses-
tially unknown. A well-known classical difficulty to which
we shall again refer later, is the presence of small di-
visors, which give rise to periodic terms of very large
magnitude. The burning question in any application of the
calculus of perturbations is this: over what interval can

we expect our calculations to hold?  This leads at once to an examination of the convergency of the series used in perturbation theory.  It may be noted that a discussion of this question, important as it is from an astronomical standpoint, is wholly indispensable from the point of view of atomistics.  For while the astronomer can always verify by direct observation whether his theory is in accordance with the facts, no such method of direct verification is at the disposal of the physicist.  Should our series be semi-convergent, it is not only useless, but harmful, to push approximations farther, and any too general application of such expansion is totally meaningless.

Lindstedt, in 1882 [(2)] showed that the canonical equations  $\dot{p}_k = -\dfrac{\partial H}{\partial q_k}$,  $\dot{q}_k = \dfrac{\partial H}{\partial p_k}$  can be formally satisfied by series of the form

$$q_k = q_k^o + \mu\, q_k^{\prime} + \mu^2\, q_k^2 + \ \cdots\cdots$$

$$p_k = w_k + \mu\, p_k^{\prime} + \mu\, p_k^2 \ \cdots\cdots \tag{1}$$

where $p_k^j$ and $q_k^j$ are periodic functions of the angle variables  $w_k = \nu_k t + \delta_k$  $(k = 1,2\ldots n)$, so that we have

$$p_k^j \ \text{or} \ q_k^j = A_o + \sum_{m_1\cdots m_n = -\infty}^{\infty} A_{m_1\ldots m_n}\, e^{\, j\sum_{k=1}^{h} m_k w_k} \tag{2}$$

Poincaré [,(3)] examined the following questions:

1) Are the "partial" series (2) for $p_k^j$ or $q_k^j$ convergent, if so, are they absolutely and uniformly convergent?

2) If they do not converge uniformly, is it possible to so to group them together that semi-convergent series can be obtained?

3) If the partial series (2) are convergent, does it follow that the total series converge absolutely and uniformly?

He then proceeded to prove that the sum of the terms of a partial series (2) cannot be less than half any one of its coefficients, hence that in order that this series converge uniformly it is necessary and sufficient that the absolute value of the coefficients A be bounded, which is so provided that the **ν**'s are incommensurable or else that A vanish whenever commensurable ratios occur. On the other hand, the total series (1) do not converge uniformly for **all** values of $\mu$ and $q_k^0$ within a finite interval, nor do they converge for any values of $\mu$ , however small, when $q_k^0$ vary within any finite interval. Poincaré's conclusion might also therefore, be formulated as follows: It is impossible to express the coordinates in phase-space of the n-body problem in a power expansion of the perturbing parameter $\mu$ , which converges everywhere within a domain D. Otherwise, we have shown elsewhere that if the perturbing function F can be expressed in the series:

$$F = \sum_{m_1 \cdots m_n = -\infty}^{\infty} A_{m_1 \cdots m_n} \cos \left( \sum_{k=1}^{n} m_k w_k \right)$$

then the differential equations of any element E is

$$\dot{E} = \sum_{m_1 \cdots m_n = -\infty}^{\infty} B_{m_1 \cdots m_n} \cos \left( \sum m_k w_k \right)$$

whence by integration

$$E = \sum_{m_1 \cdots m_n = -\infty}^{\infty} \frac{B_{m_1 \cdots m_n}}{\sum m_k \nu_k} \sin \left( \sum m_k w_k \right) + ct + E_o$$

we now investigate the convergence of the series in the right-hand term. For simplicity and again without much loss of generality, consider the three-body problem ($k = 1,2$). Then if $\nu$ is the ratio $\nu_1 / \nu_2$, we may write the above as

$$E = \sum_{m_1, m_2} \frac{K_{m_1 m_2}}{m_1 - m_2 \nu} \sin \left( m_1 w_1 - m_2 w_2 + D \right)$$

If $\nu$ is rational, a term in the above expansion becomes infinite, even when $\nu$ is irrational the denominator may become small enough though never vanish. From this circumstance arise the difficulties of the investigation. However, should $m_1 - m_2 \nu = 0$, a new secular term would arise, which can no longer belong to the present investigation, because it is contained in the secular term Ct. We consider the series $\sum \frac{k_{m_1 m_2}}{m_1 - m_2 \nu}$ . One of the most celebrated theorems of dynamics, discovered by Bruns in

1884,[4] establishes the following property of this
series: Let us call the values of $\nu$ for which the series
converges, "convergence points", those for which it
diverges, "divergence points", Bruns' theorem then
states <u>that the convergence and divergence points of the
series in question make up an everywhere dense set</u> through-
out the real domain of $\nu$.[5] Gylden[6] then sought to
prove that, in spite of this, the probability of a diver-
gence is infinitely small, but his considerations do not
seem to be entirely free from reproach[7].

BIBLIOGRAPHY.

1) Bohlin, Acta Mathematica, Vol.10,1887.

2) Poincaré, "Les méthodes nouvelles de la mécanique cé-
   leste" Vol.2, p.15, Paris, 1893.

3) Poincaré, l.c. Ref.(2), pp.94 ff.

4) Charlier, "Mechanik des Himmels", Vol.2,p.307, Leip-
   zig,1907.

5) For the definition of "everywhere dense set" see for
   example Fraenkel, "Mengenlehre", p.106, Berlin,1923
   or Hobson "The Theory of Functions of a Real Varia-
   ble" p.74, Cambridge,1921.

6) Gylden, Comptes-Rendus, 1884.

7) Charlier, l.c. Ref.(4), pp.319/320.

A P P E N D I X .

## APPENDIX

I.  General Introduction to the Theory of Perturbations.

1) Canonic variables. Contact-transformation.

Consider a system of $2_n$-variables $q_1 \ldots q_n$, $p_1 \ldots p_n$ determined by a canonic set of equations:

$$\dot{q}_k = \frac{\partial F}{\partial p_k} \quad , \quad \dot{p}_k = - \frac{\partial F}{\partial q_k} \tag{1}$$

and a transformation

$$q_k = f_k (\xi_1 \cdots \xi_n, \eta_1 \cdots \eta_n)$$
$$p_k = g_k (\xi_1 \cdots \xi_n, \eta_1 \cdots \eta_n) \tag{2}$$

there exist an infinite number of forms of the functions f and g for which the differential equations relating the new variables $\xi$ and $\eta$ are again of the canonic form. The problem is now to determine the conditions to be satisfied by the functions f and g in order that the variables $\xi$ and $\eta$ satisfy the canonic equations.

Let F be a function of p, q and time. Adopting the summation convention, as will be done throughout this paper -- a repeated index means summation on that index from 1, 2... to n  unless otherwise expressly stated -- we have

$$\dot{\xi}_k = \frac{\partial \xi_k}{\partial q_i} \dot{q}_i + \frac{\partial \xi_k}{\partial p_i} \dot{p}_i$$

or by (1)

$$\dot{\xi}_\kappa = \frac{\partial \xi_\kappa}{\partial q_i} \frac{\partial F}{\partial p_i} - \frac{\partial \xi_\kappa}{\partial p_i} \frac{\partial F}{\partial q_i}$$

(1)

We now introduce the Poisson bracket expressions:

$$\left(a, b\right) = \frac{\partial a}{\partial x_s} \frac{\partial b}{\partial y_s} - \frac{\partial a}{\partial y_s} \frac{\partial b}{\partial x_s}$$

(3)

and write

$$\dot{\xi}_i = \left(\xi_i, F\right)$$

(4)

on the other hand, we have

$$\frac{\partial F}{\partial p_s} = \frac{\partial F}{\partial \xi_i} \frac{\partial \xi_i}{\partial p_s} + \frac{\partial F}{\partial \eta_i} \frac{\partial \eta_i}{\partial p_s}$$

and therefore

$$\frac{\partial \xi_i}{\partial q_s} \frac{\partial F}{\partial p_s} - \frac{\partial \xi_i}{\partial p_s} \frac{\partial F}{\partial q_s} = \frac{\partial F}{\partial \xi_i} \left( \frac{\partial \xi_\kappa}{\partial q_s} \frac{\partial \xi_i}{\partial p_s} - \frac{\partial \xi_\kappa}{\partial p_s} \frac{\partial \xi_i}{\partial q_s} \right) +$$

$$+ \frac{\partial F}{\partial \eta_i} \left( \frac{\partial \xi_\kappa}{\partial q_s} \frac{\partial \eta_i}{\partial p_s} - \frac{\partial \xi_\kappa}{\partial p_s} \frac{\partial \eta_i}{\partial q_s} \right)$$

whence

$$\dot{\xi}_i = \left( \xi_i, F \right) = \left( \xi_\kappa, \xi_i \right) \frac{\partial F}{\partial \xi_i} + \left( \xi_\kappa, \eta_i \right) \frac{\partial F}{\partial \eta_i}$$

(5)

and likewise

$$\dot{\eta}_\kappa = \left( \eta_\kappa, F \right) = \left( \eta_\kappa, \xi_i \right) \frac{\partial F}{\partial \xi_i} + \left( \eta_\kappa, \eta_i \right) \frac{\partial F}{\partial \eta_i}$$

(6)

If it is required that

$$\dot{\xi}_\kappa = \frac{\partial F}{\partial \eta_\kappa} \qquad \dot{\eta}_\kappa = - \frac{\partial F}{\partial \xi_\kappa}$$

(1a)

we must, therefore, have the following relations satis-

fied

$$( \xi_i, \xi_r ) = 0 , \qquad ( \xi_i, \eta_r ) = 0 \qquad\qquad i \neq r$$
$$( \eta_i, \eta_r ) = 0 , \qquad ( \xi_i, \eta_i ) = 1 \qquad i,r = 1,2....n \tag{7}$$

From this method of derivation, these conditions are _suf-

ficient_. Whether they are or not _necessary_ is immaterial

for our purpose, since we are only concerned with test-

ing out given transformations. But, if we compare (5)

and (6) with (1a), we see that, should no _special_ assump-

tion be made with regard to the function F, these condi-

tions are also necessary. Variables satisfying conditions

(7) will be called here _canonic_ variables.

Consider again the transformation $q, p \rightarrow \xi, \eta$

If the equations connecting the two sets of variables are

such that the differential form

$$\eta_r \, d\xi_r \quad - p_k \, dq_k$$

is the perfect differential of a function of $(q,p)$, when

expressed in terms of q, p and their differentials, the

transformation $q,p \rightarrow \xi, \eta$ is called a _contact-transformation_.

This definition is in a certain sense more restricted,

but in another sense more general, than the geometric de-

finition, viz. (cf.Lie, [(1)] ): A contact-transformation

is a transformation from a set of $2n+1$ variables $(Z, p_k, q_k)$

to another set $(Z\ P_k Q_k)$ such that the relation

$$dZ - P_k dQ_k = \rho\ (dz - p_k\ dq_k)$$

is satisfied, $\rho$ being a function of $(q,\ p,\ z)$. In particular if the n variables $Q_k$ are functions of $q_k$ only, the transformation $Q \to q$ is said to be a <u>point-transformation</u>. From the definition, it follows at once that the result of n contact-transformations is itself a contact-transformation. Likewise, if $q, p \to \xi, \eta$ is a contact-transformation, $\xi, \eta \to q, p$ is also a contact-transformation. Therefore the theorem: A contact-transformation has the properties of a group. The following theorem is also easily established (Whittaker, l.c. Note 1). A transformation $q, p \to, \xi, \eta$ for which conditions (7) are satisfied, is a contact-transformation and we have just shown that a canonic system of equations is invariant for a contact-transformation.

From a geometrical standpoint, the change from the system of variables p, q to the set $\eta, \xi$ is a transformation from a hypersurface S in $2n - 1$ -space to a hypersurface $\sigma$ . If two surfaces S and S' touch at a point, the corresponding surfaces $\sigma$ and $\sigma'$ also touch at the corresponding point. For this reason, this type of transformation was called by Sophus Lie a contact-transformation.

### 2. Introduction to the n-body problem. Semicanonic equations. Canonic elements.

We consider a holonomic reversible conservative system of 3n degrees of freedom made up of n mass-points moving under central forces. The canonic equations of motion are

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k} \qquad k = 1,2,\ldots 3n$$

where $H = T + V = \frac{1}{2} \sum_{\kappa=1}^{n} \frac{p_k^2}{m_k} + V(q)$.

As a consequence of the properties of the characteristic function, these equations admit _ten_ algebraic integrals, as follows:

1) H does not contain the time explicitly. Hence the existence of an energy integral.

2) H is independent of the origin of coordinates. Hence the _six_ integrals of the uniform motion of the center of gravity of the system.

3) V is independent of a rotation of the coordinate axes. Hence the _three_ integrals of angular momentum and the existence of an invariable plane.

If we restrict ourselves, for simplicity only, and without any loss of generality, to the three-body problem, the existence of these ten algebraic integrals enables us to reduce the order of the equations of motion from the 18th to the 8th, viz. to a problem of 4, instead of 9, de-

grees of freedom. Practically it is seldom possible to reduce the system to below the 12th order.[2] Two more integrals can be found by introducing Jacobi's canonic variables.[3] So the system can be finally reduced, in the present state of our knowledge, to the 6th order (3 degrees of freedom). The general case of n-bodies is likewise reduced to a system of (6n - 12)th order.

By introducing "relative" coördinates,[4] expressing T as a function of the latter and defining $p_k$ as usual by the equation $p_k = \dfrac{\partial T}{\partial q_k}$ , the Newtonian equations of motion $m_k \ddot{\xi}_r = -\partial V/\partial \xi_r$ can again be brought to the canonic form. If on the other hand the p's are defined by $p_k = m\dot{q}_k$ we obtain the so-called "semi-canonic" equations of motion[5] originally introduced by Poincaré and characterized by the fact that each body of the system has its own Hamiltonian function. Equations of this type will be largely made use of in what follows.

An important astronomical case of the n-body problem, but which unfortunately finds but a limited application in atomic problems, is that in which the mass of one of the bodies is very large compared to the mass of the other. In this case, the influences of the latter on one another may be neglected, at least for a certain interval of time. To a first approximation the paths of the small masses -- for shortness called "planets" -- are conics of which the

large mass -- the "sun" -- occupies one of the foci.
If now it is required to take into account the mutual
actions of the planets, we proceed as follows:  we con-
sider the elements of the conic obtained above as a first
approximation to be _variable_ and determine their variations
in such a manner that the actual motion is correctly de-
scribed.  This is the principle of Lagrange's "method of
variation of constants".  Mathematically, it can be for-
mulated as follows:  In order to integrate the system of
canonic equations

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k} \qquad k = 1,2 \ldots n \qquad (1)$$

we take first a part H' of the Hamiltonian function and
integrate the system

$$\dot{q}_k = \frac{\partial H'}{\partial p_k} \qquad \dot{p}_k = -\frac{\partial H'}{\partial q_k} \qquad k = 1,2 \ldots n - 1 \qquad (2)$$

We thus get the coördinates $q_k$ and $p_k$ as a function of
t(time) and $2(n-1)$ parameters.  This path is called the
"intermediate" orbit.  If now these parameters are looked
upon as variables, we may derive from the former a system
of $2(n-1)$ equations of the first order, which correspond
completely to the original system.  In particular, if e-
quations (2) are integrated through the  Hamilton-Jacobi
method, the parameters are, by Jacobi's theorem, themselves
canonic.  Such parameters are called canonic elements.

As an example, we choose the three body problem. Expressed in Jacobi coördinates, the Hamilton function (6) is

$$H = \frac{1}{2\mu_b} (p_1^2 + p_2^2 + p_3^2) + \frac{1}{2\mu_a} (p_4^2 + p_5^2 + p_6^2) - k^2 \left( \frac{m_b m_c}{r_{bc}} + \frac{m_c m_a}{r_{ca}} + \frac{m_a m_b}{r_{ab}} \right)$$

where

$$\mu_a = \frac{m_a (m_b + m_c)}{m_a + m_b + m_c}$$

$$\mu_b = \frac{m_b m_c}{m_b + m_c}$$

and $m_a$, $m_b$, $m_c$ are the masses of the three bodies.

Let $q_1$, $q_2$, $q_3$ be the relative coördinates of $m_b$ referred to $m_c$ and $q_4$, $q_5$, $q_6$, the relative coördinates of $m_a$ referred to the center of gravity. If one of the masses, say $m_c$ is very large compared to the two others, an easy algebraic computation gives, if we write

$$H = H' + H''$$

$$H' = \frac{1}{2\mu_b} (p_1^2 + p_2^2 + p_3^2) + \frac{1}{2\mu_a} (p_4^2 + p_5^2 + p_6^2) - k^2 \left( \frac{m_b m_c}{\sqrt{q_1^2 + q_2^2 + q_3^2}} + \frac{m_a m_c}{\sqrt{q_4^2 + q_5^2 + q_6^2}} \right)$$

$$H'' = \frac{k^2 m_a m_c}{r_{ga}} - \frac{k^2 m_a m_c}{r_{ca}} - \frac{k^2 m_a m_b}{r_{ab}}$$

The system (2) can be split into two separate systems: the first for $k = 1,2,3$, the second for $k = 4, 5, 6$. We consider the equations

$$\dot{q}_k = \frac{\partial H_b}{\partial p_k} \quad , \qquad \dot{p}_k = - \frac{\partial H_b}{\partial q_k} \quad k = 1,2,3.$$

where $H_b = \dfrac{1}{2\mu_b} (p_1^2 + p_2^2 + p_3^2) - \dfrac{k\, m_b\, m_c}{\sqrt{q_1^2 + q_2^2 + q_3^2}}$

and the integral of the system is, therefore, an integral of the Hamilton-Jacobi equation

$$\left(\frac{\partial W}{\partial q_1}\right)^2 + \left(\frac{\partial W}{\partial q_2}\right)^2 + \left(\frac{\partial W}{\partial q_3}\right)^2 + \frac{2k^2 \mu_b\, m_b\, m_c}{\sqrt{q_1^2 + q_2^2 + q_3^2}} - 2h_1 \mu_b = 0$$

$h_1$ being the energy-constant. Transforming this equation to spherical coördinates we obtain

$$\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \phi}\right)^2 + \frac{1}{r^2 \cos^2\phi}\left(\frac{\partial W}{\partial \theta}\right)^2 - \frac{2k^2 \mu_b\, m_b\, m_c}{r} - 2h_1 \mu_b = 0$$

which, as seen by substituting in the Levi-Civita conditions (7), is integrable by separation of variables. We have

$$r^2 \left[ -\left(\frac{\partial W}{\partial r}\right)^2 + \frac{2\beta^2}{r} + 2\mu_b h_1 \right] = h_2^2 \,,$$

$$\cos^2\phi \left[ -\left(\frac{\partial W}{\partial \phi}\right)^2 + h_2^2 \right] = h_3^2$$

$$\left(\frac{\partial W}{\partial \theta}\right)^2 = h_4^2$$

$$h_2 + h_3 + h_4 = h_1$$

whence $\quad W_r \; = \; \int \sqrt{\dfrac{2\,\beta^2}{r} + 2\mu_b h_1 - \dfrac{h_2^2}{r^2}} \quad dr \; = \int \sqrt{R} \; dr$

$$W_\phi \; = \; \int \sqrt{h_2^2 - \dfrac{h_3^2}{\cos^2\phi}} \quad d\phi \; = \int \sqrt{\bar\Phi}\, d\phi$$

$$W_\theta \; = \; h_4\,\theta$$

$$h_2 + h_3 + h_4 = h_1$$

and by Jacobi's theorem

$$t + \beta_1 \; = \; \mu_b \int_{r_1}^{r} \frac{dr}{\sqrt{R}} \quad , \quad \beta_2 \; = \; -h_2 \left[ \int_{r_1}^{r} \frac{dr}{r^2\sqrt{R}} + \int_{0}^{\phi} \frac{d\phi}{\sqrt{\bar\Phi}} \right]$$

$$\beta_3 \; = \; -h_3 \int_{0}^{\phi} \frac{d\phi}{\cos\phi \, \sqrt{\bar\Phi}} \; + \; \theta$$

The path is therefore a conic. The intermediate integrals
are

$$\mu \,\dot r \; = \; \sqrt{R} \qquad \mu\, r^2\, \dot\phi \; = \; \sqrt{\bar\Phi} \qquad \mu\, r^2 \cos^2\phi\, \dot\theta \; = \; h_3$$

and the duration of the period T of r is

$$T \; = \; \mu \int_{r_{min}}^{r_{max}} \frac{dr}{\sqrt{R}}$$

the mean motion being therefore $\; n = \pi/T \;$.

first two equations we get

$$\tan \Omega = \tan \Omega'$$

therefore,
$$\Omega = \Omega' + \pi$$

and we obtain the important theorem due to Jacobi:

The increasing node of the orbit of one of the planets on the invariable plane coincides with the decreasing node of the other planet.

The third integral is one of the most important theorems of celestial mechanics, viz. the stability-proof of Laplace. Laplace proved that, <u>if only terms of the lowest degree in the masses are kept</u>, the semi-major axes a and a' can only oscillate about mean values $a_0$ and $a'_0$ , and also that the eccentricities and inclinations oscillate by small amounts. The assumptions for the validity of this last part of the theorem which should be clearly brought out in view of future developments, are,

1) that the semi-axes describe oscillations of small amplitude.

2) that $\beta \sqrt{a}$ and $\beta' \sqrt{a'}$ are of the same order of magnitude.

It should also be clearly brought out that Laplace's stability theorem holds for any number of bodies, n.

### 3. Elimination of the Nodes. Laplace's stability Theorem.

We denote the Jacobi coördinates of B by $q_1$ $q_2$ $q_3$, those of A by $q_4$ $q_5$ $q_6$. The initial point of the coördinates lies in the former case at C, in the latter at the center of gravity of B and C. We set (three-body problem)

$$p_k = \mu_b \, q_k \qquad\qquad k = 1, 2, 3$$

$$p_k = \mu_a \, q_k \qquad\qquad k = 4, 5, 6$$

the canonic equations of the intermediate path are

$$\dot{q}_i = \frac{\partial H_b}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H_b}{\partial q_i} \qquad i = 1, 2, 3$$

$$\dot{q}_i = \frac{\partial H_a}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H_a}{\partial q_i} \qquad i = 4, 5, 6$$

where, as in Par. 2),

$$H_a = \frac{1}{\mu_a} (p_4^2 + p_5^2 + p_6^2) + \frac{k^2 \, m_a \, m_c}{\sqrt{q_4^2 + q_5^2 + q_6^2}}$$

$$H_b = \frac{1}{\mu_b} (p_1^2 + p_2^2 + p_3^2) + \frac{k^2 \, m_b \, m_c}{\sqrt{q_1^2 + q_2^2 + q_3^2}}$$

The conservation of moment of momentum gives, for B,

$$\mu_b (q_2 \dot{q}_3 - q_3 \dot{q}_2) = c_1$$

$$\mu_b (q_3 \dot{q}_1 - q_1 \dot{q}_3) = c_2$$

$$\mu_b (q_1 \dot{q}_2 - q_2 \dot{q}_1) = c_3$$

Vectorially $\qquad \mu_b q_\kappa \times r_\kappa = C$

and likewise for A. The intermediate motion is therefore in a plane. Let the inclination of this plane be i; $\Omega$ the longitude of the nodal point; we have:

$$c_1 = c \sin i \sin \Omega$$

$$c_2 = - c \sin i \cos \Omega$$

$$c_3 = c \cos i$$

with $c = \sqrt{c_1^2 + c_2^2 + c_3^2}$

and likewise for the mass A. If the XY-plane is laid in the orbit of B, it is easily shown that

$$c = \beta \sqrt{a(1 - e^2)}$$

when $\beta^2 = \dfrac{k^2 m_b^2 m_c^2}{m_b + m_c}$ , a is the semi-major axis and

e the eccentricity; so that the above equations can be written

$$\mu_b (q_2 \dot{q}_3 - q_3 \dot{q}_2) = \beta \sqrt{a(1 - e^2)} \; \sin i \sin \Omega$$

$$\mu_b (q_3 \dot{q}_1 - q_1 \dot{q}_3) = -\beta \sqrt{a(1 - e^2)} \; \sin i \cos \Omega$$

$$\mu_b (q_1 \dot{q}_2 - q_2 \dot{q}_1) = \beta \sqrt{a(1 - e^2)} \; \cos i$$

and likewise for A. Since the law of conservation of angular momentum holds also for the true path, we have,

$$a(1-e^2) \sin i \sin \Omega + \beta' \sqrt{a'(1-e'^2)} \sin i' \sin \Omega' = c_1''$$

$$a(1-e^2) \sin i \cos \Omega + \beta' \sqrt{a'(1-e'^2)} \sin i' \cos = -c_2''$$

$$a(1-e^2) \cos i + \beta' \sqrt{a'(1-e'^2)} \cos i' = c_3''$$

and in the particular case where the XY-plane is chosen as the invariable plane, $c_1'' = c_2'' = 0$, so that from the

### 4. Delaunay's and Poincare's variables. The Perturbation Function and its Expansion.

The elliptic elements of an orbit, which we shall call $a, e, i, t_\pi, \pi$ and $\Omega$ are, as known, the semi-major axis, eccentricity, inclination, time of passage through perihelion, angular distance along equatorial plane of perihelion from increasing nodal point, longitude of nodal point, respectively. Instead of these, we have used canonic variables, $L, G, H, l, g, h$ which are related to the elliptic elements by the equations

$$L = \beta\sqrt{a} \qquad G = \beta\sqrt{a(1-e^2)} \qquad H = \beta\sqrt{a(1-e^2)}\cos i$$

$$l = n(t+\gamma) \qquad g = \pi - \Omega \qquad h = \Omega$$

we now perform a contact-transformation to Delaunay variables $\Lambda, \Gamma, z, \lambda, \gamma, \zeta$ defined by

$$\Lambda = L \qquad \Gamma = L - G \qquad z = G - H$$

$$\lambda = l + g + h \qquad \gamma = -g - h \qquad \zeta = -h$$

That this is a contact-transformation is readily verified by using the Poisson bracket expressions. A new contact transformation from variables $\Gamma, z, \gamma, z$ to Poincare's variables $\xi, \eta, p, q$ is defined by

$$\xi = \sqrt{2\Gamma}\,\cos\gamma \qquad\qquad p = \sqrt{2z}\,\cos\zeta$$

$$\eta = \sqrt{2\Gamma}\,\sin\gamma \qquad\qquad q = \sqrt{2z}\,\sin\zeta$$

and we have
$$(\xi,\xi) = (\eta,\eta) = 0 \qquad (\xi,\eta) = 1$$
$$(p,p) = (q,q) = 0 \qquad (p,q) = 1$$

$\xi, \eta$ are therefore canonic variables. We shall now use the following mixed system of Delaunay-Poincaré variables: $\Lambda, \lambda, \xi, \eta, p, q.$ We have the relations between these and the elliptic elements:

$$\Lambda = \beta \sqrt{a}, \qquad \lambda = \lambda + \pi \qquad \xi = \sqrt{2\Lambda (1 - \sqrt{1-e^2})} \cos \pi$$

$$\eta = \sqrt{2\Lambda (1 - \sqrt{1-e^2})} \sin \pi, \qquad p = \sqrt{2\Lambda (1-e^2)^{\frac{1}{2}}(1-\cos i)} \cos \Omega$$

$$q = -\sqrt{2\Lambda (1-e^2)^{\frac{1}{2}} (1 - \cos i)} \sin \Omega$$

From which, by power expansion,

$$\left(\frac{\xi}{\sqrt{\Lambda}}\right)^2 + \left(\frac{\eta}{\sqrt{\Lambda}}\right)^2 = 2(1 - \sqrt{1-e^2}) = e^2 + \frac{e^4}{4} + \cdots\cdots$$

hence $e^2$ can be expanded in powers of $\left[ \left(\frac{\xi}{\Lambda^{\frac{1}{2}}}\right)^2 + \left(\frac{\eta}{\Lambda^{\frac{1}{2}}}\right)^2 \right] = u$ as follows

$$e^2 = u - \frac{1}{4} u^2 + \cdots\cdots$$

Further $\left[ 2(1 - (1-e^2)^{\frac{1}{2}}) \right]^{\frac{1}{2}} = e + \frac{e^3}{8} + \cdots\cdots$

therefore $\qquad \dfrac{\xi}{\Lambda^{\frac{1}{2}}} = e \cos \pi (1 + \alpha_h e^{2n})$ add ed on n.

$$\dfrac{\eta}{\Lambda^{\frac{1}{2}}} = -e \sin \pi (1 + \alpha_h e^{2n}) \qquad " \quad " \quad "$$

hence the theorem: $e \cos \pi$ and $e \sin \pi$ can be expanded in powers of $\xi / \Lambda^{\frac{1}{2}}$ and $\eta / \Lambda^{\frac{1}{2}}$ and conversely. In a similar manner we may show that $\sin i \cos \Omega$ and $\sin i \times \sin \Omega$ can be expanded in a powers of $p/\Lambda^{\frac{1}{2}}$ and $q/\Lambda^{\frac{1}{2}}$.

The function $F = \dfrac{\beta^{\prime}}{2\mu_a L^{\prime 2}} + \dfrac{\beta^{\prime\prime}}{2\mu^{\prime} L^{\prime 2}} - H^{\prime\prime}$

or $F = \dfrac{\beta^{\prime}}{2\mu L^{2}} + \dfrac{\beta^{\prime\prime}}{2\mu L^{\prime 2}} + \dfrac{K^{2} m_a m_b}{r_{ab}} + \dfrac{K^{2} m_a m_c}{r_{ac}} - \dfrac{K^{2} m_a m_c}{r_{ga}}$

is known as the perturbation function of the three-body problem. It is now required to show that the perturbation function can be expanded in powers of $e \cos\pi$, $e \sin\pi$, $\sin i \cos\Omega$, $\sin i \sin\Omega$, etc. Since this perturbation function is an analytic function of the coördinates, it is only required to prove that the latter can be expanded in power series of the quantities in question. We shall not take up the formal proof here.[8] The more important question of the convergency of the expansion is taken up later. Suffice it to state that the perturbation function can be expanded as follows:

$$F = \sum_{i,j,k,\dots=0}^{\infty} A_{i,j,k,l,\dots} \left(\frac{\xi}{\Lambda^{\frac{1}{2}}}\right)^{i} \left(\frac{\eta}{\Lambda^{\frac{1}{2}}}\right)^{j} \left(\frac{p}{\Lambda^{\frac{1}{2}}}\right)^{k} \left(\frac{q}{\Lambda^{\frac{1}{2}}}\right)^{l} \dots$$

and the differential equations of the canonic elements are

$$\dot\Lambda = \frac{\partial F}{\partial \lambda}, \quad \dot\lambda = -\frac{\partial F}{\partial \Lambda}; \quad \dot\xi = \frac{\partial F}{\partial \eta}, \quad \dot\eta = -\frac{\partial F}{\partial \xi};$$

$$\dot p = \frac{\partial F}{\partial q}, \quad \dot q = -\frac{\partial F}{\partial p} \quad \text{etc.}$$

No one has yet been able to carry out the rigorous integration of this system. It is not known whether the semi-axes of the osculating ellipses oscillate within finite limits, or whether the elements, $\xi, \eta, p, q$ etc. can increase

indefinitely. All that we do know, from the days of
Laplace, is that _if_ the variation of the $\Lambda$'s is _small_
so are the variations of $\xi$, $\eta$, etc. All that we can do
is to calculate the orbits for a _limited_ time. The methods
used in this connection constitute the so-called Cal-
culus of Perturbations. The whole astronomical theory
of perturbations is based upon the assumption that the
deviations of the elements are _small_. If this is true, the
changes of the elements themselves are small, at least
over certain intervals of time, and may be neglected. The
integration of the equations attained in this manner gives
the "perturbation of the first order". From this, a
repetition of the process gives the "expansion in powers
of the masses". It is emphasized here that such power
expansions are _not_ always convergent, but may be suffi-
ciently approximate over certain finite intervals.

Let  E  be any element. We have for E a differential
equation of the form

$$E = f\left(\Lambda, \lambda, \xi, \eta, p, q \ldots\right)$$

If we look upon the system planet-planet for a short time
as a two-body problem, we know that  f  is a periodic
function of time, because the two-body problem is condi-
tioned periodic. We may therefore write

$$\dot{E} = \sum \beta^{(i,i')} \cos\left[i\lambda + i'\lambda' + D^{(i,i')}\right]$$

and for each element we have an equation of the same form.

In order to obtain the perturbation of the <u>first</u> order,
we set,

$$\lambda = \lambda_o = n_o(t + \gamma_o)$$

$$\lambda' = \lambda_o' = n_o'(t + \gamma_o')$$

and set the other elements $\Lambda, \xi, \eta$ etc. equal to constants
$\Lambda_o, \xi_o, \eta_o$   etc.  We get then

$$\dot{E} = \sum B_o^{(i,i')} \cos(i\lambda_o + i'\lambda^o + D^{(i,i')})$$

where $B_o^{(i,i')}$ , $D_o^{(i,i')}$   are independent of time.  The

discussion of the convergence of this series will be left till
later.   Integrating we get

$$E = \sum \frac{B^{i,i'}}{in_o + i'n_o'} \sin(i\lambda_o + i'\lambda_o' + D_o^{i,i'}) + Ct + E_o$$

where $C = B^{o,o}$   and $E_o$ is an integration constant.  $\Lambda_o, \gamma_o, \xi_o,$
etc. are so chosen that at a given instant -- so-called
"epoch" -- they determine an osculating-element system.
The above expression for E consists of the following terms:

   a) A term ct, non-periodic in t, which is the so-
called <u>secular</u> perturbation

   b) The terms $\sum \dfrac{B^{i,i'}}{in_o + i'n_o'} \sin(i\lambda_o + i'\lambda_o' + D_o^{i,i'})$

which are the <u>periodic</u> perturbations.

It is apparent that secular perturbations increase inde-
finitely with time, <u>so long as we consider perturbations of</u>
<u>the first order</u>.  If perturbations of higher orders are

taken into account, it is shown, <u>although the mathe-matical proof is not free from objection,</u> that such secular perturbations are <u>also periodic</u>, of comparatively large amplitude and period.

If the sum

$$\sum_{i'=0}^{\infty} \left| \frac{B^{i,i'}}{in_0 + i'n'_0} \right|$$

is finite, periodic perturbations cannot exceed a certain upper limit. Each term in (b) is periodic and comes back to its same value after a certain time, except when $in_0 + i'n'_0 = 0$, i.e. <u>when the osculating motions of the two planets are commensurable.</u> In this case our differential equation can obviously not be handled in the way above. On the other hand, if $n_0$ and $n'_0$ are not commensurable, we may choose $i^0$ and $i'$ so that $in_0 + i'n'_0$ remains as small as we please. Such terms -- so-called "small divisors" -- play an important part in the theory of perturbations and cause no inconsiderable difficulties, because terms containing them may become exceedingly large. Such terms, as will be seen later, are of paramount importance in atomic dynamics.

5. <u>Secular Perturbations</u>. In previous paragraphs we have shown that, by using Jacobi coördinates and Delaunay elements, the equations of the three-body problem can be reduced to a canonic system with six degrees of freedom and, further, have pointed out that the perturbation function F may be expanded in a Fourier series

$$F = \sum_{i,i'} A \cos(il + i'l') + \sum_{i,i'} B \sin(il + i'l')$$

where, for $i = i' = 0$ we obtain the secular perturbations. Denoting the secular part of the perturbation function by S, we have, by Fourier's theorem:

$$S = \frac{1}{\pi} \int_0^\pi \int_0^\pi F dl dl' = S(L,L',G,G',H,H',g,g',h,h')$$

Since S does not depend on $l,l'$, the canonic equations yield at once

$$\dot{L} = 0 , \qquad \dot{L}' = 0$$

whence $\qquad L = \text{const.} \quad L' = \text{const.}$

which constitutes the first part of Laplace's celebrated stability theorem. The secular perturbations in G,H,g, h, etc. are determined by the canonic system

$$\dot{G} = \frac{\partial S}{\partial g} , \quad \dot{H} = \frac{\partial S}{\partial h} , \quad \dot{G}' = \frac{\partial S}{\partial g'} , \quad \dot{H}' = \frac{\partial S}{\partial h'}$$

$$\dot{g} = -\frac{\partial S}{\partial G}, \quad \dot{h} = -\frac{\partial S}{\partial H} , \quad \dot{g}' = -\frac{\partial S}{\partial G'} , \quad \dot{h}' = -\frac{\partial S}{\partial H'}$$

If we place $\quad G = \Gamma \qquad G' = \Gamma'$

$$H = \frac{c}{2} + \frac{1}{2c} (\Gamma^2 - \Gamma'^2)$$

$$H' = \frac{c}{2} - \frac{1}{2c} (\Gamma^2 - \Gamma'^2)$$

the canonic system above may be reduced to one of two degrees of freedom and, if the motion is plane, the former may still be reduced to a system of one degree of freedom having the integral S = const. and which, therefore, can be integrated by quadratures. So far as is known to me, the actual integration has never been carried out. [9]

For the numerical calculation of the perturbation, we may make use of the following expansions:

$$ S = \frac{\beta^4}{2\mu\Lambda^2} + \frac{\beta'^4}{2\mu'\Lambda'^2} + \frac{k^2}{2}m_a m_b A_0 + k^2 m_a m_b \left[ \frac{B_1}{8} \left( \frac{\xi^2 + \eta^2}{\Lambda} + \right. \right. $$

$$ + \frac{\xi'^2 + \eta'^2}{\Lambda'} \left) - \frac{B_2}{4} \left( \frac{\xi\xi'}{\sqrt{\Lambda\Lambda'}} + \frac{\eta\eta'}{\sqrt{\Lambda\Lambda'}} \right) - \frac{B_1}{8} \left( \frac{p^2 + q^2}{\Lambda} + \right. $$

$$ \frac{p'^2 + q'^2}{\Lambda'} - \frac{2(pp' \; qq')}{\sqrt{\Lambda\Lambda'}} \right) $$

in which $A_0$, $B_1$, $B_2$ are the Laplace coefficients.

In the case of n-bodies of masses $m_1$ $m_2$ .... $m_n$ whose Poincaré elements are $\xi_i, \eta_i$ etc. (i = 1,2...n), we may express S as follows:

$$ S = R_0 + \sum R_2'(\xi_i, \xi_j) + \sum R_2'(\eta_i, \eta_j) - \sum R_2''(p_i, p_j) $$

$$ - \sum R_2''(q_i, q_j) $$

where

$$ R_0 = \sum \frac{\beta_i^4}{2\mu_i \Lambda_i^2} + \frac{1}{2}\sum k^2 m_i m_j A_0(a_i, a_j) $$

$$ R_2'(x_i, x_j) = k^2 m_i m_j \left[ \frac{B_1}{8}(a_i, a_j)\left( \frac{x_i^2}{\Lambda_i} + \frac{x_j^2}{\Lambda_j} \right) - \right. $$

$$ - \frac{B_2}{4}(a_i, a_j) \frac{x_i x_j}{\sqrt{\Lambda_i \Lambda_j}} \right] $$

with $B_1(a_i, a_j) = \dfrac{2}{\pi} \displaystyle\int_0^\pi \dfrac{a_i a_j \cos\omega\, d\omega}{\left[a_i^2 + a_j^2 - 2a_i a_j \cos\omega\right]^{3/2}}$

$B_2(a_i, a_j) = \dfrac{2}{\pi} \displaystyle\int_0^\pi \dfrac{a_i a_j \cos 2\omega\, d\omega}{\left[a_i^2 + a_j^2 - 2a_i a_j \cos\omega\right]^{5/2}}$

The secular perturbations in eccentricities and longitude of the perihelions are given by the canonic equations:

$$\dot{r} = -k_1 s + k_2 s' \quad , \quad \dot{s} = k_1 r - k_2 r'$$

$$\dot{r}' = -k' s' + k_2' s' \quad , \quad \dot{s}' = k_1' r' - k_2' r$$

where $\zeta = \sqrt{\Lambda}\, e \cos\pi = r\sqrt{\Lambda'}$ approx.

$p = \sqrt{\Lambda} \sin i \cos\Omega = u\sqrt{\Lambda}$ approx.

$\eta = -\sqrt{\Lambda}\, e \sin\pi = -s\sqrt{\Lambda}$ approx.

$q = -\sqrt{\Lambda} \sin i \sin\Omega = u - v\sqrt{\Lambda}$ approx.

and $k_1 = \dfrac{k^2 m_a m_b}{4\Lambda} B_1 \qquad k_2 = \dfrac{k^2 m_a m_b}{4\Lambda} B_2$

$k_1' = \dfrac{k^2 m_a m_b}{4\Lambda'} B_1 \qquad k_2' = \dfrac{k^2 m_a m_b}{4\Lambda'} B_2$

Expressed in variables r, s, u, v, etc. the perturbation function is

$$S = \frac{\beta^2}{2\mu\Lambda^2} + \frac{\beta'^2}{2\mu'\Lambda'^2} + \frac{k^2}{2} m_a m_b A_0$$

$$+ K_1^2 m_a m_b \left[ \frac{B_1}{8}(r^2 + s^2 + r'^2 + s'^2) - \frac{B_2}{4}(rr' + ss') \right.$$

$$- \frac{B_1}{8} \left( u^2 + v^2 + u'^2 + v'^2 \right) + \frac{B_1}{4} \left( uu' + vv' \right)$$

We may write the above as

$$k'_2 \left( r\dot{r} + s\dot{s} \right) + k_2 \left( r'\dot{r}' + s'\dot{s}' \right) = 0$$

whose integral is

$$k'_2 \left( r^2 + s^2 \right) + k_2 \left( r'^2 + s'^2 \right) = c$$

or $\quad k'_2 \, e^2 + k_2 \, e'^2 = c$

an equation which immediately shows that the essen-
tricities cannot increase indefinitely, provided $k'_2$

and $k_2$ have the same sign as must be the case if both
planets revolve in the same direction. If the former
are small at a given epoch, they must stay small.
The integrals of the above linear equation are of the
form

$$r = N \cos \left( gt + \beta \right) \qquad r' = N' \cos \left( gt + \beta' \right)$$
$$s = N \sin \left( gt + \beta \right) \qquad s' = N' \sin \left( gt + \beta' \right)$$

Setting these solutions in our differential equations,
we see that the condition

$$\begin{vmatrix} g - k_1 & k_2 \\ k'_2 & g - k_1 \end{vmatrix} = 0$$

must be satisfied. This equation has two roots $g_1$
and $g_2$ which must be real and positive provided both
planets move in the same direction. If we let $g_1 = N_1 / N'_1$

and $g_2 = N_2/N_2'$ , we may write the above solutions as follows:

$$r = N_1 \cos (g_1 t + \beta_1) + N_2 \cos (g_2 t + \beta_2)$$

$$s = N_1 \sin (g_1 t + \beta_1) + N_2 \cos (g_2 t + \beta_2) \qquad \text{etc.}$$

$N_1$, $N_2$, $\beta_1$, $\beta_2$ may be regarded as integration constants. The following theorems follow readily,

1) If $N_1 < N_2$ , the longitude of the perhelion has the mean motion $g_2$ .

2) If $N_1 > N_2$ , the perihelion has the mean motion

$$\tfrac{1}{2}(g_1 + g_2)$$

Now, the equations of motion of the orbital plane are:

$$\dot{u} = k_1(v-v') , \quad \dot{v} = -k_1(u-u') , \quad \dot{u}' = -k_1' (v-v')$$

$$\dot{v}' = k_1'(u-u')$$

whence
$$k_1' (u\dot{u} + v\dot{v}) + k_1(u'\dot{u}'+v'\dot{v}') = 0$$

$$k_1' \dot{u} + k_1 \dot{u}' = 0$$

$$k_1' \dot{v} + k_1 \dot{v}' = 0$$

whence by integration we obtain immediately

$$k_1' u + k_1 u' = c_1 , \quad k_1' v + k_1 v' = c_2$$

$$k_1' (u^2 + v^2) + k_1 ( u'^2 + v'^2) = c_3$$

That these equations correspond to the general areal integrals can be readily shown. If now we locate the fundamental plane along the invariable plane, $c_1 = c_2 = 0$ so that $u/u' = v/v' = -k_1/k_1'$

we get: $\tan \Omega = \tan \Omega'$

which is Jacobi's well-known theorem, that the nodal lines of both planets fall together on the invariable plane, which, as shown by him, holds for all powers of u,v,etc. Again from

$$u\dot{u} + v\dot{v} = 0, \qquad u'\dot{u}' + v'\dot{v}' = 0$$

we obtain the theorem that the inclination of either orbit with respect to the invariable plane is constant. The single variable to be determined is therefore that which determines the motion of the common nodal line on the invariable plane. This is given by

$$\dot{\Omega} = -(k_1 + k_1')$$

and we have the theorem: the common nodal line moves backwards with uniform velocity $k_1 + k_1'$ .

# Bibliography

1. Lie-Scheffers: Berührungstransformationen.
   also: Whittaker "Analytical Dynamics", Ch. XI
      specially p.298.

2. Whittaker, l.c. Ch. XIII.

3. Charlier; Mechanik des Himmels, Vol. 1, p.237.

4. Charlier, l.c. p. 230.

5. Charlier, l.c. p.234.

6. Charlier, l.c. p. 243.

7. Levi-Civita: Mathematische Annalen, Vol. 58, p.383, 1904.

8. Charlier, l.c. p. 296.

9. Charlier, l.c. p.338.
   also: K. Bohlin "Till frågan om sekulära störingar"
      Kgl. Svenska Vetenskab Akad. 1891.

II)Periodic Solutions.

1) <u>Simplest example of a rigorous solution of</u>
<u>the three-body problem.</u> No general integral of the
three-body problem is known, but for a long time certain
configurations of the three bodies ~~are~~ <sub>^</sub>known for which
(have been)
rigorous solutions can be obtained. These solutions,
discovered by Lagrange (1 )  seem to be unimportant, at
first sight -- Lagrange says textually "cette recherche
n'est a la verite que de pure curiosite" -- but later
investigations have shown them to be of fundamental im-
portance, because they stand in a close relationship to
the singularities of the general integrals. Lagrange's
solutions constitute the starting point of a family of
periodic solutions of the n-body problem and may be looked
upon as the simplest known periodic solutions. The latter
are the <u>only</u> <u>rigorous</u> (known) solutions of the n-body
pboblem.

Suppose that we have n bodies in a <u>plane</u>, so arranged
that the resultant of the forces acting on each body goes
through their common center of mass G and that this re-
sultant is proportional to the distance from G, it is
obvious, that the masses will stay forever in the same re-
lative position if we give the whole system a spin around
G such that the centrifugal force arising from this ro-
tation is equal to the corresponding resultant force. This

type of solution has been often utilized in atomic models.

A similar stable configuration arises when the initial velocities of all the masses have an oblique direction with respect to the line joining the mass in question with G and are proportional to the distance of this mass from G. The problem is to find the required configuration of the mass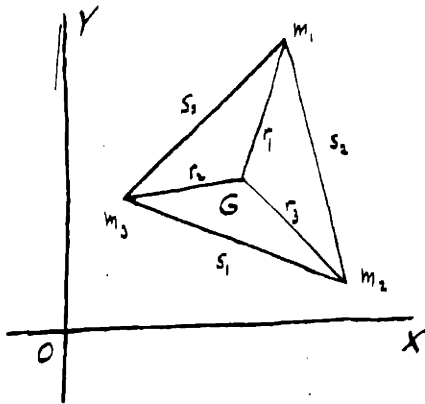es. Let $m_1$ $m_2$ $m_3$ (Fig. 1) be the three masses; $s_1$ $s_2$ $s_3$ their mutual distances; $r_1$ $r_2$ $r_3$ their distances from the center of gravity. If $X_1$ $X_2$ $X_3$ are the accelerations of $m_1$ $m_2$ $m_3$ respectively parallel to the x-axis and $\varphi(s)$ the law of force, we have:



Fig. 1

$$X_1 = m_2 s_3^{-1} \varphi(s_3)(x_2-x_1) + m_3 s_2^{-1} \varphi(s_2)(x_3-x_1)$$

$$X_2 = m_3 s_1^{-1} \varphi(s_1)(x_3-x_2) + m_1 s_3^{-1} \varphi(s_3)(x_1-x_2)$$

$$X_3 = m_1 s_2^{-1} \varphi(s_2)(x_1-x_3) + m_2 s_1^{-1} \varphi(s_1)(x_2-x_3)$$

and likewise for $Y_1$, $Y_2$, $Y_3$. Since the resultant of the forces acting on $m_1$ $m_2$ $m_3$ goes through G we have:

$$X_1 = k_1 x_1 \qquad X_2 = k_2 x_2 \qquad X_3 = k_3 x_3$$

$$Y_1 = k_1 y_1 \qquad Y_2 = k_2 y_2 \qquad Y_3 = k_3 y_3$$

The magnitudes of the ^resultants are:

$$k_1(x_1^2 \quad y_1^2)^{\frac{1}{2}} \qquad k_2(x_2^2 \quad y_2^2)^{\frac{1}{2}} \qquad k_3(x_3^2 \quad y_3^2)^{\frac{1}{2}}$$

and since they must be proportional to their distances to G, therefore $k_1 = k_2 = k_3$. Eliminating $x_3$, $y_3$ we get $x_1 : x_2 = y_1 : y_2$, provided $\varphi(s_2)\, s_2^{-1} -$

$$- \varphi(s_3)\, s_3^{-1} \neq 0. \quad \text{If such is the case:}$$

$$s_1 = s_2 = s_3 = \frac{(m_1 + m_2 + m_3)\, r}{(m_1^2 + m_2^2 + m_3^2)^{\frac{1}{2}}}$$

In particular, if the force is Newtonian, it is readily seen that each mass describes a conic, of which one of the foci lies at the common center of mass. The masses are at the vertices of an equilateral triangle, and if the conic is a ^parabola or hyperbola, their mutual distances may increase indefinitely. Exceptionally the three masses may lie on a straight line.

Following Gylden, we shall call these points in which there is a Lagrangian rigorous solution of the three-body problem, "librations centers".

2) <u>Periodic solutions in the vicinity of li-</u>
<u>bration centers.</u>  The <u>general</u> solutions of the three-
body problem, containing the required number of arbi-
trary constants $(\mathbf{n})$ is as yet unknown.  Nevertheless
various <u>particular</u> solutions, containing less than n
arbitrary constants, have been found.  Periodic solu-
tions of this type play a predominant part in recent
astronomical investigations, but it is only quite re-
cently that their application to atomic systems has been
considered.  The investigation of these periodic orbits
is now in fact so far developed that it can even serve
for numerical computation of orbits.

Periodic solutions of the type we are about to con-
sider were introduced by G. W. Hill [3] and their gen-
eral theory was first developed by H. Poincaré, who
applied to this task all the powerful mathematical equip-
ment which stood at his disposal.  His first memoir on
this subject dates back to 1889 [4] and a good deal of
his classical work "Les Méthodes nouvelles de la mécani-
que céleste" is devoted to the consideration of this
problem.

Periodic orbits are defined by the statement that they
are such that the configuration of the system of n bodies
repeats itself at fixed intervals. [5]  They are usually
divided into two classes:  In the <u>first</u> class, the de-
partures from the initial configuration is <u>infinitely</u>

small, in the second class, the former are _finite_.
Periodic orbits of the second class can be found in gen-
eral for any continuous region of space.  Periodic so-
lutions of the first class on the other hand, occur only
for certain quite particular configurations of the bodies.
Orbits of the latter class are of course easier to find
and handle analytically.

Consider the special case of the three-body problem
in which the mass of one of the three bodies is infini-
tesimal and besides both finite masses move in a circle
around the common center of gravity with uniform velocity.
Such a case is known in astronomy as "the asteroidal three-
body problem". (French: "Problème restreint".).  Although
of restricted importance in connection with atomic problems,
we shall discuss it briefly for the sake of illustration.

Let $m_1$ and $m_2$ be the masses of the sun and planet
respectively,  P the asteroid (planetoid), $r_1, r_2$ distances
of $m_1, m_2$ from center of gravity G, respectively.  We place

$$\mu = m_2 / m_1 < 1$$

and have,

$$r_1 + r_2 = 1$$

$$r_1 = \mu r_2$$

Further choose the time unit so that the gravitational con-
stant is 1.  The time for a revolution of $m_1$ and $m_2$ around
G, is  $T = 2\pi / \sqrt{1+\mu}$   and the angular velocity is
$n = \sqrt{1+\mu}$  .  If, as above, the asteroid moves in
the plane of $m_1, m_2$, we have,   choosing G for origin of

coordinates and referring the cartesian coordinates of P (x,y) to axes rotating at a constant velocity n we have the equations of motion of P:

$$\ddot{x} - 2n\dot{y} - n^2 x = -\frac{\partial V}{\partial x}$$

$$\ddot{y} + 2n\dot{x} - n^2 y = -\frac{\partial V}{\partial y}$$

where $V = \dfrac{-1}{\left[(x-r_1)^2 + y^2\right]^{1/2}} - \dfrac{\mu}{\left[(x+r_2)^2 + y^2\right]^{1/2}} = -\dfrac{1}{\rho_1} - \dfrac{\mu}{\rho_2}$

after an easy algebraic transformation, writing

$$\Omega = \tfrac{1}{2}\left[\rho_1^2 + \frac{2}{\rho_1} + \mu\left(\rho_2^2 + \frac{2}{\rho_2}\right)\right]$$

we see at once that the above equations admit the Jacobi integral

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C$$

where C is Jacobi's constant.

Placing now

$$q_1 = x, \quad q_2 = y, \quad p_1 = \dot{x} - ny, \quad p_2 = \dot{y} + nx$$

$$H = \tfrac{1}{2}\left[p_1^2 + p_2^2 + 2n(p_1 q_2 - p_2 q_1)\right] + V \quad \text{(Hamilton function)}$$

we get the equations of motion in the canonic form:

$$\dot{q}_1 = \frac{\partial H}{\partial p_1}, \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1}, \quad \dot{q}_2 = \frac{\partial H}{\partial p_2}, \quad \dot{p}_2 = -\frac{\partial H}{\partial q_2}$$

If $x = a$, $y = b$ are the coördinates of an arbitrary point which does not coincide either with $m_1$ or $m_2$, we

see that $\Omega$ and all its derivatives with respect to
x and y can be expanded in positive powers of x - a
and y - b. If x - a and y - b are sufficiently small,
the expansion will involve terms of lower powers only.
We now investigate the <u>existence</u> of points enjoying the
property that, points in their immediate neighborhood
admit periodic solutions of our differential equations,
so that P <u>can stay forever in the vicinity of these points,</u>
because it moves along a path which repeats itself after
fixed intervals of time. Let $x = a + \xi$ , $y = b + \eta$
From the equations obtained by using $\Omega$ instead of V, we
have:

$$\ddot{\xi} - 2\eta\dot{\eta} = -\left( \frac{\partial \Omega}{\partial a} + \xi \frac{\partial^2 \Omega}{\partial a^2} + \eta \frac{\partial^2 \Omega}{\partial a \partial b} + \cdots \right)$$

$$\ddot{\eta} - 2\eta\dot{\xi} = -\left( \frac{\partial \Omega}{\partial b} + \xi \frac{\partial^2 \Omega}{\partial a \partial b} + \eta \frac{\partial^2 \Omega}{\partial b^2} + \cdots \right)$$

Following our initial assumptions, we neglect higher
powers of $\xi$ and $\eta$ . In order that this may be so, we
must have

$$\frac{\partial \Omega}{\partial a} = \frac{\partial \Omega}{\partial b} = 0 \quad .$$

and the point (a,b) is determined by these equations.
We have $\quad \rho_1^2 = (a - r_1)^2 + b^2, \quad \rho_2^2 = (a + r_2)^2 + b^2$

whence $\quad \dfrac{\partial \Omega}{\partial \rho_1} \dfrac{a - r_1}{\rho_1} + \dfrac{\partial \Omega}{\partial \rho_2} \dfrac{a + r_2}{\rho_2} = \dfrac{\partial \Omega}{\partial a} = 0$

$$\frac{\partial \Omega}{\partial \rho_1} \quad \frac{b}{\rho_1} + \frac{\partial \Omega}{\partial \rho_2} \quad \frac{b}{\rho_2} = \frac{\partial \Omega}{\partial b} = 0$$

In order that these may be satisfied, we must have either

$$\frac{\partial \Omega}{\partial \rho_1} = \frac{\partial \Omega}{\partial \rho_2} = 0 \qquad (1)$$

or

$$\begin{vmatrix} \dfrac{a - r_1}{\rho_1} & \dfrac{a \ r_2}{\rho_2} \\ \underline{b} & \underline{b} \end{vmatrix} = - \frac{b}{\rho_1 \rho_2}(r_1 + r_2) = - \frac{b}{\rho_1 \rho_2} \qquad (2)$$

Equations (1) yield $\rho_1 - \rho_1^{-1} = \rho_2 - \rho_2^{-2} = 0$

$$\text{or} \qquad \rho_1 = \rho_2 = 1$$

and the point (a,b) lies át the vertex of an equilateral triangle whose side is equal to $m_1 \ m_2$. In calculating the roots of (2) we must distinguish three cases

1) $-r_2 < a < r_1$

2) $a < -r_2$

3) $r_1 < a.$

$\rho_1$ or $\rho_2$ are then determined by the quintics

1) $(1+\mu)\rho_2^5 - (3 + 2\mu)\rho_2^4 + (3+\mu)\rho_2^3 - \mu \rho_2^2 + 2\mu\rho_2 - \mu = 0$

2) $(1+\mu)\rho_2^5 + (3 + 2\mu)\rho_2^4 + (3+\mu)\rho_2^3 - \mu \rho_2^2 - 2\mu\rho_2 - \mu = 0$

3) $(1 +\mu)\rho_1^5 + (2 + 3\mu)\rho_1^4 + (1+3\mu)\rho_1^3 - \rho_1^2 - 2\rho_1 - 1 = 0$

$$( \ 6 \ )$$

For $\mu$ very small, the roots are

1) $\rho_2 = (\mu/3)^{1/3}$ , 2) $\rho_2 = (\mu/3)^{1/3}$, 3) $\rho_1 = 1 - 7\mu/12.$

Now, if we compare the quintics just obtained with

the equation which determines the relative position of
the three bodies in the collinear case of Lagrange's
solution $^{(7)}$ we see at once that points where vicinity
periodic solutions can be found fall together with the
points for which a rigorous Lagrangian solution can be
found.

We close this paragraph with a summary investigation
of the curve $2\Omega - c = 0$. From Jacobi's integral

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - c$$

we see that the point $P$ can move only in that part of the
plane for which $2\Omega - c > 0$. If the curve $2\Omega - c = 0$
is open, it divides the real plane into two parts and
motion is possible only in one of them. If it is closed,
or has one or more closed branches, motion can take place
only within such bounded regions of the plane. $2\Omega - c = 0$
is therefore called the "limiting curve" or also, "Hill's
curve". It is very important in discussions connected with
the stability of a dynamic system. A discussion and cal-
culation of Hill's curves was given first by Darwin $^{(8)}$
also by Bohlin $^{(9)}$. For any given $\mu$ the curve depends
only on the parameter c. If $c < 0$ then evidently
$2\Omega - c \neq 0$. c has therefore a positive minimum value
and if c is smaller than this minimum, there exists no
limiting curve. This minimum is $c_{min} = 3(1 + \mu)^{(10)}$.
The limiting curves for $\mu = 0.1$ are shown in Fig. 2. where

$L_1$, $L_2$, $L_3$ etc. are the libration centers.

From the fact that the equations for $\xi$, $\eta$ are linear with <u>constant</u> coefficients in the neighborhood of the libration centers, it follows that they admit solutions

$$\xi = A e^{\lambda t} \qquad \eta = B e^{\lambda t}$$

and the equation determining $\lambda$ is

$$\begin{vmatrix} \lambda^2 - \dfrac{\partial^2 \Omega}{\partial a^2} & -2n\lambda + \dfrac{\partial^2 \Omega}{\partial a \, \partial b} \\[4mm] 2n - \dfrac{}{a \; b} & \lambda^2 - \dfrac{\partial^2 \Omega}{\partial b^2} \end{vmatrix} = 0$$

The nature of the motion depends on the value of the roots. If $\lambda^2$ is real and negative, periodic solutions exist.

The investigation of periodic solutions in the vicinity of the masses $m_1$ and $m_2$ is complicated by the fact that the potential function cannot be expanded there in powers of     distances. This case presents scanty interest in those applications in which we are mainly concerned and will not be considered here.

# Bibliography

1. Lagrange: Oeuvres. Vol. IV, p.230.

2. See for ex. Whittaker: Analytical Dynamics. Chapter XIII, Cambridge, 1917.

3. G. W. Hill: American Journal of Mathematics, "Researches in Lunar Theory". 1878.

4. H. Poincare: Sur le probleme des trois corps et les equations de la dynamique" (Stockholm Preisschrift), Paris, 1889.

5. Whittaker, l.c. p. 386.

6. Charlier: Mechanik des Himmels, Vol. II, p.110.

7. Whittaker, l.c. p. 392.

8. C. G. Darwin: Acta Mathematica. Vol. 21, p.99., 1897.

9. K. Bohlin: Acta Math. Vol. 10, p.102, 1887.

10. Charlier, l.c. Vol. 2, p.112.

## Periodic Solutions of the First, Second and

## Higher Classes

In his classical investigation of periodic solutions of the dynamical equations, Poincare starts with a consideration of the <u>osculating</u> <u>elements</u> and is thereby led to the consideration of the following three classes of periodic solutions of the three-body problem:

1) For the first class, the inclinations with respect to the invariable plane are zero and the eccentricities vanish for vanishing perturbations.

2) For the second class, the inclinations likewise vanish, but the eccentricities are finite for vanishing mass.

3) For the third class, neither the inclinations nor the eccentricities vanish.

Solutions of the first class are distinguished from those of the second class by an essential point: Suppose we consider two planets moving uniformly in circular paths around the main mass, then the motion of these three masses is <u>always</u> periodic and its period is equal to the synoidal time of both planets. Matters are different if the planets move in elliptical orbits around the central body. The motion can also be here periodic, but only under the condition that the mean motions of both planets shall be commensurable.

In many cases, Poincare proceeds as follows:
Suppose that for $\mu = 0$ the inclinations are zero but
the eccentricities are finite. The question is now:
when does a system of three bodies, of which two move
in given Keplerian orbits around the third, make up a
periodic system? Evidently such is the case when the
mean motions of the two planets are commensurable. Then
the motion is always periodic. Let $n/n' = p/q$ where
p and q are prime integers. If N is the greatest common
devisor of p and q, the period T is $2\pi/N$. Now, when do
periodic motions of the same period T exist when $\mu \neq 0$?
We have seen already how the general equations of the
three-body problem can be reduced to a system of three
degrees of freedom. Taking the canonic elements as

$$L = \beta \sqrt{a} \qquad l = \text{mean anomaly of } m,$$
$$L' = \beta' \sqrt{a'} \qquad l' \quad '' \quad '' \quad '' \ m'$$
$$K = \beta \sqrt{a(1-e^2)} \qquad K = \pi - \pi'$$

The perturbation function in terms of these elements
will be called P. We have

$$P = P_0 + \mu P_1 + \mu^2 P_2 + \ldots$$

In our previous notations:

$$P_0 = \frac{\beta^2 m}{2 L^2} + \frac{\beta'^2 m'}{2L'^2}$$

(1)

and $P_1 = \sum A \cos (il + i'l' + jK)$

where the A's depend only on L,L' and K.  Our canonic equations are

$$\dot{L} = \frac{\partial F}{\partial l} \ , \quad \dot{L'} = \frac{\partial F}{\partial l'} \ , \quad \dot{K} = \frac{\partial F}{\partial K} \ , \quad \dot{l} = -\frac{\partial F}{\partial L}$$

$$\dot{l'} = -\frac{\partial F}{\partial L'} \ , \quad \dot{k} = -\frac{\partial F}{\partial K}$$

We investigate periodic solutions of these equations of period T.  By Poincaré's theorem we must have:

$$H(P_o) = \frac{q\,\beta^a\beta'^b\,m\,m'}{L^4\,L'^4}$$

and besides

$$\frac{\partial[P_1]}{\partial l} = \frac{\partial[P_1]}{\partial l'} = \frac{\partial[P_1]}{\partial K} = 0, \qquad \frac{\partial[P_1]}{K} = 0$$

We can easily find, if $l_o, l'_o$ are the value of $l,l'$ for $t = 0.$  $\mu = 0$

$$[P_1] = \sum A\cos\left[s(ql_o - pl'_o) + jk_o\right] \qquad s = 1,2,3\ldots$$

So that our conditions are

$$\sum sq\,A\sin\left[s(ql_o - pl'_o) + jk_o\right] = 0$$

$$\sum sp\,A\sin\left[s(gl_o - pl'_o) + jk_o\right] = 0$$

$$\sum j\,A\sin\left[s(gl_o - pl'_o) + jk_o\right] = 0$$

p and q are <u>given</u> numbers, so the first and second of these equations are identical, and we may choose one of the two quantities $l_o$, $l'_o$ arbitrarily.  We put $l'_o = 0$, i.e. the mass m' is at its perihelion at $t = 0$.  The equations to be satisfied are now:

$$\sum s\ A \sin (sql_o + jk_o) = 0$$

$$\sum j\ A \sin (sql_o + jk_o) = 0$$

which are satisfied for $k_o$, $l_o = i\pi$ or 0, where i is
an integer. Geometrically, this means that at $t = 0$
and for $\mu = 0$, both bodies are either in conjunction or
in opposition on the line of apsides, which obviously
has the same direction for both planets. The longitudes
of the perihelions can either fall together or differ
by $180^o$. Poincaré refers to this case as "symmetrical
conjunction" or opposition. To be sure this is not the
general solution, for Schwarzschild[(2)] first pointed out
that, in order that our equations be satisfied, it is
not necessary that $l_o$ be a multiple of $(180^o)$ but only
that $gl_o$ be such a multiple. We have then the solutions

$$\alpha)\quad k_o = \pi - \pi' = 0 \qquad l_o = \frac{r}{q}\ 180^o$$

$$)\quad k_o = \pi - \pi' = 180^o \qquad l_o = \frac{r}{q}\ 180^o$$

where $r = 0,1,2, \ldots 2q - 1$. In each one of these cases
there are 2q values of $l_o$ which together make up a per-
iodic solution. These 4q values of $l_o$ do not have to be
all different. For example if $n/n' = 1/5$ , $q = s$ and
$l_o$ may have any one of the values.

$$l_o = 0, \quad 60^o, \quad 120^o, \quad 180^o, 240^o, 300^o.$$

In closing this paragraph we shall mention two im-
portant theorems, for a proof of which the reader is re-
ferred to either Poincaré of Charlier. These are:

1) In any periodic solution of the second kind of the asteroidal (restricted) three-body problem, the perihelion of the small planet is <u>fixed</u>.

2) The eccentricity of the orbit of the small planet is always <u>smaller</u> than that of the perturbing planet.

Periodic solutions of the third class arise also for $\mu = 0$, when the mean motions n and n' are commensurable. An important point in this connection which to the writer's knowledge was first pointed out by Charlier[3] is that no circular periodic orbits exist for $\mu = 0$.

In Poincaré's investigation, periodic orbits corresponding to <u>large</u> values of $\mu$ are excluded from the start. It should be borne in mind that $\mu$ is a factor in the expressions for the secular variations of perihelions and nodes of the periodic orbits, and therefore the expansion in powers of $\mu$ is likewise at the same time in powers of t. Therefore we cannot expect to obtain in this way such periodic orbits for which the period T is larger than a certain maximum $T_o$.

Again, we have assumed with Poincare that the period T is the <u>same</u> for $\mu = 0$ and for $\mu \neq 0$. A more general solution would be, if $T_\mu$ is the period for $\mu \neq 0$ and $T_o$ that for $\mu = 0$, obtained by putting $T_\mu = f(\mu)$ such that $\lim_{\mu \to 0} T_\mu = T_o$.

As pointed out by Poincaré[4], the importance of periodic solutions lies in the fact that they can be used as

the starting point of a method of successive approxi-
mations and also has opened up the way to the analytic
investigation of the nature of the integrals of the n-
body problem. The fundamental memoirs of Poincaré are a
precious source of information to the mathematician, the
astronomer and the physicist as well. Indeed it might
be said that the interest of such solutions, since the
advent of Bohr's theory, has moved over from the astro-
nomer to the physicist. It is in this field that their
most important applications ought to be found.

# Bibliography

1) Charlier, Mechanik des Himmels, Vol. II, p. 205, Leipzig, 1907.

2) Schwarzschild, Astronomische Nachrichten, Vol. 147, P. 17, 1898.

3) Charlier, l.c. (1), p. 238.

4) Poincaré, Les méthodes nouvelles de la mécanique céleste, Vol. 1, p. 152, Paris 1892.

Bohr's Atomic Model from the Standpoint of the

General Theory of Relativity and of the

Calculus of Perturbations
by M. S. Vallarta.
<span style="text-decoration: underline">Abstract</span>

I.   In the first part of this thesis, Bohr's

atomic model is studied in detail from the stand-

point of the general theory of relativity. A)Lenard's

arguments:  1) that the theory of relativity stands

in no causal relationship to Sommerfeld's theory of

fine structure, and 2) that it is not clear why the

restricted theory of relativity is sufficient to ac-

count for the motion of the electron in the atom,

whereas the general theory is required to explain the

anomaly of the perihelion of Mercury, are examined at

length.  The former objection naturally leads to an

examination of the more general question:  Does the

metric of space-time stand in a causal relationship to

the advance of the perihelion  of Mercury?  It is shown

that

$\alpha$)  The Newtonian equations of dynamics are ob-

tained from the geodesic equations in a four-dimension-

al continuum by cancellations which, so far as our an-

alysis can show, are purely arbitrary.

$\beta$)  Complementing Le Reux's and Painleve's objec-

tions to the arbitrariness of Schwarzschild's so-
lution of Einstein's gravitational equations $G_{\mu\nu} = 0$
in a static sphero-symmetrical gravitational field,
it is proved that Schwarzschild's result is arrived
at by a series of suitable restrictive arbitrary assum-
ptions.  There exist in fact an infinite number of so-
lutions of the gravitational equation consistent with
the definition of static, spherical and symmetrical
field and the choice of the Schwarzschildian form
must be regarded as purely arbitrary.  It follows that
Einstein's equation is insufficient to fix uniquely
the relativistic Keplerian orbit.

$\gamma$)  Von Gleich's and Zaremba's analysis are dis-
cussed in detail and found to hold.  The argument may
be summed up as follows:  For motions in a central
field the dynamic equations of classical mechanics must
be supplemented by a "relativistic" equation, but this
last equation is arbitrary because several forms of
this equation lead to the same formula for the peri-
helion advance.  Hence it cannot be maintained that the
metric of space-time stands in a necessary causal re-
lationship to the anomaly of the perihelion.  The gen-
eral conclusion is that Lenard's first objection holds.

2)  Using Nordström's fundamental form in a static
sphero-symmetrical material field and Weyl-Eddington's
equation of motion of the electron in this field it is

shown: $\alpha$)that the motion is plane, $\beta$) that the cur-
vature of the space-time in the region around the
nucleus almost vanishes (Laue's condition), $\gamma$) that
the field of the nucleus is very near static, $\delta$)that
it is justified to treat hydrogen or hydrogenic atoms
as a relativistic one-body problem, $\varepsilon$) that our more
general equation of the orbit reduces to Sommerfeld's
equation if conclusions ($\alpha$) to ($\delta$) are taken into ac-
count, $\gamma$) that Einstein's equation for planetary mo-
tion follows from our equation of the orbit if the
terms depending on electric charges vanish, $\Theta$) that
condition ($\beta$) is not satisfied for planetary motion but
condition ($\gamma$) is. $\zeta$) that the general relativistic
Keplerian orbit can be obtained by the familiar method
of the calculus of perturbations and yields results in
accordance with rigorous analysis. In closing, the weak-
ness of the Weyl-Eddington equation of motion is dis-
cussed.

B. An attempt is made to bring the quantum theory
into relativity. Following on the work of Mecke, it is
shown that the two fundamental postulates of the quantum
theory: minimal principle, and continuity principle can
be brought in harmony with relativity and lead to the
following results: $\alpha$) an explanation of the existence
of unmechanical orbits in the atom, $\beta$) a unified inter-
pretation of Bohr's second postulate (monochromatic radia-

tion) and his principle of correspondence. Mecke's "integral principle" (atomistic conception of action) can also be brought into the body of the theory of relativity, but, if it is to have any usefulness, privileged coördinate systems must be introduced. On the author's view, this is due entirely to the artificiality of our present quantum conditions and shows no weakness of the main argument. The fundamental meaning of the relativistic two-body problem, in which probably the periodic solutions are the natural quantized states, irrespective of any quantum rules, is brought out. It is shown next that the electron problem is not yet suffidently developed to warrant an attack on the quantum problem from that side. In closing, Mohorovicic's objection that "different geometries hold at the same spot of the universe" (ex. atom in gravitational field) examined and found to be untenable.

II. A. In the second part of this investigation, the applicability of the calculus of perturbations to the Bohr atom with more than one electron is studied in detail. The methods of quantization of Epstein, Bohr and Born and Pauli are examined. It is shown that it follows directly from the form of the integrals of the equations of dynamics, as determined by the classical theorems of Poincaré and Bruns, that neither of these methods can converge. In Epstein's case nothing is known

as to the remainder, and he disregards altogether a
possible libration of the momenta of the intermediate
orbit, a case which cannot be handled by Delaunay's
method, which he uses. The contention of Born and
Pauli, that the semi-convergency of the Newcomb-Lind-
stedt or Bohlin expansion of the perturbation function
is sufficient for practical atomic problems, is shown
to be untenable. Stress is laid on the fundamental
significance of commensurabilities in periodic times in
those methods that utilize the Newcomb-Lindstedt ex-
pansion and the impossibility of introducing angle var-
iables in certain cases. For continuity reasons, it
is concluded that neither the solution of Epstein nor
that of Born and Pauli are compatible with the adiabatic
hypothesis or the correspondence principle. The decisive
advantages of periodic solutions and the appropriateness
of Bohr's method as a first approximation, in those
cases to which it applies, is brought out clearly. The
general conclusions: For purely analytic reasons, the
calculus of perturbations is not in general applicable
to the atomic system with more than one electron.

B. Following Fermi, Poincaré's theorem is discussed
from a geometrical-statistical standpoint. We establish
the following theorem: Except for the dynamical system
of one degree of freedom, no continuous analytic hyper-
surface exists in phase-space, besides the energy-surface,

which can wholly contain a dynamical trajectory issued from one of its points. This holds under conditions which are explicitly stated.

C. An attempt is made to extend the adiabatic theorem to non-conditioned-periodic systems. The method of attack is geometrical-statistical and the result is in the main negative. We establish the following theorems: $\alpha$) Any dynamical system admitting a single integral, uniform and independent of time, admits an adiabatic transformation leaving this characteristic adiabatically invariant. As a particular corollary we have the well-known theorem that the energy of a quasi-ergodic system is adiabatically invariant, and the adiabatic invariant of harmonic motion. $\beta$) If a dynamical system admits more than one integral uniform and independent of time, none of these integrals is _in general_ an adiabatic invariant. An exception is, for instance, the phase-integrals of a conditioned periodic system.

The results of Krutkow and Fock and Kneser are analyzed and discussed. In closing, the difficulties standing in the way of a generalization of the concept of adiabatic invariance to systems of higher degree of freedom are discussed, following Ehrenfest. It is pointed out that the theorem of the conservation of dimensions for the transformation from W, $\zeta$-phase-space to

p,q-phase-space involves only differentiability conditions, which for physical reasons are satisfied in all likelihood, but no rigorous proof is given. Ehrenfest and Breit's example of the rotating dipole is explained from the standpoint of our general discussion.

D.  A discussion is given of the separation of variables in Hamilton-Jacobi's equation, embodying the condition of separability of Levi-Civita, the latter's theorem on the separability of the variables in the kinetic energy and Dall'Acqua's proof of the separable forms of the potential.

E.  For purposes of ready reference, Cauchy's existence proof and its generalization by Poincaré, Poincaré's periodic solution, the convergency of the expansion of the perturbation function and the theorems of Poincaré and Bruns are given at the end.

III.  An appendix is added collecting certain classical results of celestial dynamics which are made use of in the text.