Synthesis of Electric Networks by Means of the Fourier Transforms of Laguerre's Functions

by

YUK-WING LEE

S.B., Massachusetts Institute of Technology, 1927

S.M., Massachusetts Institute of Technology, 1928

Submitted in Partial Fulfillment of the Requirement for the Degree of

DOCTOR OF SCIENCE

from the

Massachusetts Institute of Technology

1930

Signature of Author Signature Redacted

Certification by the Department of Electrical Engineering

Professor in Charge of Research Signature Redacted

Chairman of Departmental Committee on Graduate Students Signature Redacted

Head of Department ______________________

ACKNOWLEDGMENT

Dr. Vannevar Bush, Acting Head of the Department of Electrical Engineering of the Massachusetts Institute of Technology, suggested this investigation to the writer in the fall of 1928. It is based upon an ingenious idea conceived by Dr. Norbert Wiener, Associate Professor of Mathematics of the same institution.

The most willing and careful general guidance, unfailing enlightenment, and intense interest of Dr. Bush have made this work unusually enjoyable to the writer.

Dr. Wiener has contributed profusely to the mathematical theory of this research. His profound knowledge of both mathematics and physics, and the patient and friendly manner with which he imparts it to the writer is indeed remarkable.

The writer considers his association with these men who excel in their respective fields of pursuit to have been the most valuable privilege which he has ever had during his many years as a student at the Massachusetts Institute of Technology.

TABLE OF CONTENTS

CHAPTER I

Synthesis of Electric Networks

a. Types of electric circuit problems and the ~~subject~~ purpose of the present research.

In the study of electric circuits, there are generally three types of problems. First, the applied voltage and the properties of the network are given and we wish to find the response. In other words, the cause and the network are given and we seek the effect. Usually the cause is a voltage, and the effect a current; both may be either a function of time or frequency. Second, the network and the response are known and we seek the cause. Third, we know the cause and the effect and seek the network. The first type of problem is the ordinary type which has been treated very extensively. The second type has received very little attention. It involves operational methods. The last type has a considerable amount of technical importance aside from scientific interests, but is very seldom dealt with in the literature owing, of course, to its difficulties.

In this thesis, we are interested in the third type of problems. Apparantely there are two stages in the solution of a problem of this sort. First, knowing the effect, we find the operator which connects the cause and the effect. This process is treated in the Operational

Calculus and consists of the evaluation of the Fourier transform[1]. The operator gives the characteristic of the network as a function of frequency. The next step in the solution is to find the network possessing such a frequency characteristic.

The second step in the solution of the third type of electric circuit problems is the main problem of the present investigation. Stated more clearly, our problem is to develop a method for the design of electric networks directly from assigned admittances or impedances. It is assumed, of course that these networks are constructable from the available circuit elements, namely, positive resistances, inductances, and capacitances.

This problem is an exceedingly important one particularly in electrical communication engineering. The design of transmission networks of which wave filters, balancing networks, artificial lines, and phase correction networks are examples, is of prime importance in all forms of electrical communication engineering. Mathematical difficulties and the limitation of the use of generally only three varieties of circuit elements, namely, positive resistances, inductances, and capacitances, have prevented

---

1. Mr. J. B. Russell of the M.I.T. is developing a method based upon a very ingenious idea of Dr. N. Wiener for the evaluation of the Fourier transform of a function.

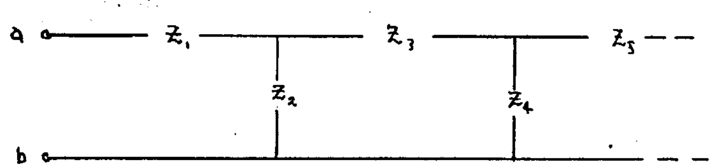
a rapid advancement in this branch of engineering. There is a great opportunity for research on this subject which is certainly as fascinating as it is important.

b. Present state of the problem.

We will not go into the details of the present methods of network design, but we can get a general idea from the following quotation from a recent paper by O. J. Zobel.[1]

> "It would be most gratifying to be able to obtain directly from a desired propagation characteristic the corresponding form of network. This is generally a difficult problem and it becomes necessary to resort to simplifying methods somewhat similar to those employed in the design of electric wave-filters. - - - - - We would, therefore, begin with the known form of networks whose general propagation characteristics have been determined and choose from them one or more whose combination offers the possibility of giving a satisfactory desired result".

The existing direct method of attack on this problem is one that involves the use of continued fractions.[2] It is known that in a ladder type network such as shown



- -

1. O. J. Zobel, Distortion Correction in Electrical Circuits with constant Resistance Recurrent Networks, Bell System Technical Journal, Vol. VII, No. 3, July 1928, p. 466.

2. See foot note of following page.

where Z stands for impedance, the impedance looking into the network from the terminals a b is given by the continued fraction

$$Z = Z_1 + \cfrac{1}{\cfrac{1}{Z_2} + \cfrac{1}{Z_3 + \cfrac{1}{\cfrac{1}{Z_4} + \cdots}}}$$

If Z is given and can be expanded in the above form with the signs of Z_i positive, and further, if Z_i are known networks, then evidently Z can be constructed. Such an expression has been studied for the more general case where Z_i is of the form $\frac{a\,j\omega + b}{c\,j\omega + d}$ which has a physical meaning when a, b, c, and d are positive constants. This method has several applications, but besides other limitations, the evaluation of the coefficients of the expansion is a long and tedious process. It is not applicable when the desired network characteristic is given as a graph which is usually the case.

---

2. T.C.Fry, The Use of Continued Fractions in the Design of Electrical Networks, Bulletin of the American Mathematical Society, vol. XXXV, pp. 463-498, July-August 1929, U.S.Patent 1,570,215, 1926, filed June 11, 1921.
W. Cauer, Die Verwirklichung von Wechselstromwiderständen vorgeschriebener Frequenzabhängiagkeit, Archiė für Elektrotechnik, vol. 17, 1926, pp. 355-388.
A. C. Bartlett, A Note on the Theory of Artificial Telephone and Transmission Lines, Philosophical magazine, vol. 48, 1924, p. 859, Properties of the Generalized Artificial Line, Phil. Mag. , Vol. 1, 1926, p. 553, British Patent 290,701, 1928, filed January 1927.

CHAPTER II

THE ADMITTANCE FUNCTION

a. The Converse of the Ordinary Problem

Since our problem is to find directly from a desired transfer admittance function, the corresponding form of network, it is important for us to know some of the properties of the admittance function, before going into the actual problem. In the ordinary problem, the network is given, and it involves no particular difficulty to obtain the admittance function. But if we are to write down an expression to represent the admittance function of an unknown network, we cannot accomplish our task as readily as before. One reason is that the real and imaginary parts of an admittance function are related. If either part is arbitrarily chosen, the other must satisfy a certain relation in order that our expression represents a physically realizable network. Likewise, the absolute magnitude and the phase of an admittance function are related. Another reason is that our expression may require negative circuit parameters.

b. The Implicit Relation Between the Real and Imaginary Parts of the Admittance Function

In the study of circuit transients, we have noticed the implicit relation between the real and imaginary parts of the admittance function. The explicit expression for the indicial admittance obtained by means of the Fourier integral is:[1]

$$A(t) = P(o) + \frac{2}{\pi}\int_0^\infty \frac{Q(\omega)}{\omega}\cos\omega t\, d\omega$$
$$= \frac{2}{\pi}\int_0^\infty \frac{P(\omega)}{\omega}\sin\omega t\, d\omega \qquad (1)$$

where

$A(t)$ = Indicial admittance

$P(\omega)$ = Real part of the admittance function

$Q(\omega)$ = Imaginary part of the admittance function

t = Time

$\omega = 2\pi$ frequency

$P(o) = P(\omega)$ at zero frequency (a constant)

Differentiating under the integral signs, we have the implicit relation between $P(\omega)$ and $Q(\omega)$.

$$\int_0^\infty Q(\omega)\sin\omega t\, d\omega = -\int_0^\infty P(\omega)\cos\omega t\, d\omega \qquad (2)$$

This equation states that the Fourier cosine transform of the real part of the admittance function is equal to minus the sine transform of the imaginary part of the same function.

It seems that in electrical engineering literature, the properties of the real and imaginary parts of the admittance function beyond what is just stated, have not appeared. In mathematical literature, however, we find a number of papers of recent years on "conjugate integrals".[2] These are just what we need in our electrical engineering problems.

------

1. See for example, V. Bush, Operational Circuit Analysis, John Wiley and Sons, 1929, p.180
2. E.C. Titchmarsh, Conjugate Trigonometrical Integrals, Proceedings of the London Mathematical Society, 2nd series, vol. 24, 1926, pp. 109-130. E.C. Titchmarsh, On Conjugate Functions, Proceedings of the London Mathematical Society, 2nd series, vol.29, 1929, pp.49-80.

c. The Explicit Relation Between the Real and Imaginary Parts of the Admittance Function

If we take the cosine transform of equation (2), we have:[1]

$$P(\omega) = -\frac{2}{\pi}\int_0^\infty \cos u\omega \, du \left[\int_0^\infty Q(t) \sin ut \, dt\right] \quad (3)$$

Note the P is an even function and Q is an odd function. Similarly, the sine transform of (3) gives:

$$Q(\omega) = -\frac{2}{\pi}\int_0^\infty \sin u\omega \, du \int_0^\infty P(t) \cos ut \, dt \quad (4)$$

Equation (3) states that the real part of the admittance function is minus the cosine transform of the sine transform of the imaginary part of the same function. Equation (4) states that the imaginary part is minus the sine transform of the cosine transform of the real part.

d. Conjugate Integrals - Hilbert Transforms

It has been proved that:[2]

$$P(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{Q(u)}{u-\omega} du \quad (5)$$

$$Q(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{P(u)}{u-\omega} du \quad (6)$$

Or, written in another form:

$$P(\omega) = \frac{1}{\pi}\int_0^\infty \frac{Q(\omega+u) - Q(\omega-u)}{u} du \quad (7)$$

$$Q(\omega) = -\frac{1}{\pi}\int_0^\infty \frac{P(\omega+u) - P(\omega-u)}{u} du \quad (8)$$

The integrals are defined as principal values, that is, integrals of the type:

$$\lim_{\epsilon \to 0} \int_\epsilon^\infty \; .$$

Foot note continued from previous page:
F.M. Wood, Reciprocal Integral Formulae, Proceedings of the London Mathematical Society, 2nd series, vol. 29, 1929, pp. 29-48. Other papers on this subject are found in these references.
The writer is indebted to Prof. Wiener for these refer-

1. See note 1 on page 9.
2. " " 2 " " ".

A theorem* states that if $P(\omega)$ is a function whose p-th power is integrable over any finite interval ($p > 1$), that is:

$$\int_a^b [P(\omega)]^p \, d\omega \quad p > 1 \quad \text{exists,}$$

and

$$\int^{\infty} \frac{P(\omega)\, d\omega}{\omega} \quad \text{and} \quad \int^{\infty} \frac{P(-\omega)}{\omega} \, d\omega \quad \text{both exist,}$$

then $Q(\omega)$ is also a function whose p-th power is integrable over any finite interval and

$$-\frac{1}{\pi} \int^{\infty} \frac{Q(\omega + u) - Q(\omega - u)}{u} \, du$$

is summable (C,I)** for all ω. If the lower limit of the integral is taken to be the origin, then:

$$-\frac{1}{\pi} \int_0^{\infty} \frac{Q(\omega + u) - Q(\omega - u)}{u} \, du$$

exists for almost all ω, and is summable (C,I) to $P(\omega)$.

Further, if $P(\omega)$ obeys a Lipschitz condition of order a, that is:

$$|P(\omega + h) - P(\omega)| < A\, |h|^a \qquad 0 < a < 1$$

for $|h| < k$, uniformly with respect to ω over any finite interval; then $Q(\omega)$ also obeys a Lipschitz condition of order a, and

$$-\frac{1}{\pi} \int_0^{\infty} \frac{Q(\omega + u) - Q(\omega - u)}{u} \, du$$

is summable (C,I) to $P(\omega)$ for all values of ω.

Foot note continued from previous page:
ences, and for giving him considerable assistance on this subject. He has not, as yet, read this chapter so that any error contained herein cannot be attri-

* See page 9.
** " " ".

buted to him.

1. See for example, V. Bush, Operational Circuit Analysis, John Wiley and Sons, 1929, Appendix B, (by N. Wiener), P. 370.
   For application of this and the next equations to specific problems, see "Illustrative Examples", at the end of this chapter, and also chapter VI.

2. Ibid
   For application of these equations to specific problems see "Illustrative Examples" at the end of this chapter.

* loc. cit.

** From E.W. Hobson, the Theory of Functions of a Real Variable and the Theory of Fourier Series, Cambridge University Press, 1926, p.385: The Integral

$$\int_a^\infty f(t)\, dt$$

is said to exist (C,I) when

$$\lim_{x \to \infty} \frac{1}{x} \int_a^x dt_1 \int_a^{t_1} f(t)\, dt$$

has a finite value. The integral

$$\int_a^\infty f(t)\, dt$$

is then said to be summable (C,I) and its sum (C,I) is defined to be the value of the limit.
We have for the sum (C,r) of the integral

$$\int_a^\infty f(t)\, dt,$$

the expression:

$$\lim_{x \to \infty} \int_a^\infty \left(1 - \frac{t}{x}\right)^r f(t)\, dt.$$

When the integral exists in the ordinary sense, it is summable (C,O).

Equations (7) and (8) are known as Hilbert transforms. The real and imaginary parts of the admittance function are therefore Hilbert transforms of each other. They are often called conjugates.

e. Cauchy-Riemann Conditions

The admittance function of a dissipative network has all its poles located on the left half of the complex plane. This means that the roots of the equation:

$$\frac{1}{Y(\lambda)} = 0$$

where $Y(\lambda)$ is an admittance function, have either negative or zero real parts. Therefore, the admittance function of a dissipative network is analytic over the right half of the complex plane.

From our study of the theory of functions of a complex variable, we know that if:

$$w = u + j\,v$$

and

$$z = x + j\,y$$

where u and v are real functions of x and y, and if u and v satisfy the Cauchy-Riemann conditions, that is:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

then w is said to be an analytic function of z. Further, two real functions, u and v, which satisfy the Cauchy-Reimann conditions are called conjugate functions.

Since our admittance function is analytic over the right half of the complex plane, it satisfies the Cauchy-Riemann conditions over that region.

Let us take the admittance function of a series circuit of an inductance and a resistance to illustrate our point. We have:

$$Y(\lambda) = \frac{1}{1 + \lambda}$$

Now make the substitutions:

$$\lambda = x + j\,y$$

$$Y(\lambda) = w = u + j\,v$$

then

$$Y(\lambda) = w = \frac{1}{1 + x + j\,y} = \frac{(1 + x)}{(1+x)^2 + y^2} - \frac{j\,y}{(1+x)^2 + y^2}$$

$$u = \frac{1 + x}{(1+x)^2 + y^2} \qquad v = \frac{-y}{(1+x)^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(1+x)^2 + y^2 - 2(1+x)^2}{(1+x)^2 + y^2} = \frac{y^2 - (1+x)^2}{y^2 + (1+x)^2}$$

and

$$-\frac{\partial v}{\partial y} = \frac{(1+x)^2 + y^2 - 2y^2}{(1+x)^2 + y^2} = \frac{(1+x)^2 - y^2}{(1+x)^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2\,(1+x)\,y}{(1+x)^2 + y^2}$$

$$-\frac{\partial v}{\partial x} = \frac{-2\,(1+x)\,y}{(1+x)^2 + y^2}$$

The Cauchy-Riemann conditions are therefore satisfied.

f. The Relation Between the Absolute Value and the Phase of the Admittance Function

Let us take the logarithm of $w = u + j\,v$.

$$\log_e w = \frac{1}{2} \ln (u^2 + v^2) + j \tan^{-1} \frac{v}{u}$$

$$= u_1 + j\,v_1$$

(Note: $\log_e = \ln$)

where

$$u_1 = \frac{1}{2} \ln (u^2 + v^2)$$

and

$$v_1 = \tan^{-1} \frac{v}{u}.$$

If $u + j\,v$ is analytic, we inquire whether $u_1 + j\,v_1$ is also analytic. We have, by assumption:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x} \quad .$$

Differentiating,

$$\frac{\partial u_1}{\partial x} = \frac{1}{2} \; \frac{2\,u \frac{\partial u}{\partial x} + 2\,v \frac{\partial v}{\partial x}}{u^2 + v^2}$$

$$\frac{\partial v_1}{\partial y} = \frac{u \frac{\partial v}{\partial y} - v \frac{\partial u}{\partial y}}{\left(1 + \frac{v^2}{u^2}\right) u^2} = \frac{u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x}}{u^2 + v^2} = \frac{\partial u_1}{\partial x}$$

and

$$\frac{\partial u_1}{\partial y} = \frac{1}{2} \; \frac{2\,u \frac{\partial u}{\partial y} + 2\,v \frac{\partial v}{\partial y}}{U^2 + v^2}$$

$$\frac{\partial v_1}{\partial x} = \frac{u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x}}{\left(1 + \frac{v^2}{u^2}\right) u^2} = \frac{-u \frac{\partial u}{\partial y} - v \frac{\partial v}{\partial y}}{u^2 + v^2} = -\frac{\partial u_1}{\partial y} \quad .$$

Hence, if $u + j\,v$ is analytic, $\log_e (u + j\,v)$ is also analytic.

The function u corresponds to the real part of our admittance function, and v, its imaginary part. Therefore, since the admittance function is analytic over the right half of the complex plane the logarithm of the same function is also analytic over the same region. In other words:

$$P(\omega) \text{ and } Q(\omega); \text{ and } \frac{1}{2}\log_e(P^2+Q^2) \text{ and } \tan^{-1}\frac{Q}{P}$$

are conjugate functions, and are Hilbert transforms of each other. The reciprocal formulas (3) and (4) or (7) and (8) hold for $\frac{1}{2}\log_e(P^2+Q^2)$ and $\tan^{-1}\frac{Q}{P}$. Of course, in these formulas, we substitute $\frac{1}{2}\log_e(P^2+Q^2)$ for P and $\tan^{-1}\frac{Q}{P}$ for Q. Thus,[1]

$$\frac{1}{2}\ln(P^2+Q^2) = \ln|Y| = -\frac{1}{\pi}\int_0^\infty \cos\omega u\, du \int_{-\infty}^{\infty}\phi(t)\sin ut\, dt \qquad (9)$$

$$\tan^{-1}\frac{Q}{P} = \phi = -\frac{1}{\pi}\int_0^\infty \sin\omega u\, du \int_{-\infty}^{\infty}\ln|Y|\cos ut\,dt \qquad (10)$$

or

$$\ln|Y| = \frac{1}{\pi}\int_0^\infty \frac{\phi(\omega+u) - \phi(\omega-u)}{u}\,du \qquad (11)$$

$$\phi = \frac{-1}{\pi}\int_0^\infty \frac{\ln|Y(\omega+u)| - \ln|Y(\omega-u)|}{u}\,du \qquad (12)$$

------

1. See "Illustrative Samples" at end of this chapter for simple applications.

It is to be noted that while we need to avoid only infinities in the integrations involving P and Q, it is necessary to avoid the zeros also, when we deal with $\log_e(P^2 + Q^2)$, the reason being that the logarithm of zero is minus infinity. When $(P^2 + Q^2) = 0$, the solution is not unique; the phase can be changed.

The relations between the real and imaginary parts and the magnitude and phase of the admittance function are familiar to us and we need no further discussion here.

We have seen that from the real part of the admittance function, we can determine its imaginary part, and vice versa; and from the magnitude we can determine the phase - by the same equations - and vice versa, provided the conditions for the validity of equations (3) and (4) or (7) and (8) are satisfied. Therefore we can say that given any one of the four quantities, that is, the real part, the imaginary part, the magnitude, and the phase, the other three can be evaluated. ???

g. Evaluation of the Conjugate Integrals

The evaluation of the integrals is difficult. Only very simple problems can be carried through. The choice between equation (3) and (4) and (7) and (8) depends on the specific problem. Remember that in evaluating these integrals, we take their principal parts. This makes our task a little more difficult.

Mr. T. Gray, of the Massachusetts Institute of Technology, under the supervision of Dr. V. Bush, has developed an optical integrating machine. This work is Gray's doctorate thesis. Equations of the type (7) and (8) can be handled by this new device and it requires but little labor and time in the actual process of solving these equations. Thus, we have at hand the proper apparatus to deal with our integrals.

h. Parseval Theorem for Conjugate Integrals

Consider the integral:

$$\int_{-\infty}^{\infty} P_1(\omega)\, P_2(\omega)\, d\omega$$

where $P_1(\omega)$ and $P_2(\omega)$ are the real parts of two different admittance functions. The integral is assumed to exist. By the Parseval Theorem for Fourier Transforms:[1]

$$\int_{-\infty}^{\infty} P_1(\omega)\, P_2(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} P_1(u) \cos u\omega \, du\right] \cdot \left[\int_{-\infty}^{\infty} P_2(u) \cos u\omega \, du\right] d\omega$$

We have seen however, that the cosine transform of P is minus the sine transform of Q. (see equation 2) Hence:

$$\int_{-\infty}^{\infty} P_1(\omega)\, P_2(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} Q_1(u) \sin u\omega \, du\right]\left[\int_{-\infty}^{\infty} Q_2(u) \sin u\omega \, du\right] d\omega$$

1. See for example, Operational Circuit Analysis, V. Bush, John Wiley and Sons, 1929, Appendix B (by N.Wiener) p.373

where Q_1 and Q_2 are the imaginary parts for P_1 and P_2 respectively.

Again, by the Parseval Theorem for Fourier transforms we write:

$$\int_{-\infty}^{\infty} P_1(\omega)\ P_2(\omega)\ d\omega = \int_{-\infty}^{\infty} Q_1(\omega)\ Q_2(\omega)\ d\omega \qquad (13)$$

which is known as the Parseval Theorem for conjugate functions.

This is an important equation in the theory of conjugate functions. It will be seen in the next chapter, that this relation simplifies our problem tremendously.

If $P_1 = P_2$, equation (13) becomes[1]:

$$\int_{-\infty}^{\infty} [P(\omega)]^2\ d\omega = \int_{-\infty}^{\infty} [Q(\omega)]^2\ d\omega \qquad (14)$$

This has been proved in a much more rigorous manner by E.C. Titchmarsh[2]. His theorem states that if $P(\omega)$ is of integrable square over $(-\infty, \infty)$ and satisfies a Lipschitz condition of order a, uniformly in ω, that is:

$$|P(\omega + h) - P(\omega)| < A\ |h|^{a} \quad (0 < a < 1),$$

for $|h| < k$, then $Q(\omega)$ satisfies the same conditions, and equation (10) holds.

---

1. For a specific case, see (7) under "Illustrative Example" at the end of this chapter.
2. E.C. Titchmarsh, Conjugate Trigonometrical Integrals, Proceedings of the London Mathematical Society, 2nd series, vol. 24, 1926, p. 119.

i. Illustrative Examples

1. Assume that:

$$P(\omega) = \frac{1}{1+\omega^2}$$

and find $Q(\omega)$.

By equation (4) which is equivalent to equation (6),

$$Q(\omega) = -\frac{2}{\pi}\int_0^\infty \frac{P(u)}{\omega - u}\, du = -\frac{2}{\pi}\int_0^\infty \frac{du}{(\omega - u)(1+u^2)} .$$

Integrating by partial fractions:

$$Q(\omega) = -\frac{2}{\pi}\int_0^\infty \left\{\frac{A}{\omega - u} + \frac{Bu + C}{1+u^2}\right\} du$$

$$= -\frac{2}{\pi}\left[\frac{1}{1+\omega^2}\int_0^\infty \frac{du}{\omega - u} + \frac{1}{1+\omega^2}\int_0^\infty \frac{u\,du}{1+u^2} + \frac{\omega}{1+\omega^2} \times \int_0^\infty \frac{du}{1+u^2}\right.$$

$$= -\frac{2}{\pi}\left[\frac{1}{1+\omega^2}\ln \frac{\sqrt{1+u^2}}{\omega - u}\Bigg|_0^\infty + \frac{\omega\pi}{2(1+\omega^2)}\right. = -\frac{\omega}{1+\omega^2} .$$

which we know is the imaginary part of the admittance of a resistance and an inductance in series. The assumed real part is, of course, recognized as the real part of the admittance of the same circuit.

2. Working the problem backwards, that is, assuming

that:

$$Q(\omega) = -\frac{\omega}{1+\omega^2},$$

we have, by equation (3), which is equivalent to equation (5),

$$P(\omega) = \frac{2}{\pi}\int_0^\infty \frac{Q(u)}{\omega - u}\,du = -\frac{2}{\pi}\int_0^\infty \frac{u\,du}{(\omega - u)(1+u^2)}.$$

Again, integrating by partial fractions,

$$P(\omega) = -\frac{2}{\pi}\int_0^\infty \left\{\frac{A_1}{\omega - u} + \frac{B_1 u + C}{1+u^2}\right\} du$$

$$P(\omega) = -\frac{2}{\pi}\left[\frac{\omega}{1+\omega^2}\int_0^\infty \frac{du}{\omega - u} + \frac{\omega}{1+\omega^2}\int_0^\infty \frac{u\,du}{1+u^2} - \frac{1}{1+\omega^2}\int_0^\infty \frac{du}{1+u^2}\right.$$

$$= -\frac{2}{\pi}\left[\frac{\omega}{1+\omega^2}\ln\frac{\sqrt{1+u^2}}{\omega - u}\Bigg|_0^\infty - \frac{\pi}{2(1+\omega^2)}\right] = \frac{1}{1+\omega^2}$$

which is the original assumed $P(\omega)$.

3. Let us use equations (3) and (4). Suppose:

$$P(\omega) = \frac{1}{1+\omega^2}$$

as before, then by equation (4),

$$Q(\omega) = -\frac{2}{\pi}\int_0^\infty \sin\omega u\,du \int_0^\infty \frac{1}{1+t^2}\cos ut\,dt^{*}$$

$$= -\frac{2}{\pi}\int_0^\infty \sin\omega u\,du \left[\frac{\pi}{2}e^{-u}\right]^{**}$$

(see foot note of p. 20)

$$= -\frac{\omega}{1+\omega^2}$$

------

* Bierens de Haan, Nouvelles Tables D'Intégrales Définies,

4. If Q (ω) is given as:

$$- \frac{\omega}{1 + \omega^2}$$

then by equation (3)

$$P = - \frac{2}{\pi} \int_0^\infty \cos \omega u \, d u \int_0^\infty \frac{-t}{1 + t^2} \sin u t \, d t \;*$$ (See next page)

$$= \frac{2}{\pi} \int_0^\infty \cos \omega u \, d u \left[\frac{\pi}{2} e^{-u} \right] \;**$$ (See next page)

$$= \frac{1}{1 + \omega^2}$$

5. Let $\frac{1}{2} \ln (P^2 + Q^2) = \frac{1}{2} \ln \left(\frac{1}{1 + \omega^2} \right)$ which is the absolute value of the admittance function of the resistance-reactance series circuit. The values of the resistance and reactance, as in the previous examples, are assumed to be unity for the sake of convenience. The problem is to find the phase of this circuit. By equation (10):

$$\phi = - \frac{1}{\pi} \int_0^\infty \sin \omega u \, d u \int_{-\infty}^\infty \frac{1}{2} \ln \left(\frac{1}{1 + t^2} \right) \cos u t \, d t.$$

Performing the first integration by parts, we get:

$$= \frac{1}{2\pi} \int_0^\infty \sin \omega u \, d u \left[\frac{1}{u} \ln \frac{1}{1+t^2} \sin u t \Big|_{-\infty}^{\infty} - \frac{2}{u} \times \int_{-\infty}^\infty \frac{t \sin u t \, d t}{1 + t^2} \right] \;***$$ (See next page)

Foot note continued from previous page:
Table 160, no.5, p.223. $\int_0^\infty \cos p x \frac{d x}{q^2 + x^2} = \frac{\pi}{2q} e^{-p q}$.

** Table 261, No.1, p. 383. $\int_0^\infty e^{-px} \sin q x \, d x = \frac{q}{q^2 + p^2}$.

Foot note continued from previous page:

* Ibid, Table 160, no. 4, p. 223

$$\int_0^\infty \sin p x \frac{x\,d\,x}{q^2 + x^2} = \frac{1}{2} \pi \quad e^{-pq}$$

**Table 261, no. 2, p.383

$$\int_0^\infty e^{-px} \operatorname{cox} q\,x\,d\,x = \frac{p}{p^2 + q^2}$$

*** Ibid, Table 160, No.4, p.223

$$\int_0^\infty \sin p x \frac{x\,d\,x}{q^2 + x^2} = \frac{1}{2} \pi \quad e^{-pq}$$

Hence:

$$\phi = - \int_0^\infty e^{-u} \sin \omega u \frac{du}{u} = - \tan^{-1} \omega \quad .^*$$

which we know is the correct expression for the phase of the series circuit.

6. The converse of the preceding problem is: given the phase ϕ of an admittance function, find the absolute magnitude $|Y|$ of the function. Again, consider the simple resistance and reactance series circuit.

Given:

$$\phi = - \tan^{-1} \omega \quad .$$

Inserting this in equation (9).

$$\ln |Y| = - \frac{1}{\pi} \int_0^\infty \cos \omega u \, du \int_{-\infty}^{\infty} - \tan^{-1} t \sin u t \, dt$$

Integrating by parts:

$$\ln |Y| = \frac{1}{\pi} \int_0^\infty \operatorname{Cos} \omega u \, du \left[- \frac{1}{u} \tan^{-1} t \cos u t \Big|_{-\infty}^{\infty} + \frac{1}{u} \int_{-\infty}^{\infty} \frac{\cos u t \, dt}{1 + t^2} \right] ^{**}$$

$$\ln |Y| = \int_0^\infty e^{-u} \cos \omega u \frac{du}{u} \quad .$$

Differentiating under the integral sign with respect to ω,

$$\frac{d}{d\omega} \ln |Y| = - \int_0^\infty e^{-u} \sin \omega u \, du = - \frac{\omega}{1 + \omega^2} \quad .^{***}$$

------

* Table 365, No.1, p.509 $\int_0^\infty e^{-px} \sin q x \frac{dx}{x} = \tan^{-1} \frac{q}{p}$

** Ibid, Table 160, no.5, p.223

$$\int_0^\infty \cos p x \frac{dx}{q^2 + x^2} = \frac{\pi}{2q} e^{-pq}$$

*** See next page.

Hence:

$$\ln |Y| = - \int_0^{\omega} \frac{\omega \, d\omega}{1 + \omega^2} = - \frac{1}{2} \ln (1 + \omega^2)$$

$$= \frac{1}{2} \ln \frac{1}{1 + \omega^2}$$

and

$$|Y| = \frac{1}{(1+ \omega^2)^{1/2}}$$

which we know is the magnitude of the admittance of the circuit under consideration.

7. As an illustration of equation (14), we take the same circuit:

$$P = \frac{1}{1+ \omega^2} \qquad , \qquad Q = - \frac{\omega}{1 + \omega^2}$$

By equation (14), we should have:

$$\int_{-\infty}^{\infty} \frac{d\omega}{(1+ \omega^2)^2} = \int_{-\infty}^{\infty} \frac{\omega^2 d\omega}{(1+ \omega^2)^2}$$

Let:

$$\omega = \tan x \qquad , \qquad d\omega = \sec^2 x \, dx$$

Then:

$$\int_{-\infty}^{\infty} \frac{d\omega}{(1+ \omega^2)^2} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x \sec^2 x \, dx =$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x \, dx = \frac{\pi}{2}$$

Foot note continued from previous page:

*** Ibid, Table 261, no. 1, p.383:

$$\int_0^{\infty} e^{-px} \sin qx \, dx = \frac{q}{p^2 + q^2}$$

and:

$$\int_{-\infty}^{\infty} \frac{\omega^2 d\omega}{(1+\omega^2)^2} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^2 x \cos^4 x \sec^2 x \, d x$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \, d x = \frac{\pi}{2}$$

which satisfy equation (14).

CHAPTER III

Expansion of the Transfer Admittance Function in Terms of the Fourier Transforms of Laguerre's Functions.

a. Basic Idea

In the usual electrical network problem, the network is given, and it is a matter of routine to find the admittance function (or impedance function). The admittance function may be either a driving point admittance or a transfer admittance. The processes for the evaluation of the admittance function are very familiar to us, and are exceedingly important in circuit analysis. But problems often arise where the exact reverse of the usual procedure is necessary. Such problems would be: Given a desired admittance function, required to find the network which behaves accordingly. Examples of this type of problems are the design of the wave filter, balancing network, phase correction network, artificial line, and transmission networks in general. The importance of these problems is well recognized by the electrical communication engineer. The synthesis of the electric network is indeed a tremendous and interesting problem.

One method of direct attack is obvious from the fact that a function can be expanded into a series. Our problem is solved if each term of the series corresponds to a physically realizable network both in value and in sign, for then the desired network is the combination of a series of networks whose impedances or admittances correspond to the terms into which the required function is expanded. At present, generally we are limited to the use of positive resistances, inductances, and capacitances. With these three network elements, and their combinations, we are usually unable to interpret the terms of a series. Thus we do not know any circuit element or combination or circuit elements whose impedance varies as the square of the frequency or any power higher than the first, or one that varies as the frequency to a power lower than minus one, or one that varies as the sine of the frequency, and so on.

In this chapter we shall consider a method of synthesis which is based on the expansion of the admittance function into a series which corresponds to a set of constructable networks.

b. Development of the expansion.

An expansion of the form[1]

$$f(x) \sim \sum_{n=o}^{\infty} a_n \quad L_n(x) \qquad (15)$$

is a special Laguerre's series where f (x) is defined for $x \geq o$, and

$$L_n(x) = \frac{1}{n!} e^x \frac{d^n}{d\,x^n} (x^n e^{-x}) \qquad (16)$$

which are called Laguerre's Polynomials

The sign of equivalence ($\sim$) indicates that the coefficients are obtained from the relation

$$a_n = \int_o^{\infty} e^{-t} \quad L_n(t)\, f(t)\, d\,t \qquad (17)$$

The polynomials $L_n(x)$ have the following properties of orthogonality.

---

1. Einar Hille, on Laguerre's Series, Proceedings of the National Academy of Sciences, vol.12, No.4, pp. 261-269, April, 1926; vol. 12, No. 5, pp. 348-352, May, 1926.

$$\int_0^{\infty} e^{-t} L_m(t)\, L_n(t)\, dt = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \tag{18}$$

With the transformations

$$L_n(x) = e^{\frac{x}{2}} h_n(x)$$

and

$$f(x) = e^{\frac{x}{2}} g(x) \tag{19}$$

equation (15) becomes

$$g(x) \sim \sum_{n=0}^{\infty} a_n h_n(x) \tag{20}$$

Similarly, equation (16) becomes

$$h_n(x) = \frac{1}{n!} e^{\frac{x}{2}} \frac{d^n}{dx^n} (x^n e^{-x}) \tag{21}$$

(17) becomes

$$a_n = \int_0^{\infty} h_n(t)\, g(t)\, dt \tag{22}$$

and (18) becomes

$$\int_0^{\infty} h_m(t)\, h_n(t)\, dt = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \tag{23}$$

Let $F(u)$ = Fourier transform of $g(x)$

$G_n(u)$ = " " of $h_n(x)$ (24)

Taking the Fourier transform of (20) we get

$$F(u) \sim \sum_{n=0}^{\infty} a_n G_n(u) \tag{25}$$

By the Parseval Theorem[1] for Fourier transforms, (22) becomes

$$a_n = \int_{-\infty}^{\infty} F(u)\, \overline{G}_n(u)\, du \qquad (26)$$

Where $\overline{G}_n(u)$ is the conjugate complex of $G_n(u)$. Substituting this in (25), we obtain

$$F(u) = \sum_{n=0}^{\infty} G_n(u) \int_{-\infty}^{\infty} F(u)\, \overline{G}_n(u)\, du \qquad (27)$$

If u stands for angular velocity ω, and F is an admittance function, (27) may be written as

$$Y(\omega) = \sum_{n=0}^{\infty} G_n(\omega) \int_{-\infty}^{\infty} Y(\omega)\, \overline{G}_n(\omega)\, d\omega^{*} \qquad (28)$$

This is the equation upon which our method of synthesis is based. It indicates that if $G_n(\omega)$ represent a set of constructable networks, the desired network whose frequency characteristic is $Y(\omega)$ will then be a combination of $G_n(\omega)$ whose coefficients are determined by the integral. The method of combination is assumed to be in agreement with the signs of the terms of the series. In equation (28) the admittance function $Y(\omega)$ is used because, as will be shown in the next chapter, $G_n(\omega)$ re-

---

1. See for example, V.Bush, Operational Circuit Analysis, John Wiley and Sons, 1929, p. 373. Briefly, if a (u) is the Fourier transform of f (x), and b (u) the Fourier transform of g (x), then

$$\int_{-\infty}^{\infty} f(x)\, \overline{g}(x)\, dx = \int_{-\infty}^{\infty} a(u)\, \overline{b}(u)\, du$$

* This equation is due to Dr. Norbert Wiener.

presents the transfer admittances of a series of constructable networks, and the signs of the series (28) can be realized by a certain method of connections. Equation (28) can, of course, be written to represent an impedance function. It would be another very important application of this equation if we can work with impedances similar to what we have done with transfer admittances.

We shall go into the physical interpretation of equation (28) in the next chapter, and confine our attention to some mathematical considerations here. Let us now obtain the Fourier transform, $G_n(\omega)$, of $h_n(x)$.

$$h_n(x) = \frac{1}{n!}\, e^{\frac{x}{2}}\, \frac{d^n}{dx^n}\left(x^n e^{-x}\right) \qquad \text{see (21)}$$

$$G_n(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty h_n(x)\, e^{-j\omega x}\, dx$$

$$= \frac{1}{n!\sqrt{2\pi}}\int_0^\infty e^{\left(\frac{1}{2}-j\omega\right)x}\, d\left[\frac{d^{n-1}}{dx^{n-1}}\left(x^n e^{-x}\right)\right]$$

---

1. [illegible]

Integrating by parts,

$$G_n(\omega)=\frac{1}{n!\sqrt{2\pi}}e^{(\frac{1}{2}-j\omega)x}\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\Big|_0^\infty - \frac{(\frac{1}{2}-j\omega)}{n!\sqrt{2\pi}}\Big[\int_0^\infty e^{(\frac{1}{2}-j\omega)x}\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\,dx\Big]$$

The first term on the right contains terms of the form

$$\frac{x^m}{n!\sqrt{2\pi}\;e^{(\frac{1}{2}+j\omega)x}} \qquad m = 1, 2, 3, \text{-------} n$$

which is zero when $x = o$, and when $x = \infty$, its limit is also zero.

The n-th integration gives

$$G_n(\omega)=\frac{(-1)^n(\frac{1}{2}-j\omega)^n}{n!\sqrt{2\pi}}\int_0^\infty x^n e^{-(\frac{1}{2}+j\omega)x}\,dx$$

$$=\frac{(-1)^n(\frac{1}{2}-j\omega)^n}{n!\sqrt{2\pi}}\frac{1}{-(\frac{1}{2}+j\omega)}\int_0^\infty x^n\,d\Big[e^{-(\frac{1}{2}+j\omega)x}\Big].$$

Integrating by parts again,

$$G_n(\omega)=\frac{(-1)^n(\frac{1}{2}-j\omega)^n}{n!\sqrt{2\pi}}\frac{1}{-(\frac{1}{2}+j\omega)}\Big[x^n e^{-(\frac{1}{2}+j\omega)x}\Big|_0^\infty - n\int_0^\infty x^{n-1}e^{-(\frac{1}{2}+j\omega)x}\,dx\Big]$$

$$=\frac{(-1)^n(\frac{1}{2}-j\omega)^n}{n!\sqrt{2\pi}}\frac{n!}{(\frac{1}{2}+j\omega)^n}\int_0^\infty e^{-(\frac{1}{2}+j\omega)x}\,dx$$

$$G_n(\omega)=\frac{(-1)^n(\frac{1}{2}-j\omega)^n}{\sqrt{2\pi}\,(\frac{1}{2}+j\omega)^{n+1}}=\frac{1}{\sqrt{2\pi}}\frac{(j\omega-\frac{1}{2})^n}{(j\omega+\frac{1}{2})^{n+1}} \qquad (29)$$

In order to have a control in the sizes of the circuit elements, and the rapidity of convergence of the series, we shall change the scale of our expansion, that is, instead of the expansion

$$g(x) = \sum_{n=0}^{\infty} a_n L_n(x) e^{-\frac{x}{2}}, \qquad \text{see (20)}$$

We shall change it to

$$g(x) = \sum_{n=0}^{\infty} a_{n_1} L_n(k_1 x) \; e^{-\frac{k_1 x}{2}} \tag{30}$$

where k_1 is any positive constant.

Reference to the preceding section shows that $G_n(\omega)$ becomes

$$G_n(k_1, \omega) = \sqrt{\frac{k_1}{2\pi}} \frac{(j\omega - \frac{k_1}{2})^n}{(j\omega + \frac{k_1}{2})^{n+1}} \tag{31}$$

and (28) takes its more general form

$$Y(\omega) = \sum_{n=0}^{\infty} G_n(k_1, \omega) \int_{-\infty}^{\infty} Y(\omega) \, \bar{G}_n(k_1, \omega) \, d\omega \tag{32}$$

Simplifying these expressions by the substitutions $\lambda = \frac{\omega}{k}$ and, $2k = k_1$, we have

$$G_n(\lambda) = \frac{1}{\sqrt{k\pi}} \frac{(j\lambda - 1)^n}{(j\lambda + 1)^{n+1}} \; ; \; G_n(k, \omega) = \sqrt{\frac{k}{\pi}} \frac{(j\omega - k)^n}{(j\omega + k)^{n+1}} \tag{33}$$

$$Y(\omega) = \sum_{n=o}^{\infty} G_n(k,\omega)\, k\int_{-\infty}^{\infty} Y(k\lambda)\, \overline{G}_n(\lambda)\, d\lambda \qquad (34)$$

or $$Y(\omega) = \sum_{n=o}^{\infty} a_n G_n(k,\omega) \qquad (35)$$

where $$a_n = k\int_{-\infty}^{\infty} Y(k\lambda)\, \overline{G}_n(\lambda)\, d\lambda \quad . \qquad (36)$$

c. General considerations

Since a Fourier transform is quadratically summable, the expansion holds for quadratically summable functions. By this we mean that

$$\int_{-\infty}^{\infty} |Y(\omega)|^2\, d\omega \qquad \text{exists.}$$

Naturally $Y(\omega)$ vanishes at infinity. This is only a very crude statement. To be precise, we need a good deal more refinement and rigor, but for engineering purposes, this seems to be sufficient.

The fact that equation (35) holds for quadratically summable functions does not in any way imply that our method of synthesis is limited to such functions. In fact, as will be shown in Chapter V, we have here a most general sort of method that can be used to attack almost any four terminal network design problem in a most powerful manner.

The factor k enables us to obtain the most rapidly convergent series for a particular problem. Equation

(30) can be used to represent the indicial admittance of a network and so

$$g(t) = \sum_{n=0}^{\infty} a_{n_1} L_n(k_1 t) \; e^{-\frac{k_1 t}{2}} \quad (37)$$

Suppose we have the indical admittance given and we wish to find the network. In this case the factor k_1 for a most rapidly convergent series is the one that makes the given function behave nearest to $e^{-\frac{k_1 t}{2}}$ at infinity. This is clear from the above equation. The transition from the indicial admittance to the admittance function is treated in the Operational Calculus. From this we can proceed to construct the network.

When the admittance function is given, the choice of k_1 is harder than that of the first case. Generally, we can say that the best k_1 is the one that makes the poles of the given function come closest to those of the approximating functions $G_n(\omega)$.

In both case, the question of the sizes of the circuit elements is a large engineering factor, and this involves the constant k_1, so that a compromise must be made between the convergence of the series and the sizes of circuit elements.

d. Evaluation of Coefficients

Consider the coefficients

$$a_n = k \int_{-\infty}^{\infty} Y(k\lambda)\, \overline{G}_n(\lambda)\, d\lambda \qquad \text{See (36)}$$

Separating

$$\overrightarrow{G_n}(\lambda) = \frac{1}{\sqrt{\pi k}} \; \frac{(-j\lambda - 1)^n}{(-j\lambda + 1)^{n+1}} \qquad \text{See (33)}$$

into its real and imaginary parts,

$$\overline{G}_n(\lambda) = \frac{(-1)^n}{\sqrt{\pi k}\,\sqrt{1+\lambda^2}} \cos\left[(2n+1)\tan^{-1}\lambda\right] + j\,\frac{(-1)^n}{\sqrt{\pi k}\,\sqrt{1+\lambda^2}}\Big(\sin\left[(2n+1)\tan^{-1}\lambda\right]\Big). \qquad (38)$$

Letting

$$\left.\begin{aligned}
Y(k\lambda) &= M(k\lambda) + j\,N(k\lambda) \\
\overline{G}_n(\lambda) &= P_n(\lambda) - j\,Q_n(\lambda) \\
M(k\lambda) &= \text{real part of } Y(k\lambda) \\
P_n(\lambda) &= \text{Real part of } \overline{G}_n(\lambda) = \frac{(-1)^n}{\sqrt{\pi k}\,\sqrt{1+\lambda^2}} \cos\left[(2n+1)\tan^{-1}\lambda\right] \\
N(k\lambda) &= \text{imaginary part of } Y(k\lambda) \\
Q_n(\lambda) &= \text{imaginary part of } \overline{G}_n(\lambda) = \frac{(-1)^n}{\sqrt{\pi k}\,\sqrt{1+\lambda^2}}\Big(\sin\left[(2n+1)\tan^{-1}\lambda\right]\Big),
\end{aligned}\right\} \quad (39)$$

We have

$$\int_{-\infty}^{\infty} \bar{G}_n(\lambda)\, Y(k\lambda)\, d\lambda = \int_{-\infty}^{\infty} P_n(\lambda)\, M(k\lambda) d\lambda + \int_{-\infty}^{\infty} Q_n(\lambda)\, N(k\lambda)\, d\lambda . \tag{40}$$

The integrals

$$\int_{-\infty}^{\infty} P_n(\lambda) N(k\lambda)\, d\lambda \text{ and } \int_{-\infty}^{\infty} Q_n(\lambda)\, M(k\lambda)\, d\lambda$$

vanish, since $P_n(\lambda)$ and $M(k\lambda)$ are even functions, and $Q_n(\lambda)$ and $N(k\lambda)$ are odd functions.

Abbreviating equation (40) by the substitutions

$$\left.\begin{aligned} a_n &= \int_{-\infty}^{\infty} \bar{G}_n(\lambda)\, Y(k\lambda)\, d\lambda , \\ b_n &= \int_{-\infty}^{\infty} P_n(\lambda)\, M(k\lambda)\, d\lambda , \\ \text{and} \quad c_n &= \int_{-\infty}^{\infty} Q_n(\lambda)\, N(k\lambda)\, d\lambda , \text{ we have ,} \end{aligned}\right\} \tag{41}$$

$$a_n = b_n + c_n . \tag{42}$$

Substituting the values for $P_n(\lambda)$ and $Q_n(\lambda)$ in b_n, and c_n,

$$b_n = \sqrt{\frac{k}{\pi}}\,(-1)^n \int_{-\infty}^{\infty} \frac{\cos\left[(2n+1)\tan^{-1}\lambda\right]}{(1+\lambda^2)^{\frac{1}{2}}}\, M(k\lambda)\, d\lambda \tag{43}$$

$$c_n = \sqrt{\frac{k}{\pi}}\,(-1)^n \int_{-\infty}^{\infty} \frac{\sin\left[(2n+1)\tan^{-1}\lambda\right]}{(1+\lambda^2)^{\frac{1}{2}}}\, N(k\lambda)\, d\lambda . \tag{44}$$

These can be very much simplified by the transformation $\lambda = \tan\theta$. With this,

$$b_n = \sqrt{\frac{k}{\pi}}\,(-1)^n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\,(2n+1)\,\theta}{\cos\,\theta}\, M\,(k \tan\theta)\, d\,\theta$$

$$= \sqrt{\frac{k}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[-1+\sum_{m=0}^{n} 2(-1)^m \cos 2\,m\,\theta\right] M\,(k \tan\theta\,)\, d\,\theta \quad (45)$$

since,

$$\frac{\cos\,(2\,n+1)\,\theta}{\cos\,\theta} = (-1)^n \left[-1 + \sum_{m=0}^{n} 2\,(-1)^m \cos 2\,m\,\theta\right] \quad (46)$$

which can be varified by changing cos (2n+1) θ and cos θ into exponential forms and performing the division.

In like manner,

$$c_n = \sqrt{\frac{k}{\pi}}\,(-1)^n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin\,(2n+1)\,\theta}{\cos\,\theta}\, N\,(k \tan\theta)\, d\,\theta$$

$$= \sqrt{\frac{k}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\tan\theta + \sum_{m=0}^{n} 2(-1)^m \sin 2\,m\,\theta\right] N\,(k \tan\theta)\, d\,\theta \quad (47)$$

since,

$$\frac{\sin\,(2n+1)\theta}{\cos\,\theta} = (-1)^n \left[\tan\theta + \sum_{m=0}^{n} 2(-1)^m \sin 2\,m\,\theta\right], \quad (48)$$

Writing out a few terms of b_n, we have

$$b_0 = \sqrt{\frac{k}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} M\,(k \tan\theta\,)\, d\,\theta$$

$$b_1 = \sqrt{\frac{k}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[1 - 2 \cos 2\theta \right] M(k \tan\theta)\, d\theta$$

$$= b_o - 2 \sqrt{\frac{k}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} M(k \tan\theta) \cos 2\theta\, d\theta$$

$$b_2 = \sqrt{\frac{k}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[1 - 2 \cos 2\theta + 2 \cos 4\theta \right] M(k \tan\theta)\, d\theta$$

$$= b_1 + 2 \sqrt{\frac{k}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} M(k \tan\theta) \cos 4\theta\, d\theta$$

$$b_3 = \sqrt{\frac{k}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[1 - 2 \cos 2\theta + 2 \cos 4\theta - 2 \cos 6\theta \right] M(k\tan\theta)\, d\theta$$

$$= b_2 - 2 \sqrt{\frac{k}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} M(k \tan\theta) \cos 6\theta\, d\theta \qquad (49)$$

and so on.

The first few terms of c_n are:

$$c_o = \sqrt{\frac{k}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} N(k \tan\theta) \tan\theta\, d\theta$$

$$c_1 = \sqrt{\frac{k}{\pi}} \int_{-\frac{\pi}{2}}^{\pi/2} \left[\tan\theta - 2 \sin 2\theta \right] N(k \tan\theta)\, d\theta$$

$$= c_o - 2 \sqrt{\frac{k}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} N(k \tan\theta) \sin 2\theta\, d\theta$$

$$c_2 = \sqrt{\frac{k}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\tan\theta - 2 \sin 2\theta + 2 \sin\theta\, 4\theta \right] N(k\tan\theta)\, d\theta$$

$$= c_1 (=) + 2 \sqrt{\frac{k}{\pi}} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} N(k \tan\theta) \sin 4\theta\, d\theta$$

?

$$c_3 = \sqrt{\frac{k}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\tan\theta - 2\sin 2\theta + 2\sin 4\theta - 2\sin 6\theta\right]\left[N(k \tan\theta)\right] d\theta$$

$$= c_2 - 2\sqrt{\frac{k}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} N(k\tan\theta)\sin 6\theta \, d\theta \qquad (50)$$

and so on.

We have seen in the preceding chapter that

$$\int_{-\infty}^{\infty} P_1(\omega)\, P_2(\omega)\, d\omega = \int_{-\infty}^{\infty} Q_1(\omega)\, Q_2(\omega)\, d\omega$$

(equation 13) which is the Parseval Theorem for conjugate functions. Applying this to our case, we obtain

$$b_n = c_n \qquad (51)$$

and the coefficients of the series (35) are

$$a_n = 2\, b_n = 2\, c_n \qquad (52)$$

which simplifies our problem tremendously since b_n involves the real part of the required admittance function alone and c_n likewise involves the pure imaginary part of the function alone. The coefficients of the series are then completely determined by either the real part or the imaginary part of the required admittance function.

In the preceding chapter, we have discussed the interrelations of the four values of the admittance function, namely, the real part, the imaginary part, the magnitude, and the phase. If any one of the four values is given, the other three can be determined by means of the Hilbert transforms, provided the conditions for the validity of the transforms are satisfied. Therefore, given any one of the four values, we can proceed to evaluate the coefficients of the series and construct the network. Of course, the problem is much simpler when either the real part or the imaginary part is known than when either the magnitude or the phase is known. However, to obtain either the real part or the imaginary part from either the magnitude or the phase involves no particular difficulty now that we have available an optical means[1] to handle the conjugate integrals.

It is to be noted that the determination of the coefficients a_n of the series

$$Y(\omega) = \sum_{n=0}^{\infty} a_n G_n(k,\omega) \qquad \text{See (35)}$$

is extremely simple graphically because the process is simply harmonic analysis. The function being analyzed is plotted on paper with a tangent scale on the abscissa. The period is $\frac{\pi}{2}$. Examples of this are in Fig. on p.96. The first integration can be done by a planimeter. For n terms of the series (35), only n integrations are necessary.

---

1. See Chapter II section g.

Of course the coefficients b_n can be evaluated from the integrals.

$$\int_{-\infty}^{\infty} P_n(\lambda)\, M(k\lambda)\, d\lambda \quad \text{and} \quad \int_{-\infty}^{\infty} Q_n(\lambda)\, N(k\lambda)\, d\lambda$$

see (40)

directly without any change of variables by means of the integraph. The first nine terms of P_n and Q_n have been calculated and plotted. But this sort of integration is much harder to handle than the simple harmonic analysis. One distinct disadvantage is that the limits are $(-\infty, \infty)$ which makes the process difficult for functions converging slowly.

CHAPTER IV

Fundamental Networks

a. Lattice type network terminated by an inductance.

Having expanded the admittance function in terms of the Fourier Transforms of Laguerre's functions $G_n(\omega)$, our immediate problem is to give $G_n(\omega)$ a physical interpretation. The problem is to find a set of fundamental networks whose transfer admittances are of the form

$$G_n(k,\omega) = \sqrt{\frac{k}{\pi}} \frac{(j\omega - k)^n}{(j\omega + k)^{n+1}} \text{*} \qquad \text{See (33)}$$

or, for the sake of convenience,

$$G_n(\omega) = \frac{1}{\sqrt{2\pi}} \frac{(j\omega - \frac{1}{2})^n}{(j\omega + \frac{1}{2})^{n+1}}$$

when $k = \frac{1}{2}$. As a matter of fact, our specification for a transfer admittance function is made here after having known the result.

---

* The first nine terms of these functions have been calculated and plotted. See Appendix a.

Consider one section of a lattice type network such as shown.

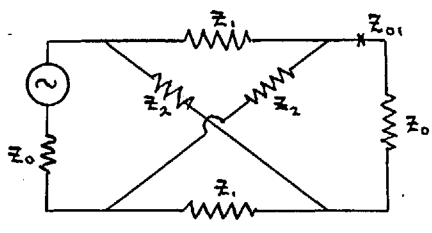


Fig. 1.

This is the same as the network in Fig. 2.

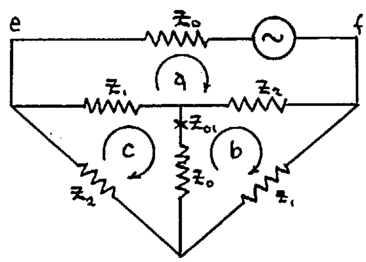


Fig. 2.

Z_{01} stands for the transfer impedance at the end of the first section as shown. If Z_{oc} is the impedance looking into the net from e f with the distant Z_0 open, and Z_{sc} the impedance looking into the net with the same Z_0 shorted, then by the usual method of evaluating impedances,

$$Z_0 = \sqrt{Z_{oc}\, Z_{se}} = \sqrt{\left[\frac{1}{2}\,(Z_1 + Z_2)\right]\left[\frac{2\, Z_1\, Z_2}{Z_1 + Z_2}\right]} = \sqrt{Z_1\, Z_2}$$

We are interested in the transfer impedance Z_{01}. The determinant for the circuit is

$$D=\begin{vmatrix} Z_{aa} & Z_{ab} & Z_{ac} \\ Z_{ba} & Z_{bb} & Z_{bc} \\ Z_{ca} & Z_{cb} & Z_{cc} \end{vmatrix} = \begin{vmatrix} Z_1+Z_2+\sqrt{Z_1Z_2} & -Z_1 & -Z_2 \\ -Z_1 & Z_1+Z_2+\sqrt{Z_1Z_2} & -\sqrt{Z_1Z_2} \\ -Z_2 & -\sqrt{Z_1Z_2} & Z_1+Z_2+\sqrt{Z_1Z_2} \end{vmatrix}$$

Multiplying out, we have

$$D=\left[Z_1+Z_2+\sqrt{Z_1Z_2}\right]\left[(Z_1+Z_2+\sqrt{Z_1Z_2})^2-Z_1Z_2\right]-Z_1\left[Z_2\sqrt{Z_1Z_2}+Z_1(Z_1+Z_2+\sqrt{Z_1Z_2})\right]$$

$$-Z_2\left[Z_1\sqrt{Z_1Z_2}+Z_2(Z_1+Z_2+\sqrt{Z_1Z_2})\right]$$

which can be readily simplified to

$$D = 2\sqrt{Z_1Z_2}\,(Z_1+Z_2)\left(\sqrt{Z_1}+\sqrt{Z_2}\right)^2$$

The minor M_{ab} is

$$M_{ab} = \begin{vmatrix} Z_{ba} & Z_{bc} \\ Z_{ca} & Z_{cc} \end{vmatrix}(-1)^4 = Z_1\sqrt{Z_1Z_2}+Z_2\left(Z_1+Z_2+\sqrt{Z_1Z_2}\right)$$

$$= \left(\sqrt{Z_1Z_2}+Z_2\right)(Z_1+Z_2)$$

Hence,

$$Z_{ab} = \frac{D}{M_{ab}} = \frac{2\sqrt{Z_1Z_2}\,(Z_1+Z_2)\left(\sqrt{Z_1}+\sqrt{Z_2}\right)^2}{\left(\sqrt{Z_1Z_2}+Z_2\right)(Z_1+Z_2)} = \frac{2\sqrt{Z_1Z_2}\left(\sqrt{Z_1}+\sqrt{Z_2}\right)^2}{\left(\sqrt{Z_1Z_2}+Z_2\right)}.$$

Similarly,

$$M_{ac} = \begin{vmatrix} Z_{ba} & Z_{bb} \\ Z_{ca} & Z_{cb} \end{vmatrix} (-1)^2 = -Z_1 \left(Z_1 + Z_2 + \sqrt{Z_1 Z_2}\right) - Z_2 \sqrt{Z_1 Z_2}$$

$$Z_{ac} = \frac{D}{M_{ac}} = \frac{-2\sqrt{Z_1 Z_2}\left(\sqrt{Z_1} + \sqrt{Z_2}\right)^2}{\left(\sqrt{Z_1 Z_2} + Z_1\right)}$$

$$Z_{o_1} = \frac{Z_{ab} Z_{ac}}{Z_{ab} + Z_{ac}} = \frac{2\sqrt{Z_1 Z_2}\left(\sqrt{Z_1} + \sqrt{Z_2}\right)^2}{Z_2 - Z_1} = \frac{2\sqrt{Z_1 Z_2}\left(\sqrt{Z_1} + \sqrt{Z_2}\right)}{\left(\sqrt{Z_2} - \sqrt{Z_1}\right)}$$

Let us now take two sections.

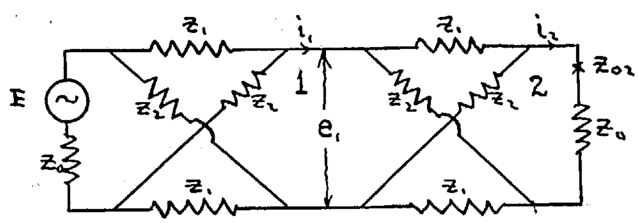


Fig. 3.

The current i_1 is

$$i_1 = \frac{E}{Z_{o_1}} = \frac{E}{2} \frac{\left(\sqrt{Z_2} - \sqrt{Z_1}\right)}{\sqrt{Z_1 Z_2}\left(\sqrt{Z_2} + \sqrt{Z_1}\right)}$$

and the voltage e_1 at 1 as indicated is

$$e_1 = i_1 \sqrt{Z_1 Z_2} = \frac{E}{2} \frac{\left(\sqrt{Z_2} - \sqrt{Z_1}\right)}{\left(\sqrt{Z_2} + \sqrt{Z_1}\right)}$$

The current i_2 at the end of the second section is

$$i_2 = e_1 \left[\frac{\sqrt{Z_2} - \sqrt{Z_1}}{\sqrt{Z_1 Z_2}\left(\sqrt{Z_2} + \sqrt{Z_1}\right)}\right] = \frac{E}{2\sqrt{Z_1 Z_2}} \left[\frac{\sqrt{Z_2} - \sqrt{Z_1}}{\sqrt{Z_2} + \sqrt{Z_1}}\right]^2$$

Therefore, Z_{o2}, the transfer impedance at the end of the second section is

$$Z_{o2} = \frac{E}{i_2} = 2\sqrt{Z_1 Z_2}\left[\frac{\sqrt{Z_2} - \sqrt{Z_1}}{\sqrt{Z_2} + \sqrt{Z_1}}\right]^{2}$$

It is clear now that Z_{on}, the transfer impedance at the n-th section is

$$Z_{on} = 2\sqrt{Z_1 Z_2}\left[\frac{\sqrt{Z_2} - \sqrt{Z_1}}{\sqrt{Z_2} + \sqrt{Z_1}}\right]^{n} \qquad (53)$$

This is almost the thing we are after. We need to make the power of the denominator one unit higher than that of the numerator. For this purpose, we use Thévenin's Theorem [1] which states that

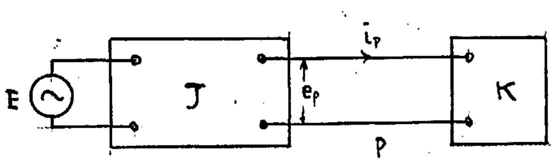


Fig. 4.

If Z_j and Z_k are the impedances looking into networks J and K respectively, and e_p is the voltage across p with K detached, then i_p, the current in the connection, is given by

$$i_p = \frac{e_p}{Z_j + Z_k} \qquad (54)$$

---

1. M. L. Thevenin, Sur un Nouveau Théorème d'Électricité Dynamique, comptes Rendus, 97, 1883, p. 159.

If J is our lattice network, then

$$e_p = i_n \sqrt{Z_1 Z_2} = \frac{E}{Z_{on}} \sqrt{Z_1 Z_2} = \frac{E}{2\sqrt{Z_1 Z_2}} \left[\frac{\sqrt{Z_2} - \sqrt{Z_1}}{\sqrt{Z_2} + \sqrt{Z_1}} \right]^n \sqrt{Z_1 Z_2}$$

$$= \frac{E}{2} \left[\frac{\sqrt{Z_2} - \sqrt{Z_1}}{\sqrt{Z_2} + \sqrt{Z_1}} \right]^n$$

and

$$i_p = \frac{\frac{E}{2} \left[\frac{\sqrt{Z_2} - \sqrt{Z_1}}{\sqrt{Z_2} + \sqrt{Z_1}} \right]^n}{\frac{1}{2} \sqrt{Z_1 Z_2} = Z_k \; +}$$

or

$$Z_{op} = \frac{\frac{1}{2} \sqrt{Z_1 Z_2} + Z_k}{\frac{1}{2} \left[\frac{\sqrt{Z_2} - \sqrt{Z_1}}{\sqrt{Z_2} + \sqrt{Z_1}} \right]^n} \tag{55}$$

If $Z_2 = j\omega L$, and $Z_1 = \frac{1}{j\omega C}$

$$Z_{op} = \frac{\frac{1}{2}\sqrt{\frac{L}{c}} + Z_k}{\frac{1}{2} \left[\frac{\sqrt{j\omega L} - \sqrt{\frac{1}{j\omega c}}}{\sqrt{j\omega L} + \sqrt{\frac{1}{j\omega c}}} \right]^n} = \frac{L \left\{ \sqrt{\frac{1}{Lc}} + \frac{2 Z_k}{L} \right\}}{\left[\frac{j\omega - \sqrt{\frac{1}{Lc}}}{j\omega + \sqrt{\frac{1}{Lc}}} \right]^n}$$

Making $Z_k = j\omega \frac{L}{2}$ and $\sqrt{\frac{1}{LC}} = \frac{1}{2}$, this reduces to

$$Z_{op} = L \frac{(j\omega + \frac{1}{2})^{n+1}}{(j\omega - \frac{1}{2})^n} \tag{56}$$

Or, if Y_{on} is the transfer admittance at the terminal inductance K with n-sections of the lattice network in J,

$$Y_{on} = \frac{1}{z_{op}} = \frac{1}{L} \frac{(j\omega - \frac{1}{2})^n}{(j\omega + \frac{1}{2})^{n+1}} \qquad (57)$$

which is the type of network characteristic we are after. These networks are shown in Fig. 5. Of course we can modify this to agree with the general expression.

$$G_n(k,\omega) = \sqrt{\frac{k}{\pi}} \frac{(j\omega - k)^n}{(j\omega + k)^{n+1}} \qquad (33)$$

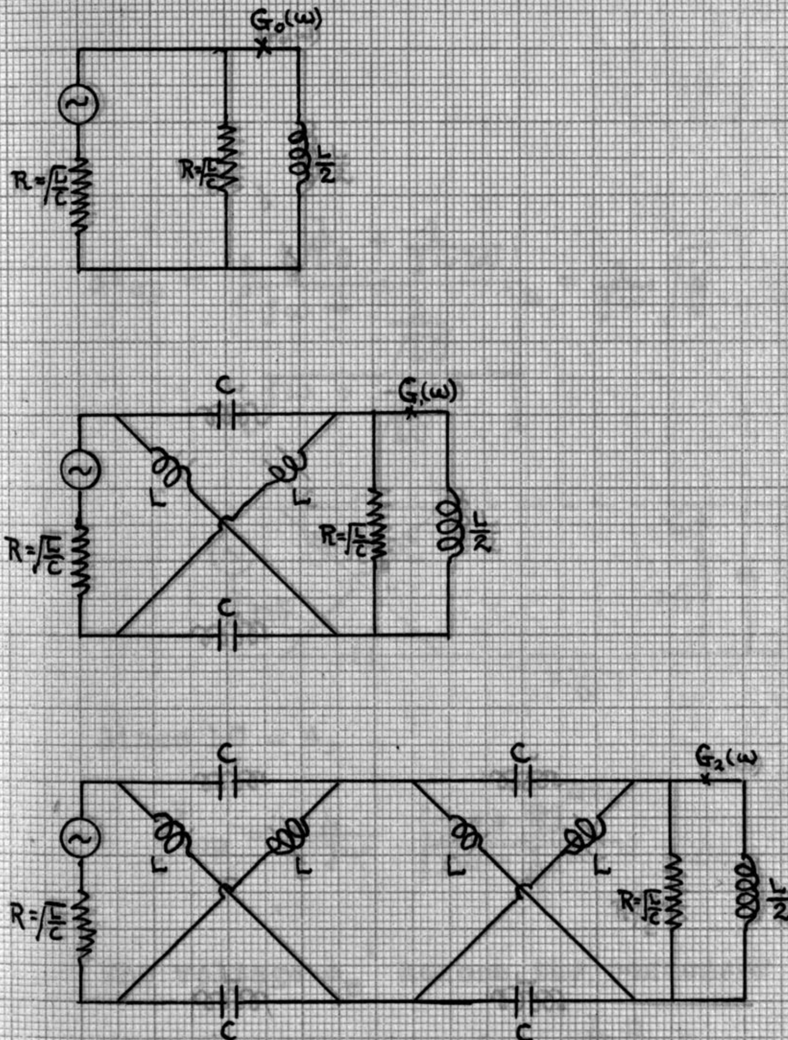
Suppose we put $z_1 = j\omega L$ and $z_2 = \frac{1}{j\omega c}$ instead of the reverse which we have just done, we can easily show that

$$Y_{on} = (-1)^n \frac{1}{L} \frac{(j\omega - \frac{1}{2})^n}{(j\omega + \frac{1}{2})^{n+1}} \qquad (58)$$

Figure 6 shows the networks of this type

b. <u>Lattice type network terminated by a capacitance</u>

Suppose now we add to the lattice network a capacitance instead of an inductance. Using equation (55) we have, with $z_{1k} = \frac{1}{j\omega 2C}$,



$G_0(\omega)$
$R=\sqrt{\frac{L}{C}}$
$R=\sqrt{\frac{L}{C}}$
$\frac{L}{2}$
C
$G_1(\omega)$
L
L
$R=\sqrt{\frac{L}{C}}$
$R=\sqrt{\frac{L}{C}}$
$\frac{L}{2}$
C
C
C
$G_2(\omega)$
L
L
L
L
$R=\sqrt{\frac{L}{C}}$
$R=\sqrt{\frac{L}{C}}$
$\frac{L}{2}$
C
C

Fig. 5.

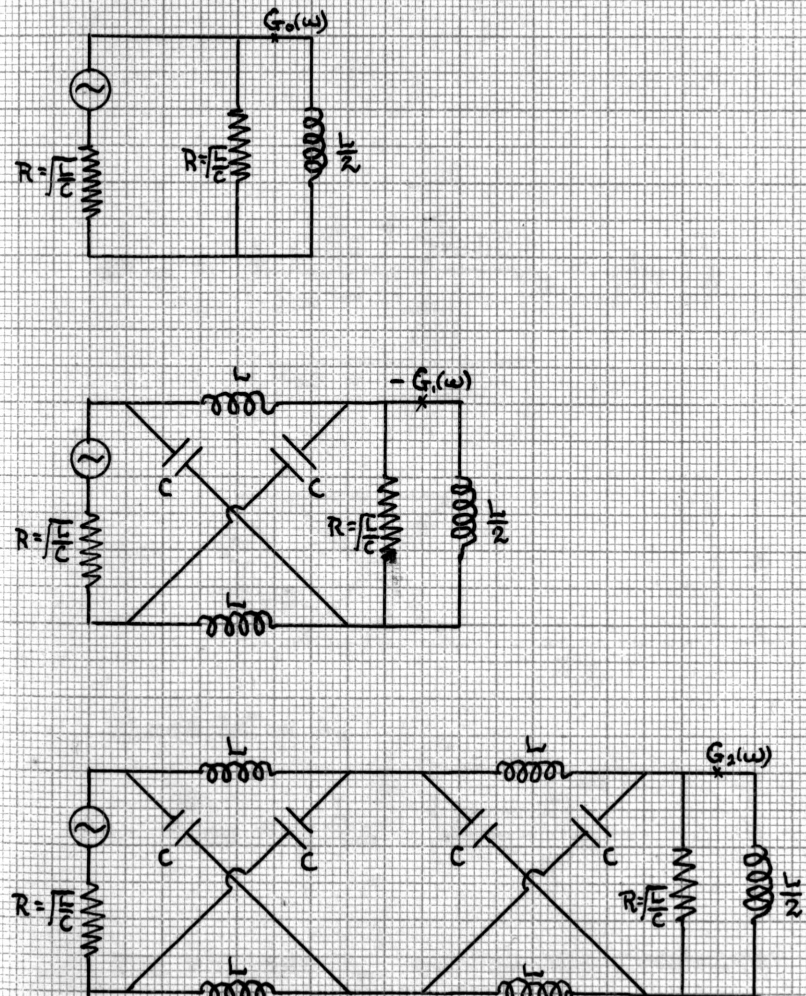


Fig. 6.

$$Z'_{op} = \frac{L\left(\frac{1}{\sqrt{LC}} + \frac{1}{j\omega LC}\right)}{\left[\dfrac{j\omega - \frac{1}{\sqrt{LC}}}{j\omega + \frac{1}{\sqrt{LC}}}\right]^n} = \frac{1}{j\omega}\sqrt{\frac{L}{C}} \; \frac{\left(j + \sqrt{\frac{1}{LC}}\right)^{n+1}}{\left(j - \sqrt{\frac{1}{LC}}\right)^n}$$

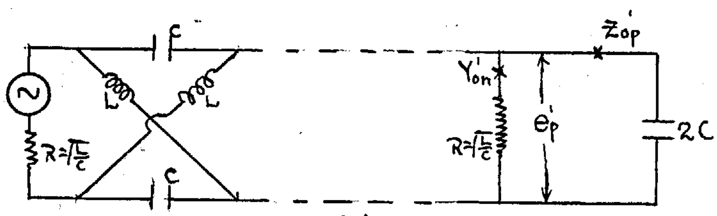


Fig. 7.

Since LC = 4,

$$Z'_{op} = \frac{L}{2\,j\omega} \; \frac{\left(j\omega + \frac{1}{2}\right)^{n+1}}{\left(j\omega - \frac{1}{2}\right)^{n}} \tag{59}$$

The voltage e_p' across the condenser is

$$e'_p = \frac{2\,j\omega}{L} \; \frac{\left(j\omega - \frac{1}{2}\right)^n}{\left(j\omega + \frac{1}{2}\right)^{n+1}} \; \frac{1}{2j\omega C} = \frac{1}{CL} \; \frac{\left(j\omega - \frac{1}{2}\right)^n}{\left(j\omega + \frac{1}{2}\right)^{n+1}}$$

The current i_R through the resistance is

$$i_R = \frac{e'_p}{\sqrt{\frac{L}{C}}} = \frac{1}{CL}\sqrt{\frac{C}{L}} \; \frac{\left(j\omega - \frac{1}{2}\right)^n}{\left(j\omega + \frac{1}{2}\right)^{n+1}}$$

$$= \frac{1}{2L} \; \frac{\left(j\omega - \frac{1}{2}\right)^n}{\left(j\omega + \frac{1}{2}\right)^{n+1}} .$$

Therefore,

$$Y'_{on} = \frac{1}{2L} \frac{(j\omega - \frac{1}{2})^n}{(j\omega + \frac{1}{2})^{n+1}} \qquad (60)$$

Which is the transfer admittance in the resistance which terminates n sections of lattice network, and which is shunted by a condenser. Notice that this transfer admittance is identical in form with $G_n(\omega)$

Thus we have seen that we can obtain $G_n(\omega)$ by shunting the last section of the lattice network with either an inductance or a capacitance. The transfer admittance functions $G_n(\omega)$ of these two cases are not found in the corresponding branches of the networks however. Remember that the factor $\frac{1}{2}$ in Y'_{on} which makes $LC = 4$ can be changed to any number k. We have discussed this before.

CHAPTER V

Methods of Connections

In the foregoing chapter, we have seen that we have at hand a set of fundamental networks corresponding to the Fourier transforms of Laguerre's functions into terms of which the transfer admittance function is expanded. Our next problem is to put these networks together precisely by the way we combine the terms of the expansion. In this chapter, we shall also give a method for designing a network to work into a terminal network. Finally we shall discuss the synthesis of networks whose admittance functions are not quadratically summable.

a. Method of connection to obtain a short circuit transfer admittance.

We have noticed that our fundamental networks are symmetrical. We have a common generator so that the networks can be paralleled at the generator end. The mid-points of the distant ends of all the networks of the set always have the same voltage both in magnitude and in phase regardless of the frequency. This is evident from the symmetry of the networks. Therefore, these points can be connected together without any disturbance among them. This is shown in Figure 8.

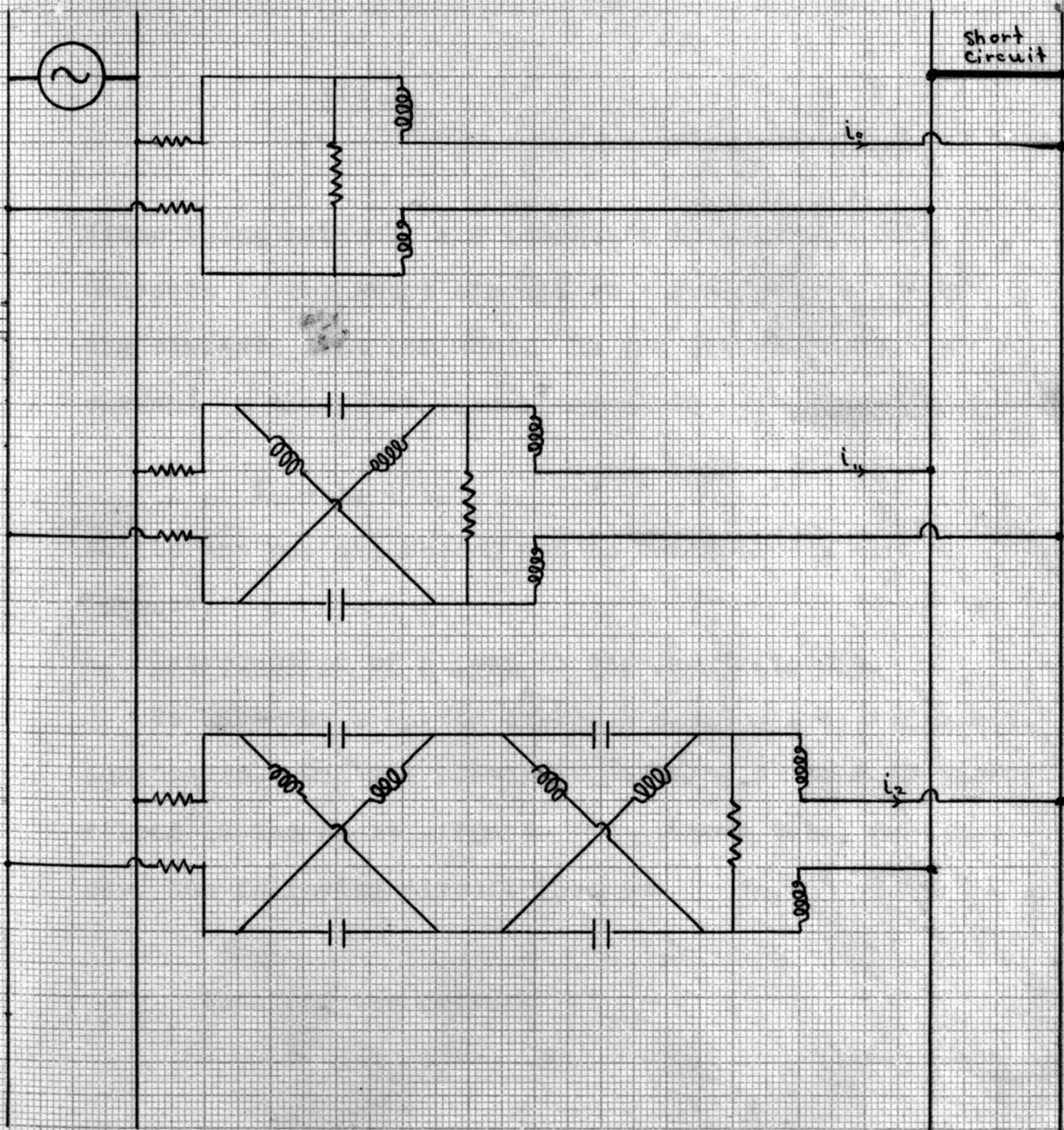


Fig. 8.

Since these networks are connected in parallel, the currents in the short circuit is the algebraic sum of the cirrents from the nets. In the figure we have a current in the short circuit equal to

$$i_0 = i_1 + i_2$$

Notice the way the second net is connected to the "short".

b. <u>Introduction of a load in the short-circuit.</u>

We have now a way to design a short circuit transfer admittance. But for practical purposes we need to insert a load in the short circuit. To achieve this, we shall need the following circuit theorem.

Consider the network J'

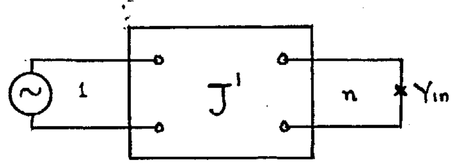


Fig.9.

The figure 1 indicates the first mesh, and n the nth mesh. The transfer admittance Y_{1n} is

$$Y_{1n} = \frac{M_{1n}}{D} = \frac{M_{1n}}{\begin{vmatrix} z_{11} & z_{12} & z_{13} & --- & z_{1n} \\ z_{21} & z_{22} & z_{23} & --- & z_{2n} \\ z_{n1} & z_{n2} & z_{n3} & --- & z_{nn} \end{vmatrix}}$$

Now if we open the short circuit and insert a network k' (see Fig.10), the new transfer admittance in the n-th mesh will be

$$Y'_{1n} = \frac{M_{1n}}{\begin{vmatrix} z_{11} & z_{12} & z_{13} & --- & z_{1n} \\ z_{21} & z_{22} & z_{23} & --- & z_{2n} \\ z_{n1} & z_{n2} & z_{n3} & --- & (z_{nn}+\frac{1}{Y}) \end{vmatrix}}$$

Where Y is the driving point admittance looking into K'.

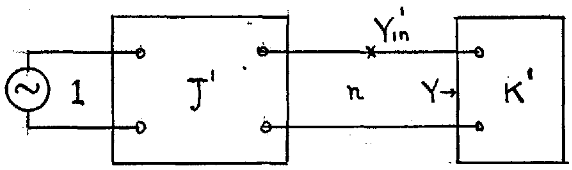


Fig. 10.

Expanding the determinant, we get

$$Y'_{in}\left[z_{n1} M_{n1} + z_{n2} M_{n2} + z_{n3} M_{n3} + --- + (z_{nn}+\tfrac{1}{Y})M_{nn}\right] = M_{in}$$

Taking the term containing Y out, the expression in the large brackets changes back to the determinant D, and so,

$$Y'_{in}\ (D + \frac{M_{nn}}{Y}) = M_{in}$$

or

$$Y'_{in} = \frac{Y \frac{M_{in}}{D}}{Y + \frac{M_{nn}}{D}} = \frac{Y\ Y_{in}}{Y+Y_{nn}} \qquad (61)$$

We may state the theorem as follows: If a driving point admittance Y be inserted into any mesh of a network, the resulting transfer admittance of that mesh at the point where the insertion is made, is equal to the product of the driving point admittance Y and the transfer admittance at that point before the alteration, divided by the sum of Y and the driving point admittance of the same mesh before the alteration.

Note that if the added admittance Y is a constant times Y_{nn}, then the new transfer admittance Y'_{in} differs from the original Y_{in} only by a constant factor, that is, if $Y = a\,Y_{nn}$, then

$$Y'_{in} = \frac{a}{a+1}\, Y_{in} .$$

Let us put this theorem in terms of impedances. Suppose

$$Z_{in} = \frac{1}{Y_{in}}, \quad Z_{nn} = \frac{1}{Y_{nn}}, \qquad Z = \frac{1}{Y}$$

and $Z'_{in} = \frac{1}{Y'_{in}}$, then (61) becomes

$$\frac{1}{Z'_{in}} = \frac{\frac{1}{Z\,Z_{in}}}{\frac{1}{Z} + \frac{1}{Z_{nn}}}$$

and

$$Z'_{in} = Z_{in}\left(1 + \frac{Z}{Z_{nn}}\right) \qquad (61)'$$

Therefore the Theorem stated in terms of impedances is: If a driving point impedance Z be inserted into any mesh of a network, the resulting transfer impedance of that mesh at the point where the insertion is made, is equal to the product of the transfer impedance of that mesh before the change and one plus the ratio of Z to the driving point impedance of the same mesh prior to the change.

Reference to equation (61)' shows that if Z is a constant times Z_{nn}, Z_{1n} differs from the original Z_{1n} by a constant factor.

As an example, let us take the following circuit.

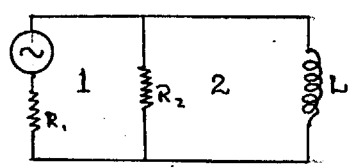


Fig. 11.

$$D = \begin{vmatrix} R_1+R_2 & -R_2 \\ -R_2 & R_2+j\omega L \end{vmatrix} = R_1R_2 + j\omega L\,(R_1+R_2)$$

$$Y_{12} = \frac{R_2}{R_1R_2+j\omega L\,(R_2+R_2)}\,;\quad Y_{22} = \frac{R_1+R_2}{R_1R_2+j\omega L(R_1+R_2)}$$

If now we insert a certain admittance Y in mesh 2 as shown,

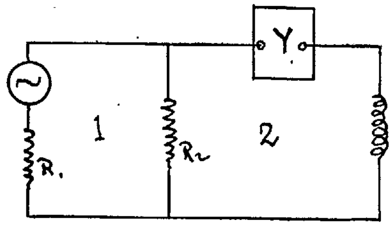


Fig. 12.

we have by our theorem,

$$Y'_{12} = \frac{Y \quad Y_{12}}{Y + Y_{22}} = \frac{Y \quad R_2}{R_1R_2+j\omega L(R_1+R_2)} \cdot \frac{1}{Y + \dfrac{R_1R_2}{R_1R_2+j\omega L(R_1+R_2)}}$$

$$= \frac{Y \; R_2}{(R_1+R_2)+R_1R_2Y+j\omega LY\,(R_1+R_2)}$$

Getting the result by determinants,

$$Y'_{12} = \frac{R_2}{\begin{vmatrix} R_1+R_2 & -R_2 \\ -R_2 & \frac{1}{Y}+R_2+j\omega L \end{vmatrix}} = \frac{R_2}{(R_1+R_2)(\frac{1}{Y}+R_2+j\omega L)-R_2^2}$$

$$= \frac{Y \; R_2}{(R_1+R_2)+R_1R_2Y+j\omega LY\,(R_1+R_2)}$$

which agrees with the previous answer.

It seems that if we apply this theorem to our problem of introducing a load in the short circuit which has a designable transfer admittance, we encounter two unknown quantities Y_{1n} and Y_{nn}. We have the desired function Y'_{1n}, know the load Y, and wish to find Y_{1n} from these so that we can proceed with the design, but the theorem requires another factor Y_{nn} which depends on Y_{1n}. However we know that looking from the short circuited end, all of the fundamental networks have driving point admittances Y_{nn} of the form

$$\frac{1}{a\left(\sqrt{\frac{L}{C}} + j\omega L\right)} = \frac{1}{a\left(\frac{L}{2} + j\omega L\right)}$$

where a is a constant. Therefore if we make the load Y of the same form, Y_{nn} will be a constant times Y, and from our theorem, Y'_{1n} will be of the same form as Y_{1n}. For example, if we terminate the network into an admittance Y equal to Y_{nn}, then

$$Y'_{1n} = \frac{1}{2} Y_{1n}$$

Therefore we can load the network as shown in Figure 13. The resistances of the networks at the generator end have common points so that these can be located and combined to make up the generator resistance. Or, we can insert a load in the generator in the same manner just described.

c. <u>Methods of analyzing a problem.</u>

It is not necessary that we treat a given transfer admittance as a whole. We can divide the function into parts whose algebraic sum is the given function. Treat each part separately and then put them together in the way discussed in section (a) of this chapter. In this way we may have a resulting network which is much simpler than the one we would have if the function were treated as a whole. Also, we can divide the function into such portions that some of them can be approximated by simple

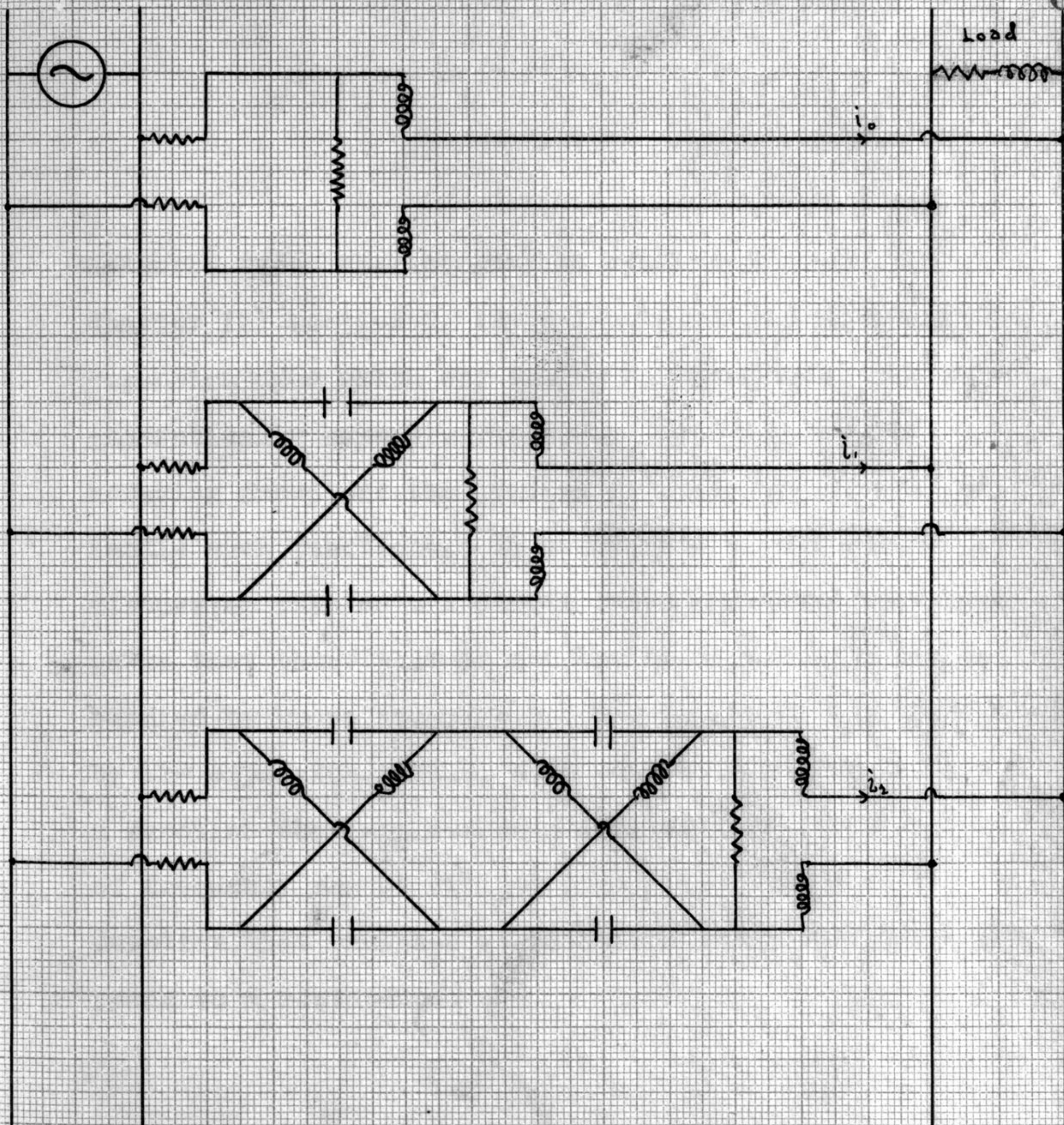


Fig. 13.

networks whose characteristics we already know. In Chapter III section C, it is mentioned that although the expansion upon which our scheme is based holds for only quadratically summable functions, that is, roughly speaking, functions that vanish at infinity, we can still handle such functions. It is clear from the above discussion that we can divide these functions into such parts that those that do not vanish at infinity can be handled by known networks and the remainder can be treated by the expansion. Thus, as a very simple illustration, a function that approaches a constant for large values of frequency can be divided into a constant plus a function that vanishes at infinity. The constant corresponds to a resistance.

CHAPTER VI

Application of the Theory to Specific Problems

In this chapter we shall work out two problems completely to show the procedure of our method of synthesis, and the way the required function is approximated. In the first case, the real part of an admittance is given, and in the second, the imaginary part. In both cases the functions are so chosen that their conjugates can be evaluated by means of the conjugate integrals so that the results may be checked.

a. Problem I. Design a net work whose transfer admittance at a certain point has its real part equal to $e^{-\omega^2}$.

Our expansion formula is

$$Y(\omega) = \sum_{n=0}^{\infty} a_n G_n(k,\omega) \qquad \text{See (35)}$$

$G_n(k,\omega)$ correspond to our fundamental networks discussed in Chapter IV.

It is shown in section (d) Chapter III that the coefficients a_n are completely determined by either the real part or the imaginary part of the admittance function, that is

$$a_n = 2b_n = 2c_n. \qquad \text{See (52)}$$

In the present case, we proceed to evaluate bn. Letting k = 1, we have from (49),

$$b_0 = \frac{1}{\sqrt{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\tan^2\theta}\, d\theta = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} \frac{dx}{1+x^2}$$

~~1. See foot-note on page~~

We can evaluate this by means of the Parseval Theorem for Fourier transforms[1]. Consider the integral

$$\int_{-\infty}^{\infty} e^{-px^2} \frac{dx}{1+x^2}$$

The Fourier transform of e^{-px^2} is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-px^2} e^{-jux} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-px^2} \cos u x \, d x \quad *$$

$$= \frac{1}{\sqrt{2p}} e^{-\frac{u^2}{4p}}$$

The Fourier transform of $\frac{1}{1+x^2}$ is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{-jux} dx}{1+x^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\cos u x \, d x}{1+x^2} = \sqrt{\frac{\pi}{2}}\, e^{-u} \quad **$$

Therefore,

$$\int_{-\infty}^{\infty} e^{-px^2} \frac{dx}{1+x^2} = \sqrt{\frac{\pi}{p}}\int_0^{\infty} e^{-\frac{u^2}{4p}} e^{-u} du = \sqrt{\frac{\pi}{p}}\, e^{p} \int_0^{\infty} e^{-\frac{(u+2p)^2}{4p}} du$$

$$= \sqrt{\frac{\pi}{p}}\, e^{p} \int_{2p}^{\infty} e^{\frac{-u^2}{4p}} du.$$

Hence,

$$b_0 = e\int_2^{\infty} e^{\frac{-u^2}{4}} du = 2e\left[\int_0^{\infty} e^{-y^2} dy - \int_0^{1} e^{-y^2} dy\right]$$

---

*Bierens de Haan, Nouvelles Tables D'Intérgrades De'finies, Table 269, No.3, p. 394.

$$\int_{-\infty}^{\infty} e^{-q^2x^2} \cos\{p(x+x)\} dx = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}} \cos p\lambda.$$

** Ibid, Table 160 No.5, p.223. $\int_0^{\infty} \cos p x \frac{dx}{q^2+x^2} = \frac{\pi}{2q} e^{-pq}$

1. See note 1. on page 9.

which can be readily evaluated.

$$bo = 2e\left(\frac{\sqrt{\pi}}{2} - 0.746.8\right) = \frac{1.3433}{\sqrt{\pi}}$$

To evaluate the succeeding coefficients, we use the identity

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\tan^2\theta} \cos m\theta \, d\theta = \left(\frac{1}{m}\right)\sin m\theta \, e^{-\tan^2\theta}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{2}{m}\times$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan\theta \sec^2\theta \sin m\theta \; e^{-\tan^2\theta} \, d\theta$$

which is obtained by integrating by parts.

When $m = 2$,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\tan^2\theta} \cos 2\theta \, d\theta = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^2\theta \; e^{-\tan^2\theta} \, d\theta =$$

$$2\int_{-\infty}^{\infty} \frac{x^2 e^{-x^2} dx}{1+x^2}$$

$$= 2\int_{-\infty}^{\infty} e^{-x^2} dx \quad -2\int_{-\infty}^{\infty} \frac{e^{-x^2} dx}{1+x^2}$$

$$= 2\left(\sqrt{\pi} - 1.3433\right) = 0.8584$$

When $m = 4$,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\tan^2\theta} \cos 4\theta \, d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan\theta \sec^2\theta \sin 4\theta \; e^{-\tan^2\theta} \, d\theta$$

The first integral on the right side without the factor 4 is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \, e^{-\tan^2\theta} \, d\theta = \int_{-\infty}^{\infty} \frac{x^2 e^{-x^2}}{(1+x^2)^2} \, dx$$

As before, this can be evaluated by Parseval's Theorem for Fourier transforms. The Fourier transform of e^{-x^2} is, as shown before, $\frac{1}{\sqrt{2}} e^{-\frac{u^2}{4}}$. The Fourier transform of $\frac{x^2}{(1+x^2)^2}$ is

$$\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \frac{x^2 e^{-jux} dx}{(1+x^2)^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x^2 \cos ux \, dx}{(1+x^2)^2} = \frac{1}{2}\sqrt{\frac{\pi}{2}} (1-u)e^{-u} \,*$$

Therefore

$$\int_{-\infty}^{\infty} \frac{x^2 e^{-x^2} dx}{(1+x^2)^2} = \frac{\sqrt{\pi}}{4} \int_{-\infty}^{\infty} (1-|u|) \, e^{-|u|} \, e^{-\frac{u^2}{4}} du$$

$$= \frac{\sqrt{\pi}}{2} \int_0^{\infty} e^{\frac{-u^2}{4} - u} du - \frac{\sqrt{\pi}}{2} \int_0^{\infty} u e^{\frac{-u^2}{4} - u} du$$

---

* Ibid. Table 170, No. 8, p. 248:

$$\int_0^{\infty} \cos px \frac{x^2 dx}{(q^2+x^2)^2} = \frac{\pi}{4q} (1-pq) \, e^{-pq}$$

The first integral on the right has already been evaluated.

$$\int_0^\infty u\, e^{-\frac{u^2}{4}-u}\, du = e \int_0^\infty u\, e^{-\frac{(u+2)^2}{4}}\, du = e \int_2^\infty (x-2)\, e^{-\frac{x^2}{4}}\, dx$$

(Let u + 2 = x)

$$= e\int_2^\infty x\, e^{-\frac{x^2}{4}} dx - 2e \int_2^\infty e^{-\frac{x^2}{4}}\, dx$$

$$= e\left[-2e^{-\frac{x^2}{4}}\right]_2^\infty - 2\, e \int_2^\infty e^{-\frac{x^2}{4}} dx = 2\left[1 - e \int_2^\infty e^{-\frac{x^2}{4}}\, dx\right]$$

So,

$$\int_{-\infty}^\infty \frac{x^2\, e^{-x^2} dx}{(1+x^2)^2} = \frac{\sqrt{\pi}}{2}\left[e \int_2^\infty e^{-\frac{u^2}{4}} du + 2e\int_2^\infty e^{-\frac{x^2}{4}}\, dx - 2\right]$$

$$= \frac{\sqrt{\pi}}{2}\left[3\, e \int_2^\infty e^{-\frac{u^2}{4}}\, du - 2\right]$$

$$= \frac{\sqrt{\pi}}{2}\left[3\left(\frac{1.3433}{\sqrt{\pi}}\right) - 2\right] = \underline{0.242,5}$$

and,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\tan\theta} \cos 4\theta\, d\theta = 4\,(0.2425) - 0.858,4 = \underline{\underline{0.111,6}}$$

When m = 6,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\tan^2\theta} \cos 6\theta \, d\theta = \frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan\theta \sec^2\theta \sin 6\theta \, e^{-\tan^2\theta} d\theta$$

$$\left[\sin 6\theta = 32\cos^5\theta \sin\theta - 32\cos^3\theta \sin\theta + 6\cos\theta \sin\theta\right]$$

$$= \frac{32}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta \, e^{-\tan^2\theta} d\theta - \frac{32}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \, e^{-\tan^2\theta} d\theta$$

$$+ 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^2\theta \, e^{-\tan^2\theta} d\theta$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta \, e^{-\tan^2\theta} d\theta = -\frac{1}{8}\left(\frac{1}{4}\sin 4\theta - \theta\right) e^{-\tan^2\theta}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$- \frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan\theta \sec^2\theta \sin 4\theta \, e^{-\tan^2\theta} d\theta + \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \theta \tan\theta \sec^2\theta \times$$

$$e^{-\tan^2\theta} d\theta$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \theta \tan\theta \sec^2\theta \, e^{-\tan^2\theta} d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\tan^2\theta} d\theta$$

Inserting this in the precee~~e~~ding equation

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta \, e^{-\tan^2\theta} d\theta = -\frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan\theta \sec^2\theta \sin 4\theta$$

$$e^{-\tan^2\theta} d\theta + \frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\tan^2\theta} d\theta$$

Collecting the various terms, we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\tan^2\theta} \cos 6\theta \, d\theta = \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\tan^2\theta} \, d\theta - \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan\theta \sec^2\theta \sin 4\theta \, e^{-\tan^2\theta} \, d\theta$$

$$+ 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^2\theta \, e^{-\tan^2\theta} \, d\theta - \frac{32}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \, e^{-\tan^2\theta} \, d\theta$$

All of the terms have been calculated so that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\tan^2\theta} \cos 6\theta \, d\theta = \frac{4}{3}(1.3433) - \frac{2}{3}(0.111,6) + 2\,(0.429,2) - \frac{32}{3}(0.242,5) = -\underline{\underline{0.0859}}$$

Reference to equations (49) shows that

$$2\,b_0 = \frac{2}{\sqrt{\pi}}(1.3433) = \underline{\underline{+0.854}}\ \sqrt{\pi}$$

$$2\,b_1 = \frac{2}{\sqrt{\pi}}\left[1.3433 - 2(0.8584)\right] = 0.238,8\ \sqrt{\pi}$$

$$2\,b_2 = \frac{2}{\sqrt{\pi}}\left[1.3433 - 2\,(0.8584) + 2(0.11,6)\right] = -\underline{\underline{0.0956}}\ \sqrt{\pi}$$

$$2\,b_3 = \frac{2}{\sqrt{\pi}}\left[1.3433 - 2(0.8584) + 2(0.111,6) + 2(0.0859)\right] = \underline{\underline{0.01369}}\ \sqrt{\pi}$$

-2-

Our result is, referring to equations (33) and (35),

$$Y(\omega) = 0.854 \frac{1}{(j\omega+1)} - 0.238,8 \frac{(j\omega-1)}{(j\omega+1)^2} - 0.0956 \frac{(j\omega-1)^2}{(j\omega+1)^3} + 0.01369 \frac{(j\omega-1)^3}{(j\omega+1)^4} \pm \cdots \qquad (I)$$

This equation is plotted and Fig. x (on p. 80) to p. 83 show how the real part of the desired function is approximated. (Tables in Appendix)

For the purpose of checking our result, we shall calculate the corresponding imaginary part of the given function by means of the conjugate integrals. We have given, $P(\omega) = e^{-\omega^2}$. Equation (4) gives

$$Q(\omega) = -\frac{1}{\pi}\int_0^\infty \sin u\omega \, du \int_{-\infty}^{\infty} P(t) \cos u t \, dt$$

$$= -\frac{1}{\pi}\int_0^\infty \sin u\omega \, du \int_{-\infty}^{\infty} e^{-t^2} \cos u t \, dt \;*$$

$$= -\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\frac{u^2}{4}} \sin u\omega \, du = -\frac{e^{-\omega^2}}{2j\sqrt{\pi}} \times \int_0^\infty \left[e^{-\frac{(u-2j\omega)^2}{4}} - e^{-\frac{(u+2j\omega)^2}{4}} \right] du$$

Let $x = u - 2j$; $y = u + 2j$.

* See foot note of page 64.

$$Q(\omega) = -\frac{e^{-\omega^2}}{2j\sqrt{\pi}}\left[\int_{-2j\omega}^{\infty} e^{-\frac{x^2}{4}}\,dx + \int_{2j\omega}^{\infty} e^{-\frac{y^2}{4}}\,dy\right]$$

$$= -\frac{e^{-\omega^2}}{2j\sqrt{\pi}}\int_{-2j\omega}^{2j\omega} e^{-\frac{u^2}{4}}\,du$$

$$= -\frac{e^{-\omega^2}}{2\sqrt{\pi}}\int_{-2\omega}^{2\omega} e^{\frac{v^2}{4}}\,dv = -\frac{e^{-\omega^2}}{\sqrt{\pi}}\int_{0}^{2\omega} e^{\frac{v^2}{4}}\,dv$$

The integral $e^{-\omega^2}\int_0^{2\omega} e^{\frac{v^2}{4}}\,dv$ can be developed into a series by integrating by parts. In so doing, we obtain

$$-\sqrt{\pi}\,Q(\omega) = \omega - \frac{\omega^3}{6} + \frac{\omega^5}{60} - \frac{\omega^7}{840} + \frac{\omega^9}{15,120} - \frac{\omega^{11}}{332,640}$$

$$+ \frac{\omega^{13}}{8,648,640} - \frac{\omega^{15}}{259,459,200} + \frac{\omega^{17}}{8,821,612,800}$$

$$- \frac{\omega^{19}}{335,221,286,400} + - - - - - - - - - -$$

This series is good up to ω equals to about 2. We need to develop another series for calculating the values of $Q(\omega)$ for large values of the argument.

Integrating by parts,

$$-\sqrt{\pi}\,Q(\omega) = \frac{2}{\omega} + \frac{2^2}{\omega^3} + \frac{2^2 \cdot 3}{\omega^5} + \frac{2^4 \cdot 3 \cdot 5}{\omega^7} + \frac{2^5 \cdot 3 \cdot 5 \cdot 7}{\omega^9} + \dots$$

which converges rapidly for larger values of ω.

From these two series, $Q(\omega)$ has been calculated and plotted. Figures on p. 86 to p. 87 show how the expansion (I) approximate the imaginary part of the given function. (Tables in Appendix)

In this particular problem the convergence is very rapid. Only three terms of the series are necessary to represent the desired function almost exactly.

b. Problem II. Design a network whose transfer admittance at a certain point has its imaginary part equal to $-\sqrt{\pi}\,\omega e^{-\omega^2}$.

Again, the expansion formula is

$$Y(\omega) = \sum_{n=0}^{\infty} a_n G_n(k,\omega) \qquad \text{See (35)}$$

We have the imaginary part given, so that

$$a_n = 2\,c_n\,.$$

Referring to equations (50), and assuming k = 1, we have the first coefficient,

$$c_0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^2\theta\, e^{-\tan^2\theta}\, d\theta = \int_{-\infty}^{\infty} \frac{x^2 e^{-x^2}\, dx}{1+x^2} = 0.429,2$$

(from the early part of problem I)

For calculating the succeeding integrals, we shall use the identity:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan\theta\, e^{-\tan^2\theta} \sin m\theta\, d\theta = -\frac{1}{m} \tan\theta\, e^{-\tan^2\theta} \cos m\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$- \frac{2}{m} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^2 \theta \sec^2 \theta \cos m \theta \, e^{-\tan^2 \theta} \, d\theta + \frac{1}{m} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^2 \theta \cos m \theta \, e^{-\tan^2 \theta} \, d\theta$$

which is obtained from integrating by parts.

When $m = 2$,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan \theta \, e^{-\tan^2 \theta} \sin 2\theta \, d\theta = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^2 \theta \sec^2 \theta \cos 2\theta \times e^{-\tan^2 \theta} \, d\theta$$

$$+ \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^2 \theta \cos 2\theta \, e^{-\tan^2 \theta} \, d\theta$$

[Note: $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$]

$$= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^2 \theta \, e^{-\tan^2 \theta} + \int_{-\frac{\pi}{2}}^{\pi/2} \tan^4 \theta \, e^{-\tan^2 \theta} \, d\theta$$

$$+ \frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\tan^2 \theta} \, d\theta - \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^2 \theta \, e^{-\tan^2 \theta} \, d\theta$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^4 \theta \, e^{-\tan^2 \theta} \, d\theta = \int_{-\infty}^{\infty} \frac{x^4 e^{-x^2} \, dx}{1+x^2} = \int_{-\infty}^{\infty} x^2 e^{-x^2} dx - \int_{-\infty}^{\infty} \frac{x^2 e^{-x^2} dx}{1+x^2}$$

$$= \sqrt{\frac{\pi}{2}} - 0.429,2 = +\underline{0.457,0}$$

Hence,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan\theta \cdot e^{-\tan^2\theta} \sin 2\theta \, d\theta = -0.429{,}2 + 0.457{,}0 - \frac{1}{2} \times (1.343{,}3) - \frac{1}{2}(0.429{,}2)$$

$$= \underline{+0.484{,}9}$$

- -

When m = 4,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan\theta \, e^{-\tan^2\theta} \sin 4\theta \, d\theta = -\frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^2\theta \sec^2\theta \cos 4\theta \times e^{-\tan^2\theta} \, d\theta$$

$$+ \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^2\theta \cos 4\theta \, e^{-\tan^2\theta} \, d\theta$$

$$\left[\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1\right]$$

$$= -4\int \sin^2\theta \, e^{-\tan^2\theta} d\theta + 4\int \tan^2\theta \, e^{-\tan^2\theta} d\theta - \frac{1}{2} \times \int \tan^2\theta \sec^2\theta \, e^{-\tan^2\theta} d\theta$$

$$+ 2\int \cos^2\theta \, e^{-\tan^2\theta} d\theta - 2\int e^{-\tan^2\theta} d\theta + \frac{1}{4}\int \sec^2\theta \, e^{-\tan^2\theta} d\theta$$

$$= -6\int \sin^2\theta \, e^{-\tan^2\theta} d\theta + 4\int \tan^2\theta \, e^{-\tan^2\theta} d\theta$$

$$= -6\left[0.242{,}5\right] + 4\left[0.429{,}2\right] = +\underline{0.261{,}8}$$

The first integral on the right can be found in the corresponding part of Problem I.

- -

When m = 6,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan\theta\, e^{-\tan^2\theta} \sin 6\theta\, d\theta = -\frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^2\theta \sec^2\theta \cos 6\theta \times e^{-\tan^2\theta}\, d\theta$$

$$+\frac{1}{6}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^2\theta \cos 6\theta\, e^{-\tan^2\theta}\, d\theta$$

$$\left[\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1\right]$$

$$= -\frac{32}{3}\int \sin^2\theta\cos^2\theta\, e^{-\tan^2\theta}\, d\theta + 16\int \sin^2\theta\, e^{-\tan^2\theta}\, d\theta$$

$$-6\int \tan^2\theta\, e^{-\tan^2\theta}\, d\theta$$

$$+\frac{1}{3}\int \tan^2\theta\sec^2\theta\, e^{-\tan^2\theta}\, d\theta + \frac{16}{3}\int \cos^4\theta\, e^{-\tan^2\theta}\, d\theta - 8\int \cos^2\theta \times e^{-\tan^2\theta}\, d\theta$$

$$+3\int e^{-\tan^2\theta}\, d\theta - \frac{1}{6}\int \sec^2\theta\, e^{-\tan^2\theta}\, d\theta$$

$$= -\frac{32}{3}\int \sin^2\theta\cos^2\theta\, e^{-\tan^2\theta}\, d\theta + 24\int \sin^2\theta\, e^{-\tan^2\theta}\, d\theta$$

$$-6\int \tan^2\theta\, e^{-\tan^2\theta}\, d\theta - 5\int e^{-\tan^2\theta}\, d\theta + \frac{16}{3}\int \cos^4\theta\, 6^{-\tan^2\theta}\, d\theta$$

From the corresponding section of Problem I,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\, e^{-\tan^2\theta}\, d\theta = -\frac{1}{16}(0.111,6)\ 2 + \frac{1}{8}(1.343,3)$$

$$= 0.153,96.$$

The rest of the integrals in the preceding equation have been evaluated in Problem I except the integral

$\int_{-\pi/2}^{\frac{\pi}{2}} \cos^4 \theta \, e^{-\tan^2 \theta} \, d\theta$. To integrate this, we use

Parseval's Theorem for Fourier transforms again.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 \theta \, e^{-\tan^2 \theta} \, d\theta = \int_{-\infty}^{\infty} \frac{e^{-x^2} dx}{(1+x^2)^3}$$

The Fourier transform of $\frac{1}{(1+x^2)^3}$ is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\cos u x \, d x}{(1+x^2)^3} = \frac{\pi}{8\sqrt{2\pi}} (3 + 3 u + u^2) \, e^{-u} . \quad *$$

The Fourier transform of e^{-x^2} is, as we have already used on several occasions, $\frac{1}{\sqrt{2}} e^{-\frac{u^2}{4}}$. Therefore,

$$\int_{-\infty}^{\infty} \frac{e^{-x^2} d x}{(1+x^2)^3} = \frac{\sqrt{\pi}}{8} \int_0^{\infty} (3 + 3 u + 3 u^2) \, e^{-\frac{u^2}{4} - u} \, d u$$

$$= \frac{\sqrt{\pi}}{8} e \int_0^{\infty} (3 + 3u + u^2) \, e^{-\frac{(u+2)^2}{4}} \, du$$

[Let x = u + 2]

---

* Bieren de Haan's, Table 170, No. 9, p. 248:

$$\int_0^{\infty} \cos p x \frac{dx}{(q^2+x^2)^3} = \frac{\pi}{16q^5} (3 + 3 pq + p^2 q^2) \, e^{-pq}$$

$$= \frac{\sqrt{\pi}}{8} e\left[\int_2^\infty e^{-\frac{x^2}{4}} dx - \int_2^\infty x\, e^{-\frac{x^2}{4}} dx + \int_2^\infty x^2 e^{-\frac{x^2}{4}} dx\right]$$

Integrating the last term by parts,

$$\int_2^\infty x^2 e^{-\frac{x^2}{4}} dx = -2x e^{-\frac{x^2}{4}}\Big|_2^\infty + 2\int_2^\infty e^{-\frac{x^2}{4}} dx = 4e^{-1} + 2\int_2^\infty e^{-\frac{x^2}{4}} dx,$$

and putting this in the preceding equation

$$\int_{-\infty}^{\infty} \frac{e^{-x^2}\, dx}{(1+x^2)^3} = \frac{\sqrt{\pi}}{8}\left[2 + 3e\int_2^\infty e^{-\frac{x^2}{4}} dx\right]$$

$$= \frac{1}{8}\left[2\sqrt{\pi} + 3\,(1.3433)\right] = \underline{+0.946{,}85}$$

The integral on the right is recognized as a multiple of the first coefficient b_0 of Problem I.

Returning now to the original integral, we write

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan\theta\, e^{-\tan^2\theta} \sin 6\theta\, d\theta = -\frac{32}{3}(0.153{,}96) + 24\,(0.242{,}5)$$

$$-6\,(0.429{,}2) - 5(1.343{,}3) + \frac{16}{3}(0.946{,}85) = -\underline{\underline{0.064{,}0}}$$

Reference to equations (50) shows that

$$2\,c_0 = \frac{2}{\pi}(0.429{,}2) = \underline{+0.484}$$
$$2\,c_1 = \frac{2}{\pi}\left[0.429{,}2 - 2(0.484{,}9)\right] = -\underline{0.610}$$
$$2\,c_2 = \frac{2}{\pi}\left[0.429{,}2 - 2(0.484{,}9) + 2(0.261{,}8)\right] = -\underline{0.0192}$$
$$2\,c_3 = \frac{2}{\pi}\left[0.429{,}2 - 2(0.484{,}9) + 2(0.261{,}8) + 2(0.064{,}0)\right] = \underline{\underline{+0.1253}}$$

The result is, referring to equations (33) and (35)

$$Y(\omega) = 0.484\,\frac{1}{(j\omega+1)} - 0.610\,\frac{(j\omega-1)}{(j\omega+1)^2} - 0.0192\,\frac{(j\omega-1)^2}{(j\omega+1)^3}$$

$$+\ 0.1253\,\frac{(j\omega-1)^3}{(j\omega+1)^4} + \text{-----------} \qquad \text{(II)}$$

This is plotted. Figures on p. 88 to p. 91 show how the imaginary part of the desired function is approximated by this series. (Tables in Appendix)

Let us find the real part of the given function.

Given: $Q(\omega) = -\sqrt{\pi}\;\omega\;e^{-\omega^2}$. Putting this in equation (3),

$$P(\omega) = \frac{\sqrt{\pi}}{\pi}\int_0^\infty \cos u\omega\;du\int_{-\infty}^{\infty} t\,e^{-t^2}\sin ut\;dt$$

Integrating by parts,

$$P(\omega) = \frac{1}{\sqrt{\pi}}\int_0^\infty \cos u\omega\;du\left[-\frac{1}{2}e^{-t^2}\sin ut\Big|_{-\infty}^{\infty} + \frac{u}{2}\times\int_{-\infty}^{\infty} e^{-t^2}\cos ut\;dt\right]^{*}$$

$$= \frac{1}{2}\int_0^\infty u\,e^{-\frac{u^2}{4}}\cos u\omega\;du$$

---

* See foot note of page 64.

Integrating by parts again,

$$P(\omega) = \frac{1}{2}\left[-2e^{-\frac{u^2}{4}} \cos u \Big|_0^{\infty} - 2\omega \int_0^{\infty} e^{-\frac{u^2}{4}} \sin u\omega \, du \right]$$

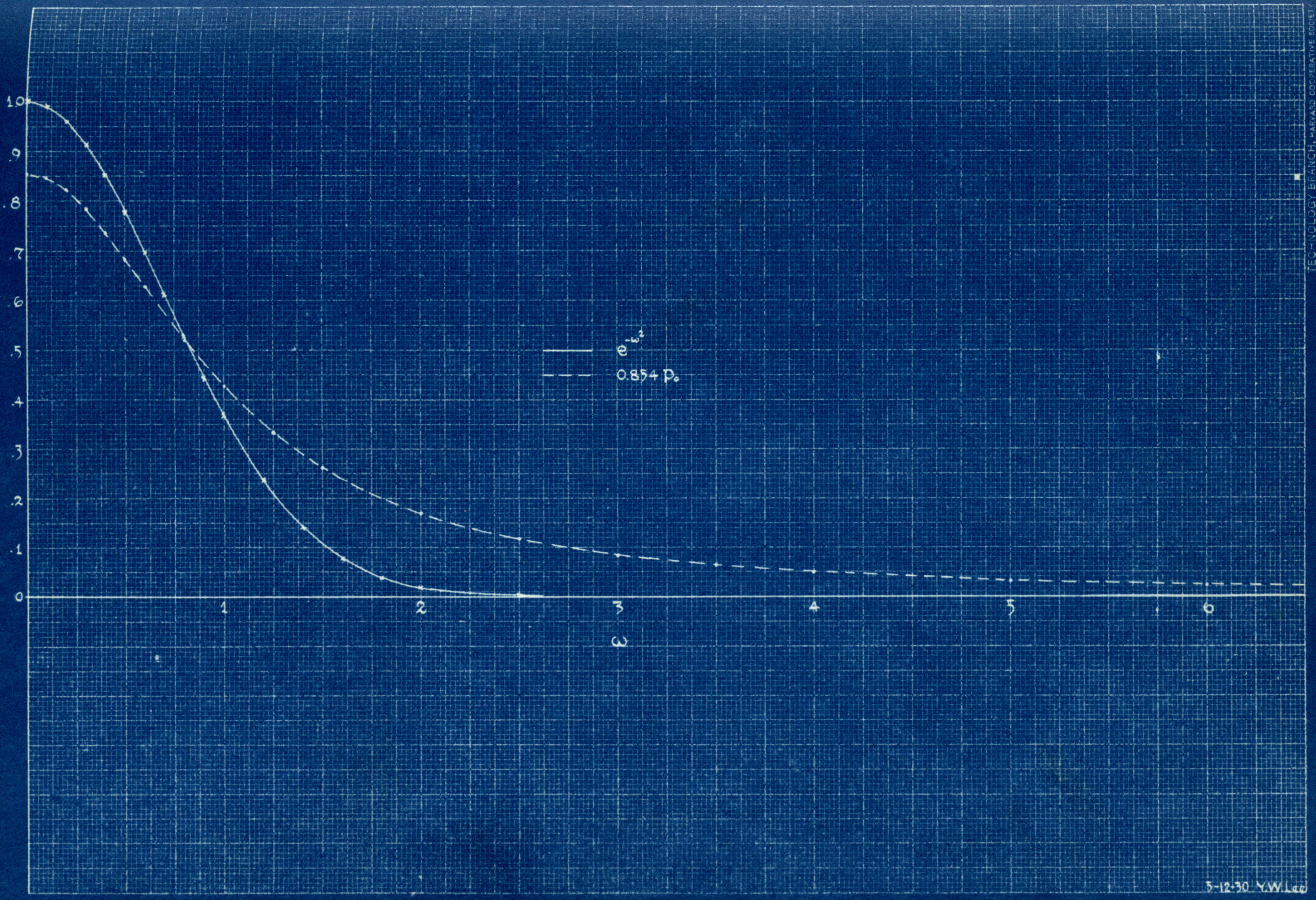
$$P(\omega) = 1 - \omega \int_0^{\infty} e^{-\frac{u^2}{4}} \sin u\omega \, du$$

This is recognized as the same sort of integral we encountered in the first problem in the evaluation of the imaginary part of the given function. Referring to this we find

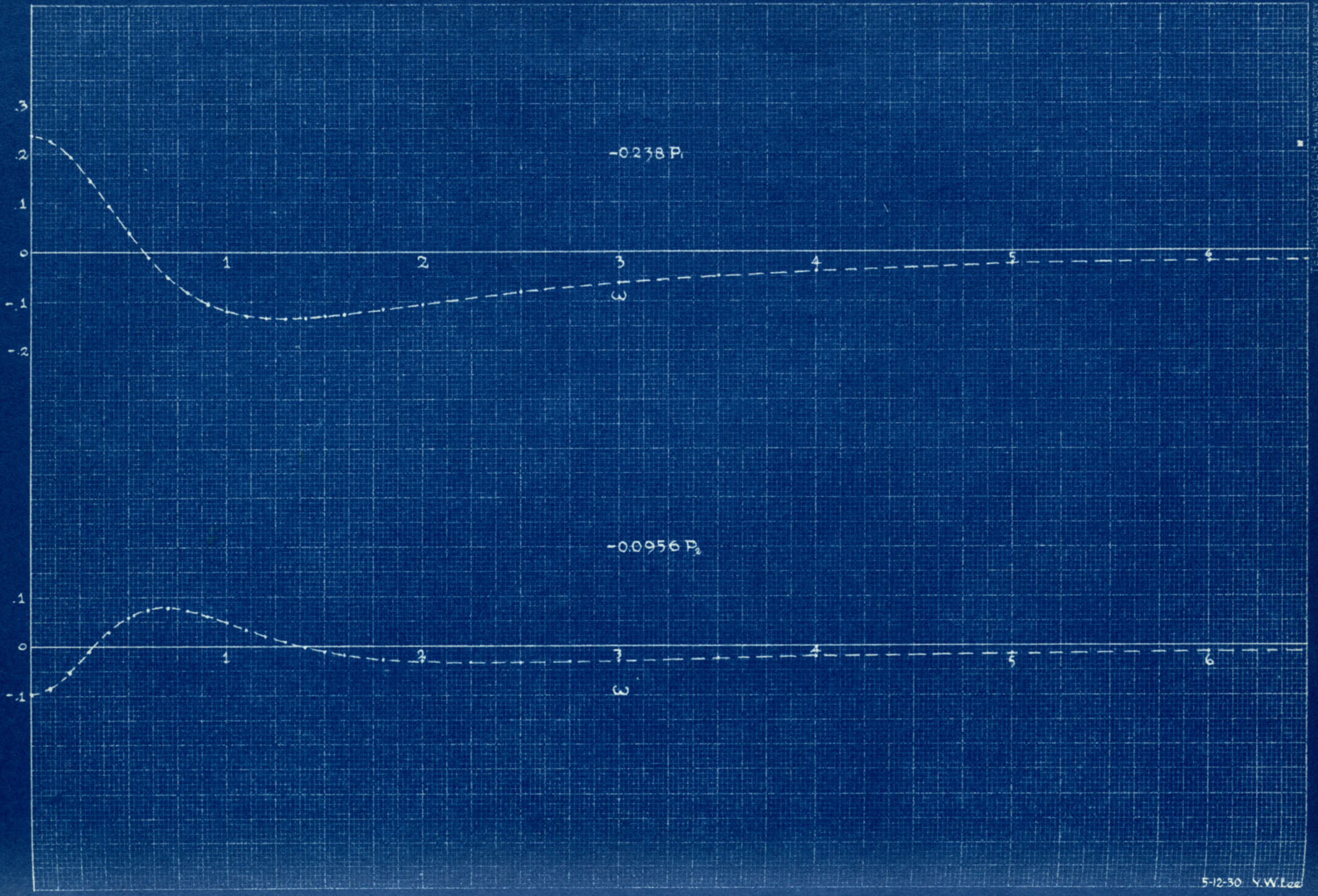
$$P(\omega) = 1 - \omega e^{-\omega^2} \int_0^{2\omega} e^{\frac{v^2}{4}} \, dv$$

The series we developed for the integral in the first problem can be used here. The above function has been plotted and Figures on p.92 to p.95 show how this is approximated by the expansion (II). (Tables in Appendix)

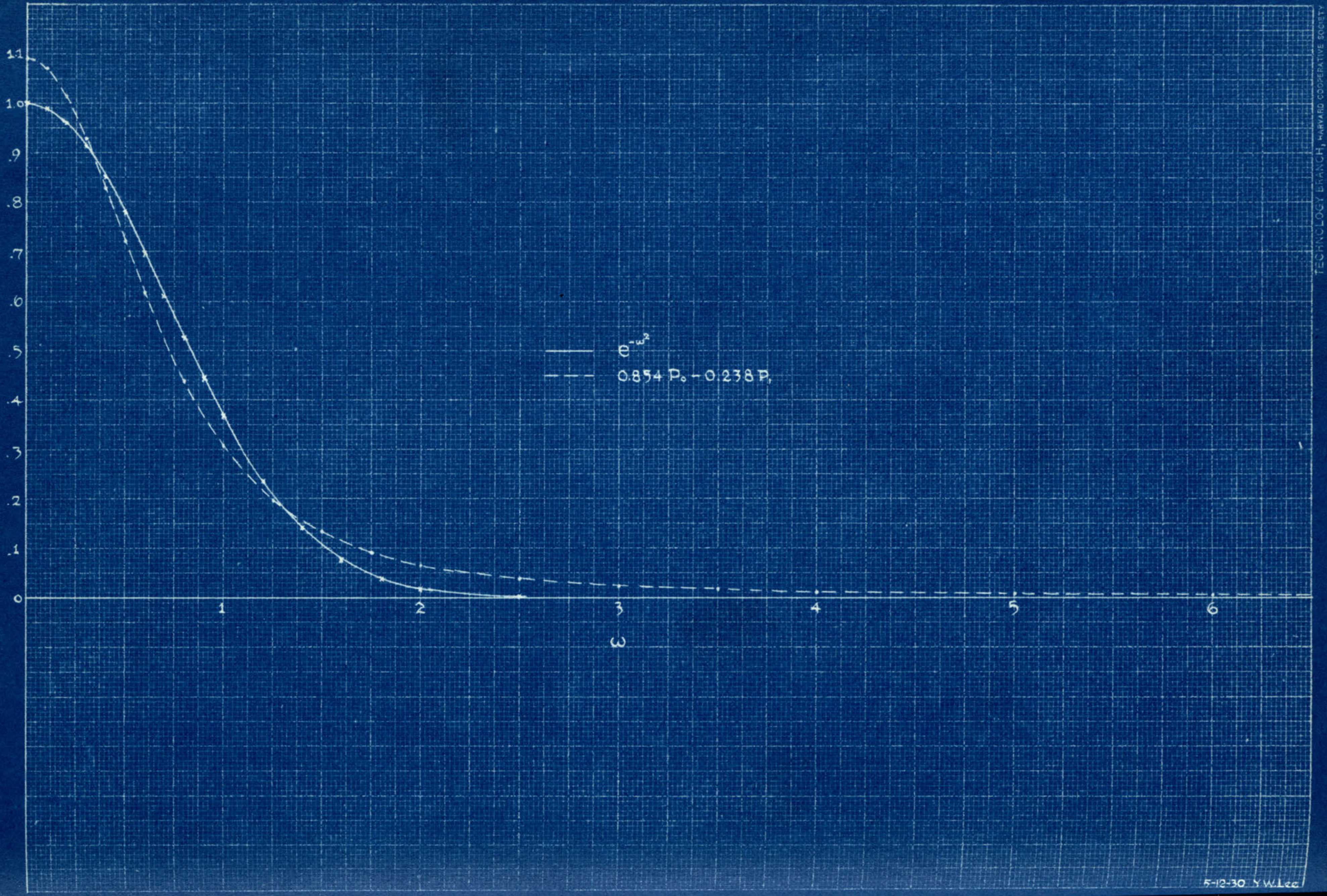
The series of this problem, like that of the first problem, is rapidly convergent. Only three terms are used for a very close approximation. The third term is excluded since it is very small.



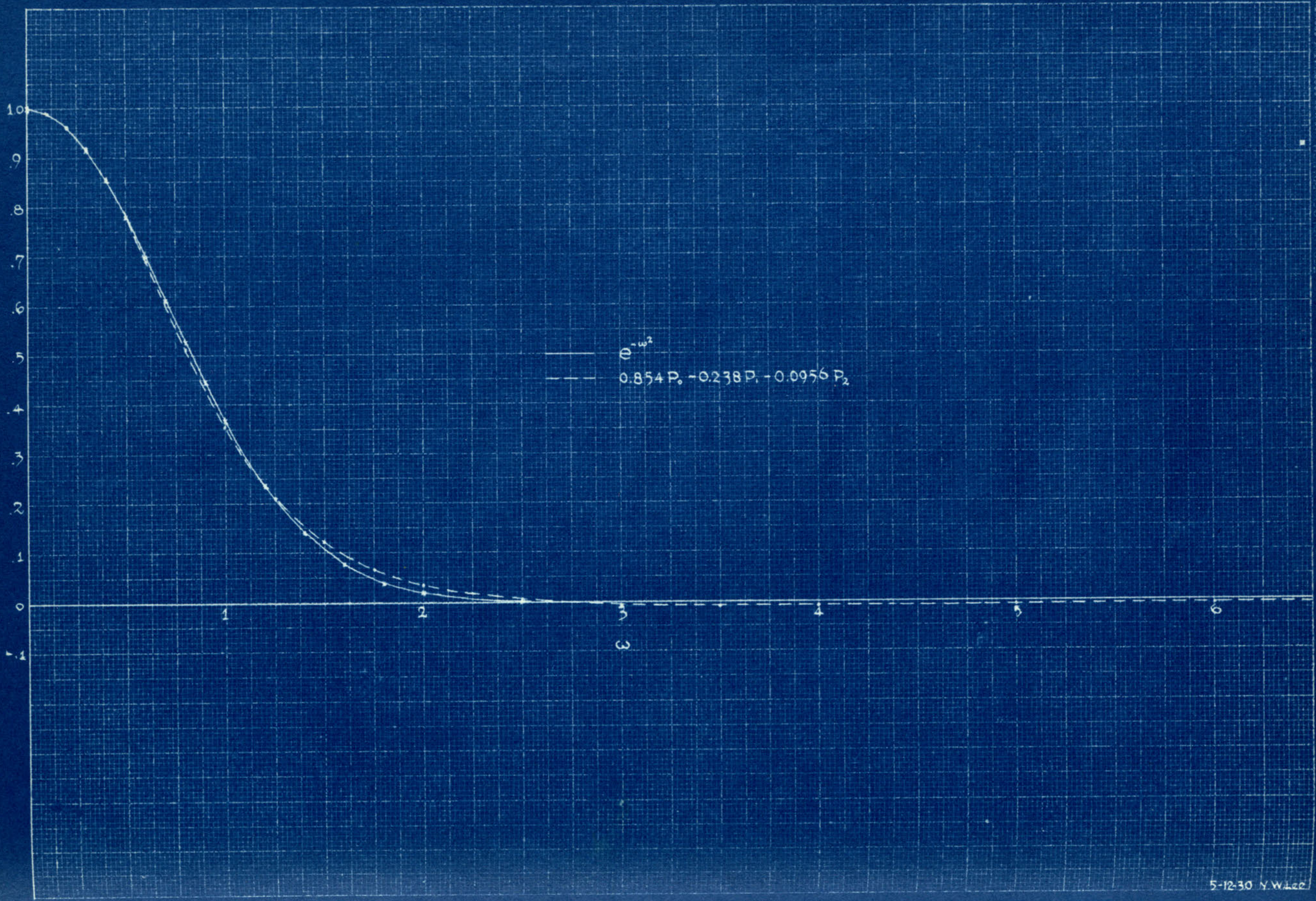
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
1
2
3
4
5
6
ω
$e^{-\omega^2}$
$0.854\,p_0$
5-12-30 Y.W. Lee
TECHNOLOGY BRANCH, HARVARD COOPERATIVE SOCIETY



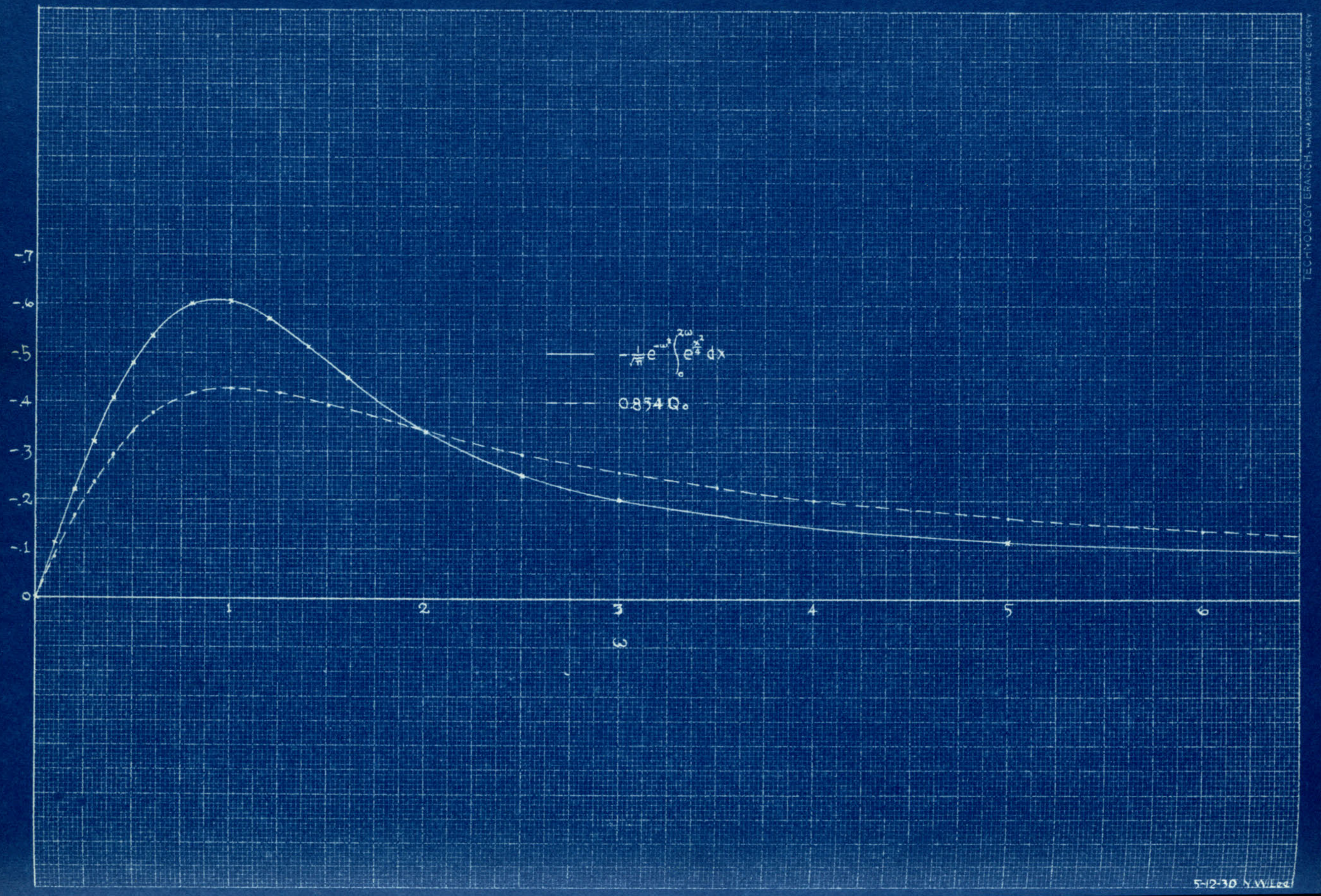
-0.238 P_1
.3
.2
.1
0
-.1
-.2
1
2
3
4
5
6
ω
-0.0956 P_2
.1
0
-.1
1
2
3
4
5
6
ω
5-12-30 Y.W.Lee
TECHNOLOGY BRANCH, HARVARD COOPERATIVE SOCIETY

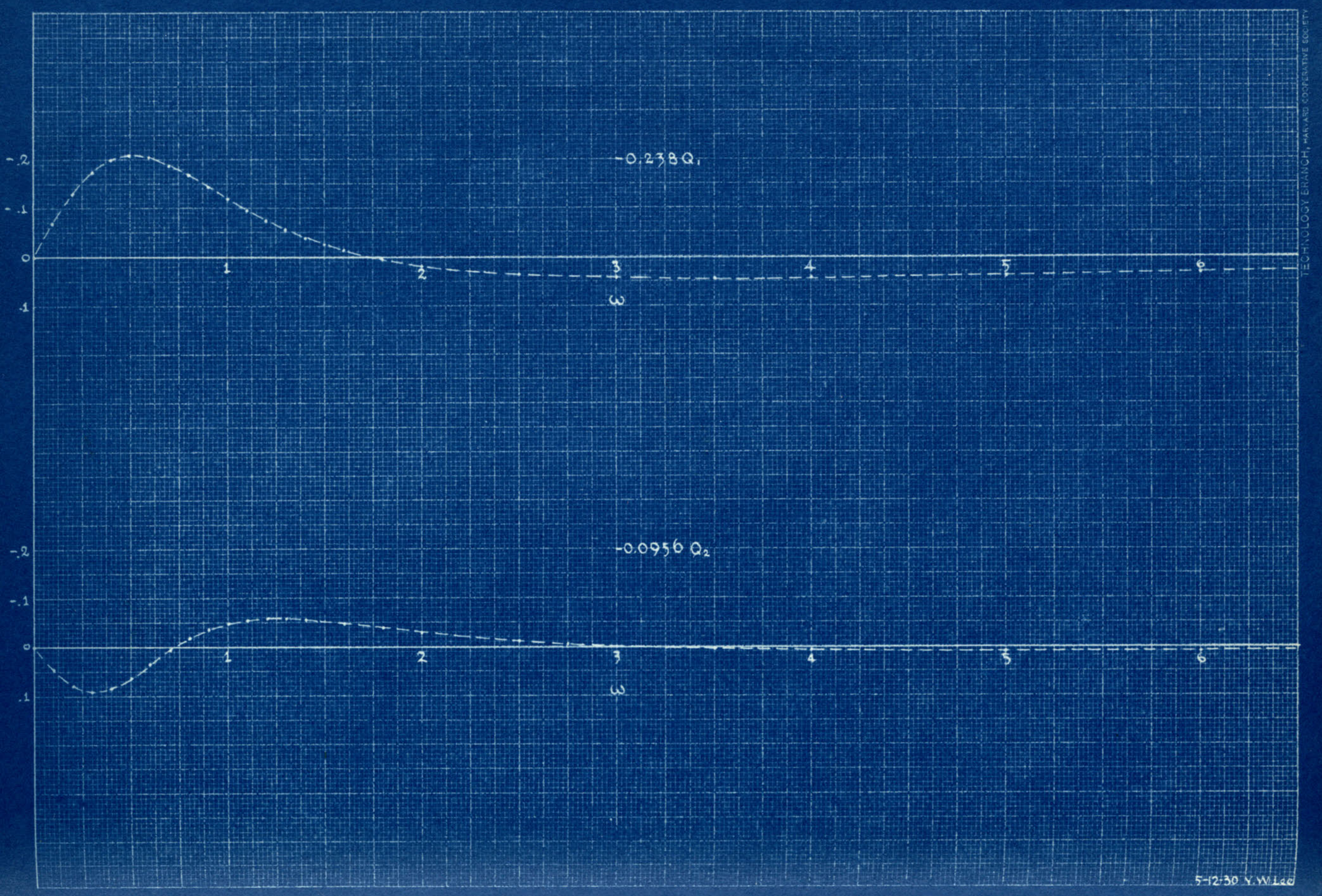


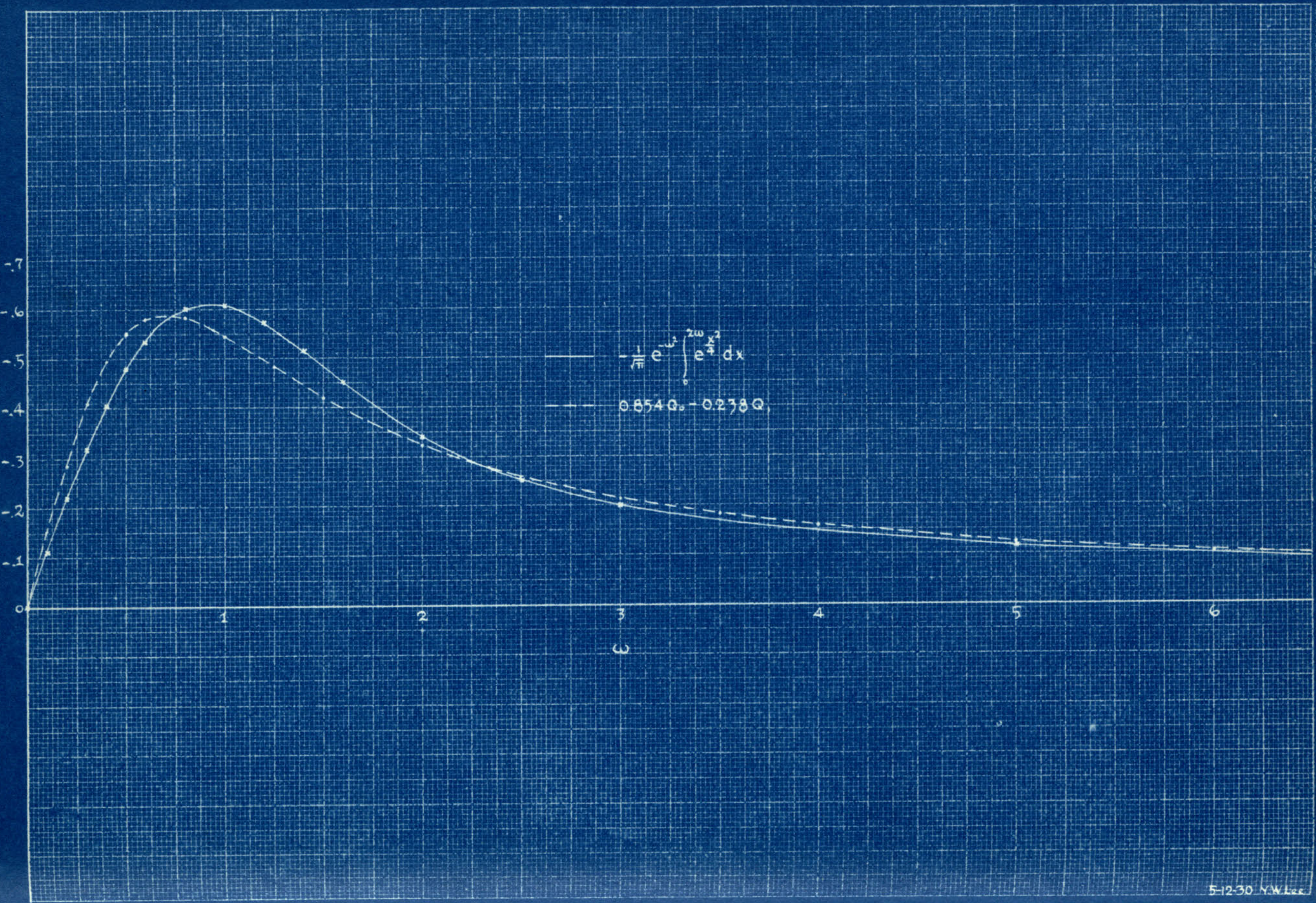
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
1
2
3
4
5
6
ω
$e^{-\omega^2}$
0.854 P₀ − 0.238 P₁
TECHNOLOGY BRANCH, HARVARD COOPERATIVE SOCIETY
5-12-30 Y.W.Lee



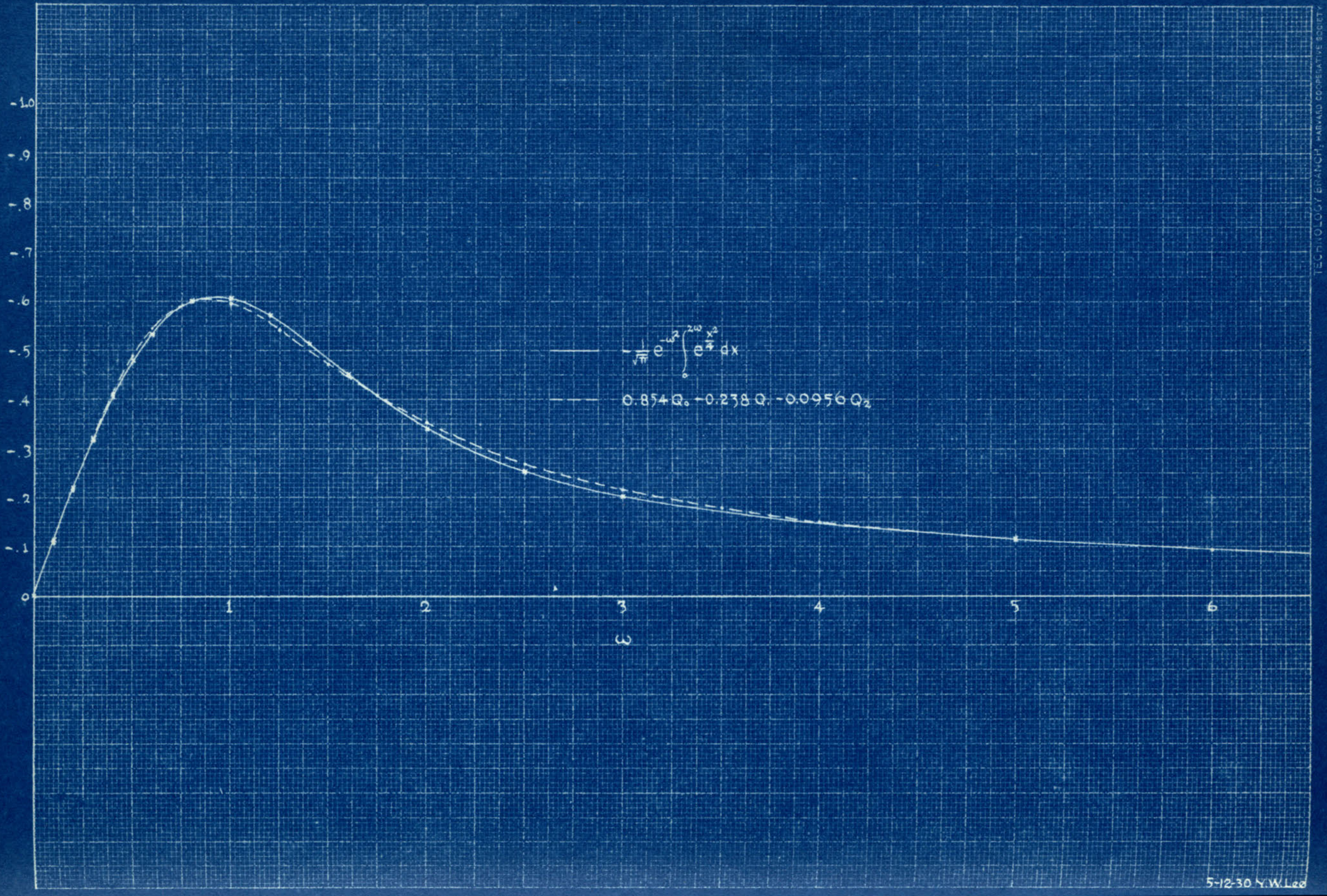
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
-.1
1
2
3
4
5
6
ω
$e^{-\omega^2}$
$0.854 P_0 - 0.238 P_1 - 0.0956 P_2$
5-12-30 Y.W.Lee

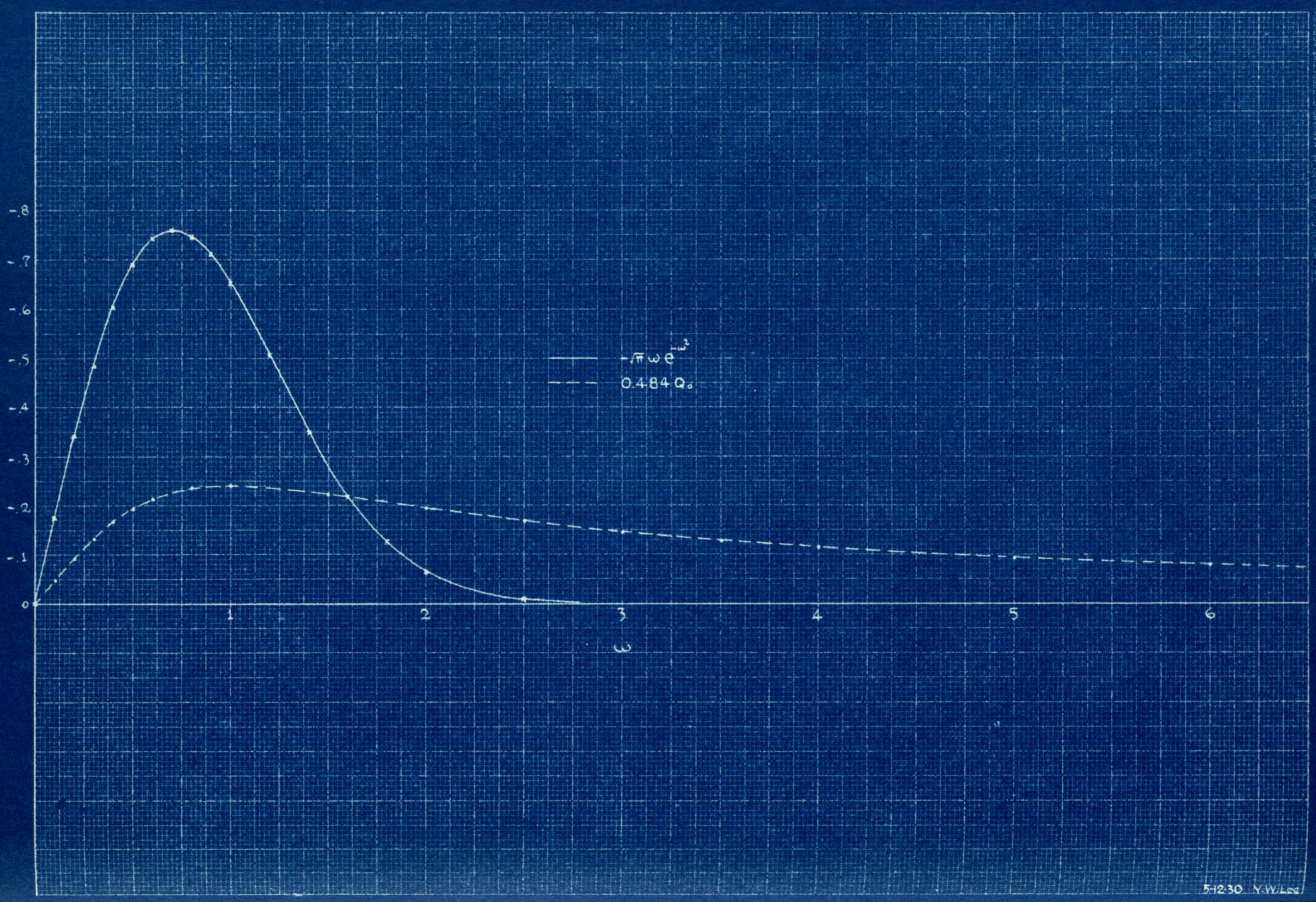

-.7
-.6
-.5
-.4
-.3
-.2
-.1
0
1
2
3
4
5
6
ω
$-\frac{1}{\sqrt{\pi}}e^{-\omega^2}\int_0^{2\omega}e^{\frac{x^2}{4}}dx$
0.854 Q_0
5-12-30 Y. W. Lee
TECHNOLOGY BRANCH, HARVARD COOPERATIVE SOCIETY

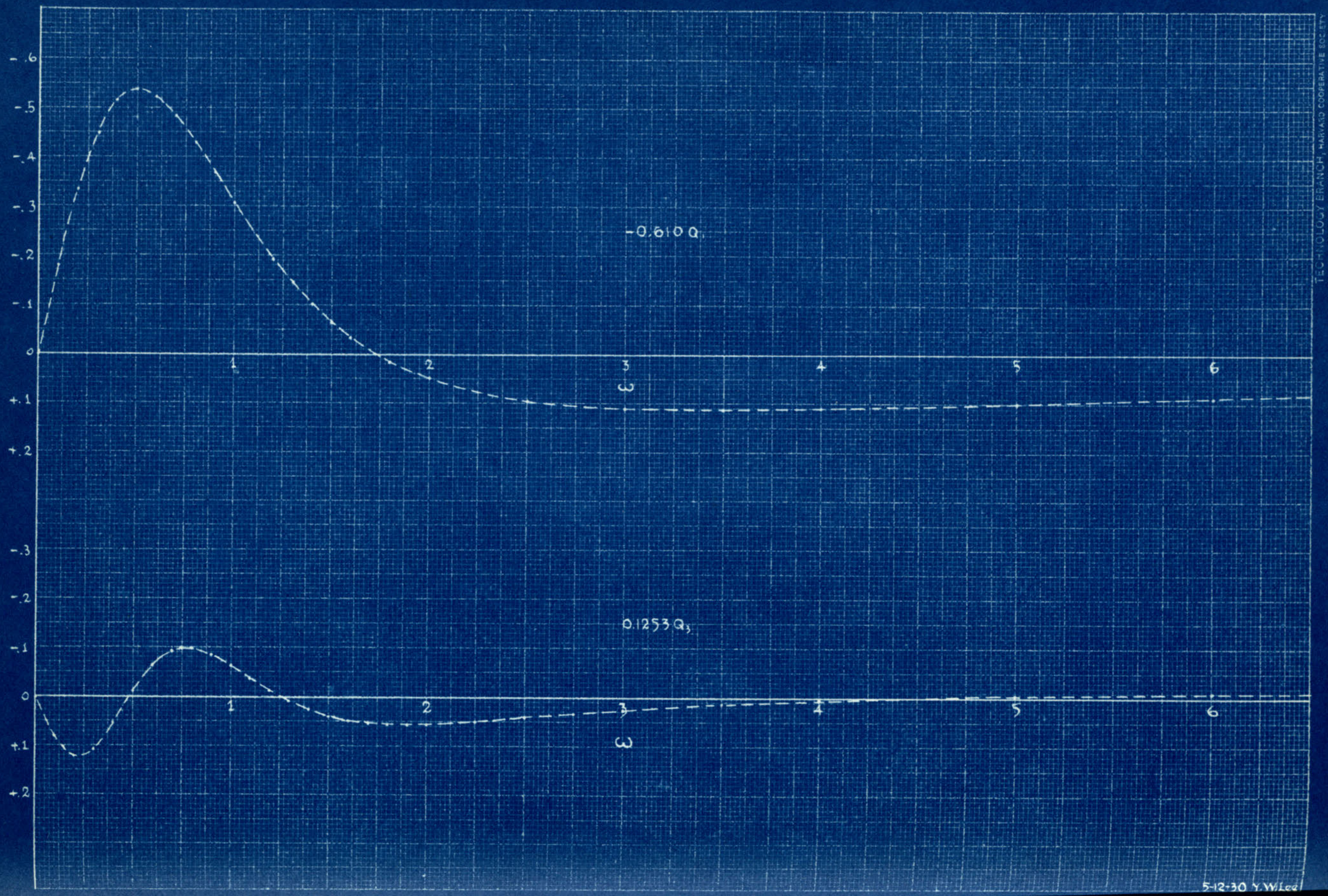

-0.238 Q₁
ω
-0.0956 Q₂
ω
.2
.1
0
1
2
3
4
5
6
5-12-30 Y. W. Lee
TECHNOLOGY BRANCH, HARVARD COOPERATIVE SOCIETY



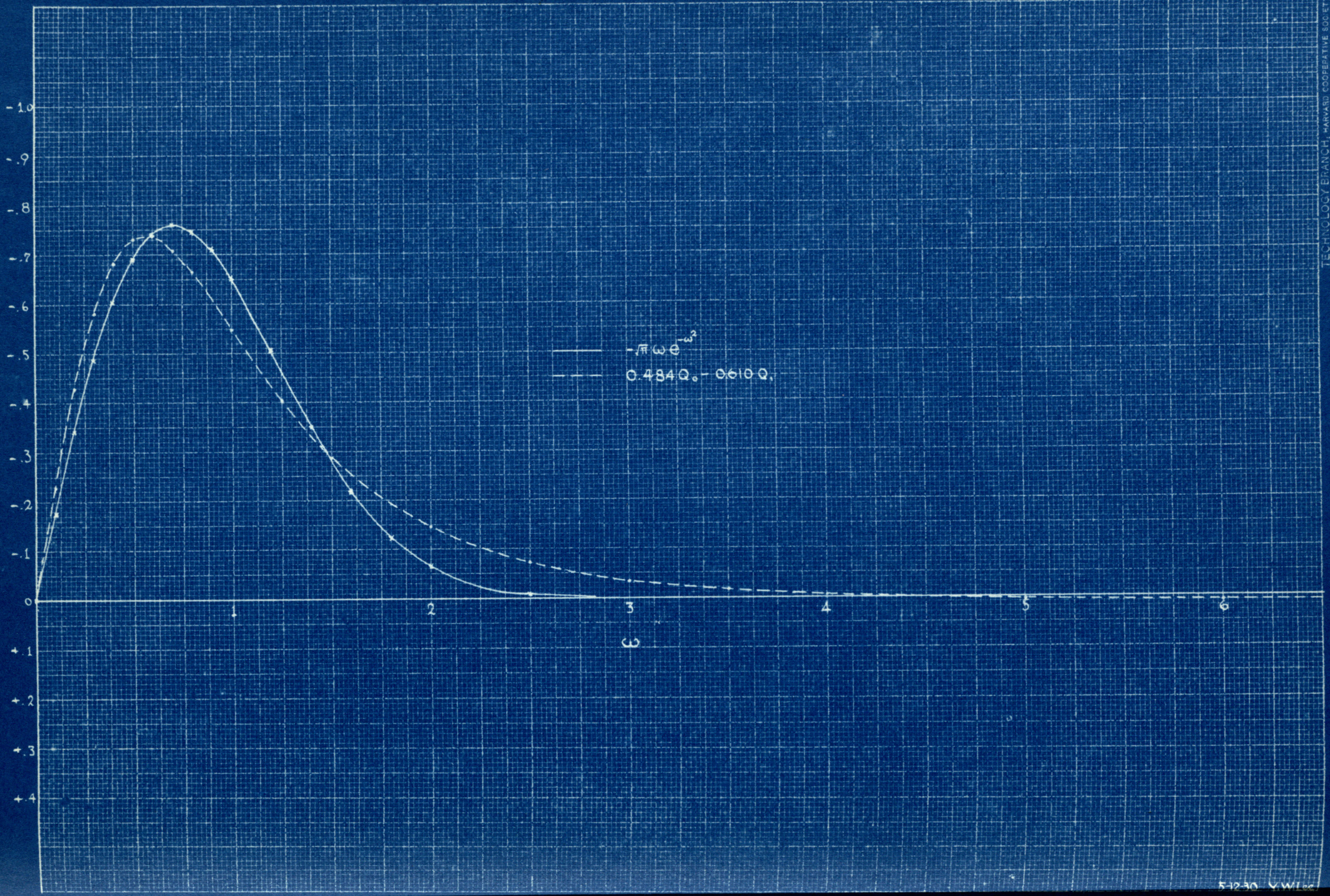
-.7
-.6
-.5
-.4
-.3
-.2
-.1
0
1
2
3
4
5
6
ω
$-\frac{1}{\sqrt{\pi}} e^{-\omega^2} \int_0^{2\omega} e^{\frac{x^2}{4}} dx$
$0.854 Q_0 - 0.238 Q_1$
5-12-30 Y.W. Lee

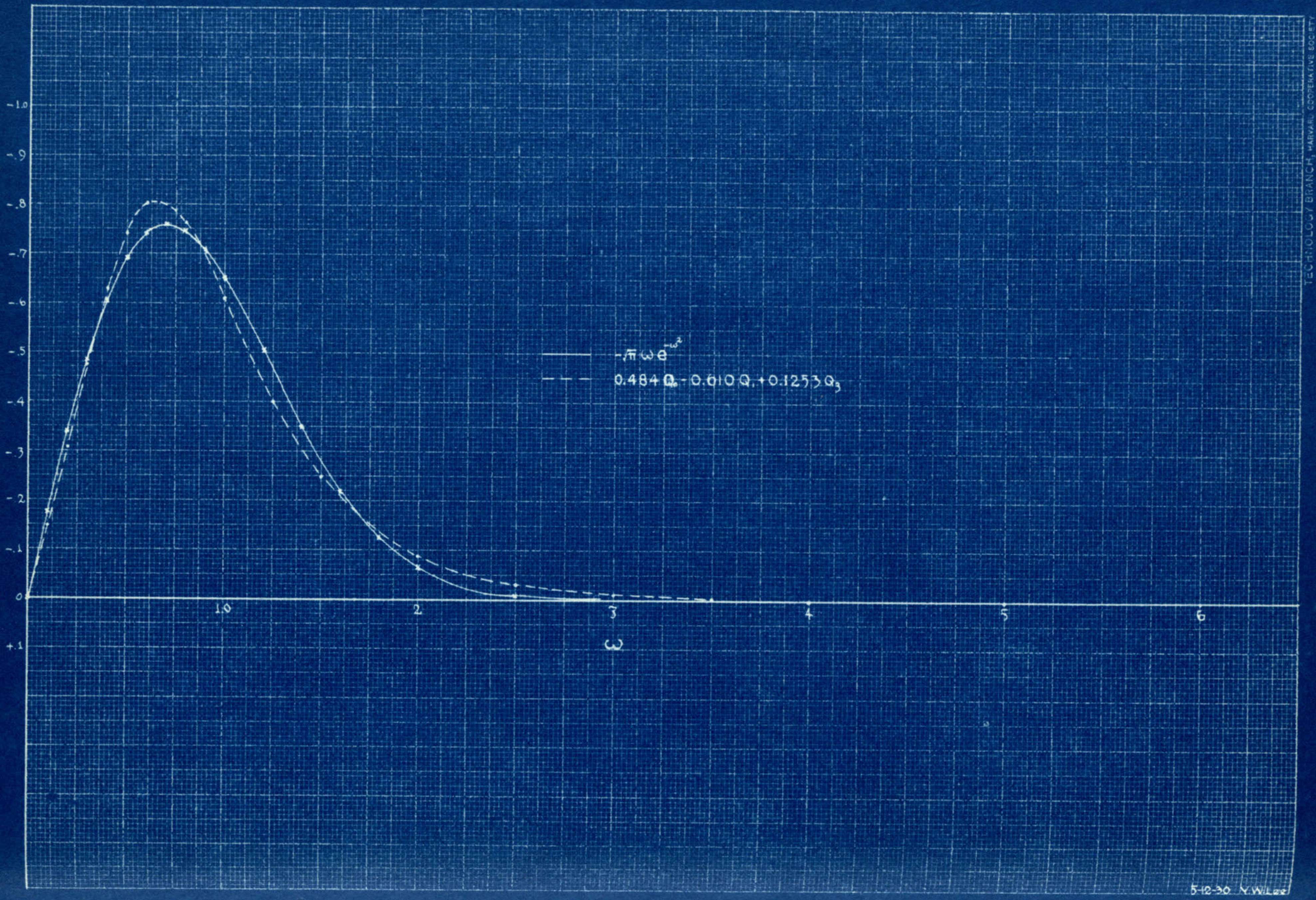

-1.0
-.9
-.8
-.7
-.6
-.5
-.4
-.3
-.2
-.1
0
1
2
3
4
5
6
ω
$-\frac{1}{\sqrt{\pi}} e^{-\omega^2} \int_0^{2\omega} e^{\frac{x^2}{4}} dx$
$0.854 Q_0 - 0.238 Q_1 - 0.0956 Q_2$
TECHNOLOGY BRANCH, HARVARD COOPERATIVE SOCIETY
5-12-30 Y.W. Lee

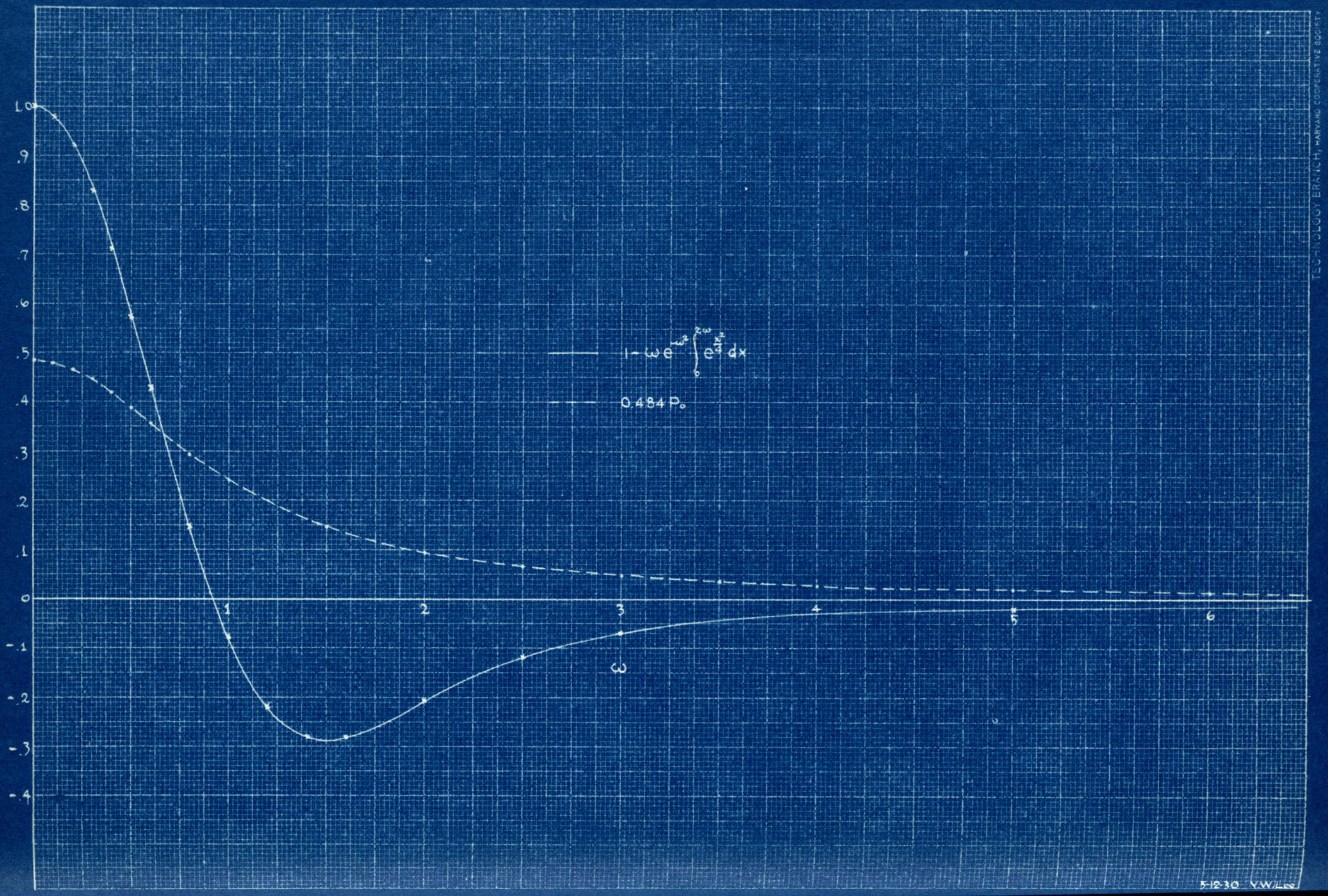

-.8
-.7
-.6
-.5
-.4
-.3
-.2
-.1
0
1
2
3
4
5
6
ω
$-\sqrt{\pi}\,\omega\, e^{-\omega^2}$
$0.484\, Q_0$
5-12-30 Y.W. Lee



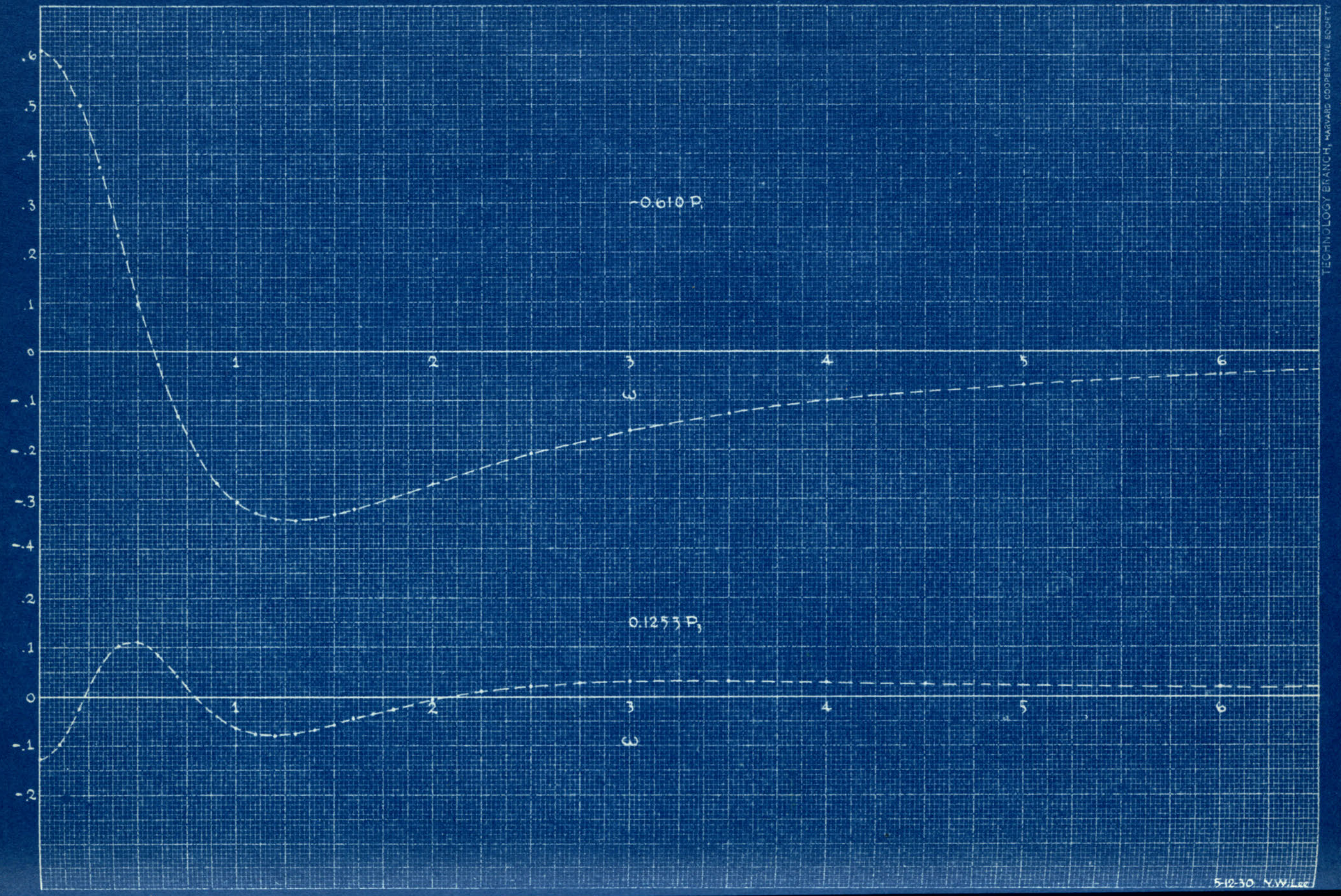
-0.610 Q1
0.1253 Q3
ω
ω
5-12-30 Y. W. Lee
TECHNOLOGY BRANCH, HARVARD COOPERATIVE SOCIETY

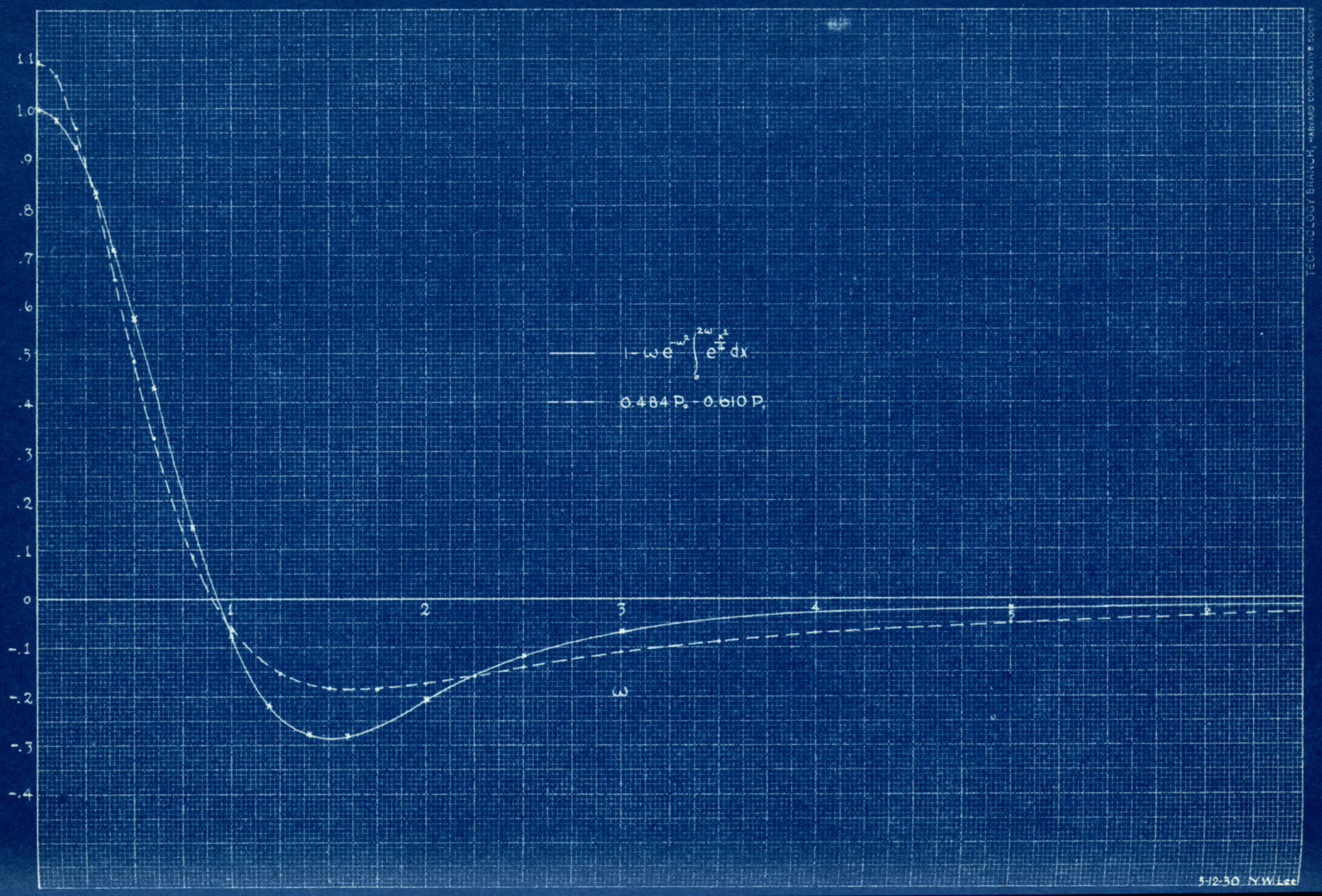

-1.0
-.9
-.8
-.7
-.6
-.5
-.4
-.3
-.2
-.1
0
+.1
+.2
+.3
+.4
1
2
3
4
5
6
ω
$-\sqrt{\pi}\,\omega\, e^{-\omega^2}$
$0.484 Q_0 - 0.610 Q_1$
5-12-30 Y.W.Lee
TECHNOLOGY BRANCH, HARVARD COOPERATIVE SOCIETY

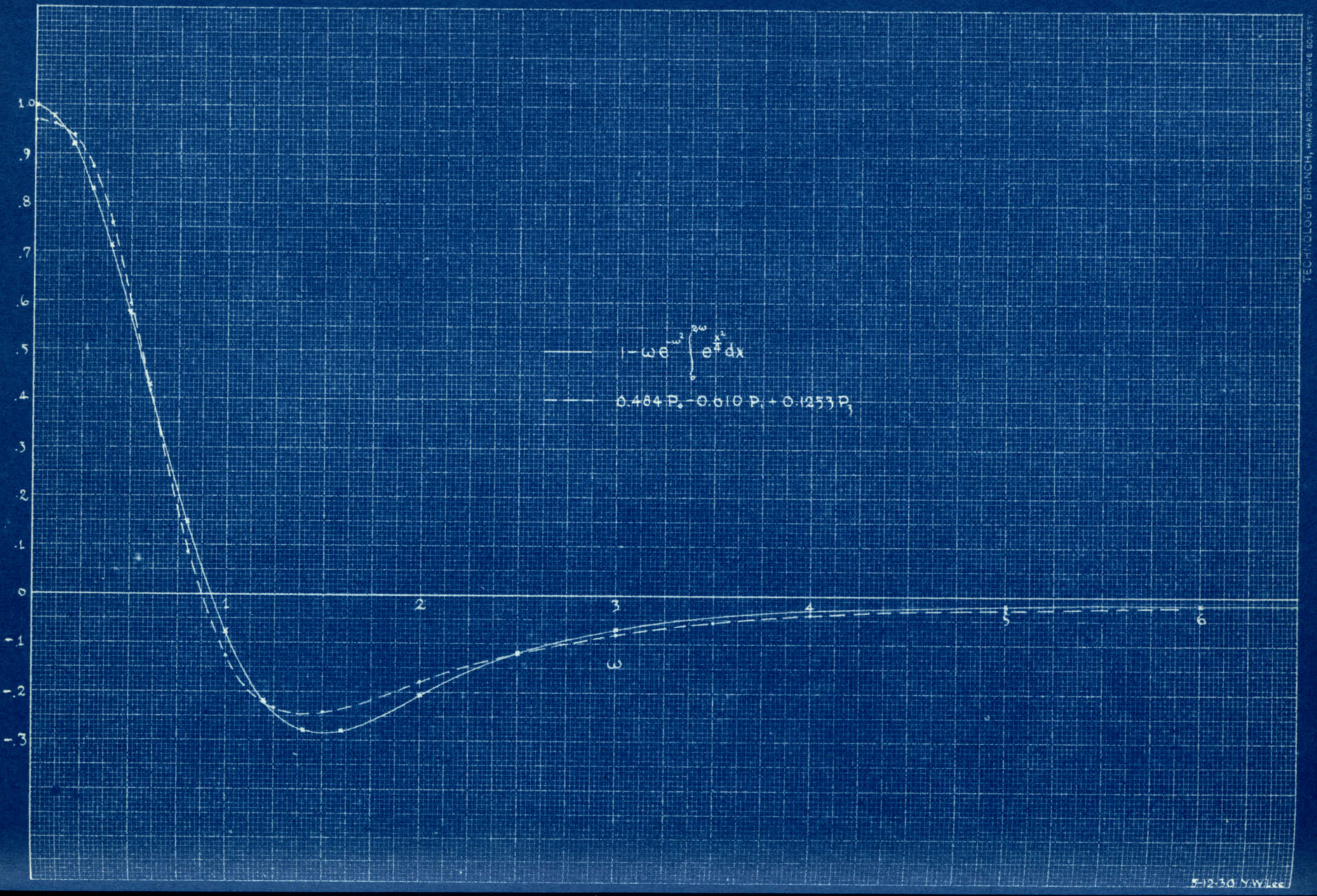

-1.0
-.9
-.8
-.7
-.6
-.5
-.4
-.3
-.2
-.1
0
+.1
1.0
2
3
4
5
6
ω
$-\sqrt{\pi}\,\omega e^{-\omega^2}$
$0.484\,Q_0 - 0.610\,Q_1 + 0.1253\,Q_3$
TECHNOLOGY BRANCH, HARVARD COOPERATIVE SOCIETY
5-12-30 Y.W. Lee



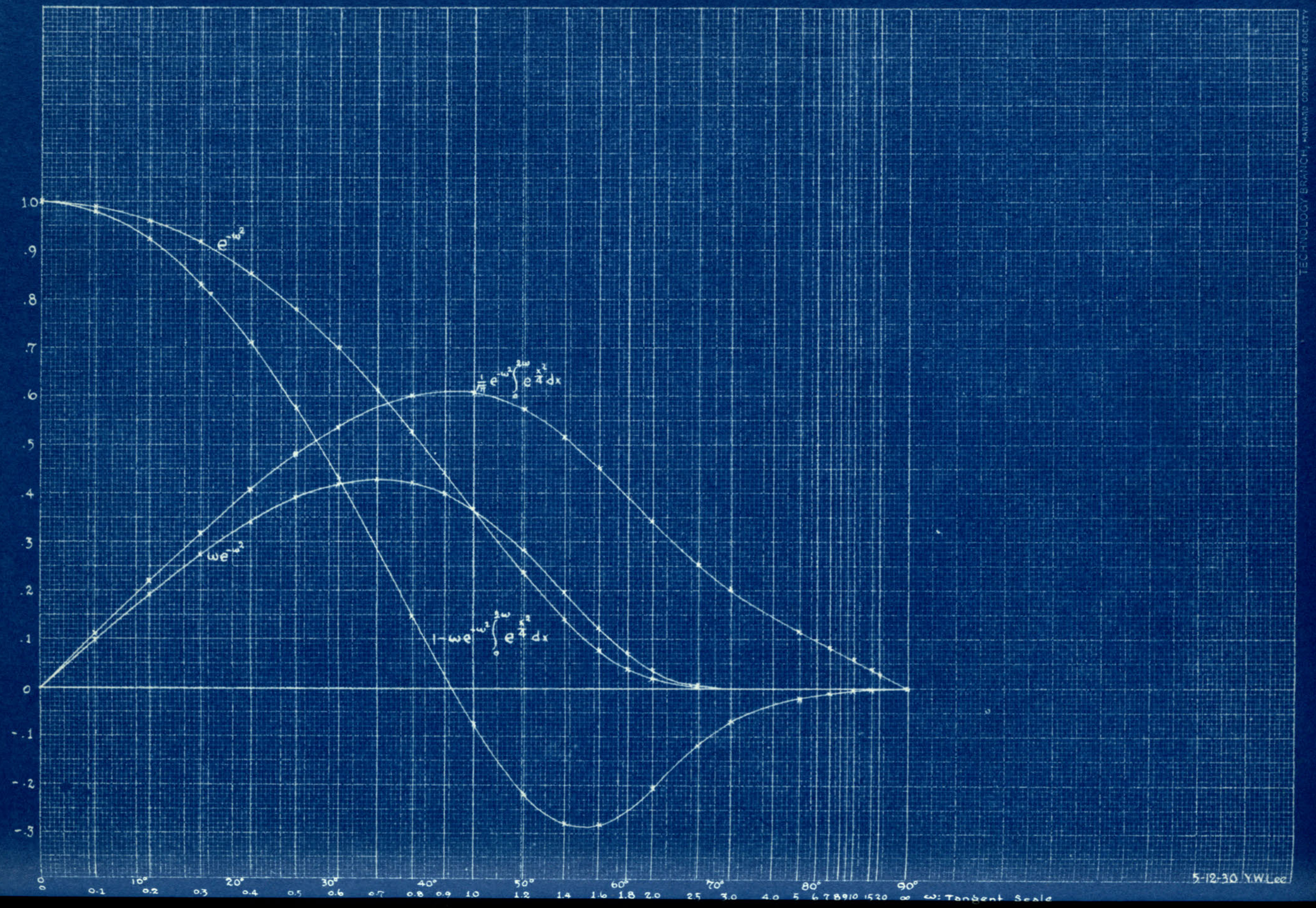
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
-.1
-.2
-.3
-.4
1
2
3
4
5
6
ω
$1-\omega e^{-\omega^2}\int_0^{2\omega} e^{\frac{x^2}{4}}dx$
0.484 P_0
5-12-30 Y.W. Lee
TECHNOLOGY BRANCH, HARVARD COOPERATIVE SOCIETY


-0.610 P₁
0.1253 P₃
ω
5-12-30. N.W. Lee
TECHNOLOGY BRANCH, HARVARD COOPERATIVE SOCIETY


1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
-.1
-.2
-.3
-.4
1
2
3
4
5
6
ω
1 - ω e^{-ω²} ∫_0^{2ω} e^{x²/4} dx
0.484 P_0 - 0.610 P_1
5-12-30 Y.W. Lee
TECHNOLOGY BRANCH, HARVARD COOPERATIVE SOCIETY



1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
-.1
-.2
-.3
1
2
3
4
5
6
ω
$1-\omega e^{-\omega^2}\int_0^{2\omega} e^{\frac{x^2}{4}}dx$
$0.484P_0 - 0.610P_1 + 0.1253P_3$
5-12-30 Y.W.Lee
TECHNOLOGY BRANCH, HARVARD COOPERATIVE SOCIETY



$e^{-\omega^2}$
$\frac{1}{\sqrt{\pi}} e^{-\omega^2} \int_0^{2\omega} e^{\frac{x^2}{4}} dx$
$\omega e^{-\omega^2}$
$1 - \omega e^{-\omega^2} \int_0^{2\omega} e^{\frac{x^2}{4}} dx$
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
-.1
-.2
-.3
0 10° 20° 30° 40° 50° 60° 70° 80° 90°
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.2 1.4 1.6 1.8 2.0 2.5 3.0 4.0 5 6 7 8 9 10 15 20 ∞
ω: Tangent Scale
5-12-30 Y.W.Lee
TECHNOLOGY BRANCH, HARVARD COOPERATIVE SOCIETY

APPENDIX

a. Graphs of the first nine of the Fourier transforms of Laguerre's functions with tables.

The tables are calculated from the formulas:

$$P_n(\omega) = \frac{(-1)^n}{(1+\omega^2)^{\frac{1}{2}}} \cos \left[(2n+1)\tan^{-1}\omega\right]$$

$$Q_n(\omega) = \frac{(-1)^n}{(1+\omega^2)^{\frac{1}{2}}} \sin \left[(2n+1)\tan^{-1}\omega\right]$$

Note that here the factor $\sqrt{\pi}$ is deleted from the definitions given by equations (39) for the sake of convenience. The values from Table 1 and the graphs should be divided by $\sqrt{\pi}$ according to the definitions. The constant factor k is taken as unity (see 33).

b. Tables for the problems of chapter VI P.131

The values of P and Q are taken from Table 1.

a.

TABLE 1

ω	P_o	Q_o
0	+1	0
0.20012	+0.9615	-0.1925
0.39997	+0.862	-0.345
0.60007	+0.735	-0.442
0.80020	+0.610	-0.488
1.00000	+0.500	-0.500
1.9999	+0.200	-0.400
3.0003	+0.1000	-0.300
4.0009	+0.0588	-0.2352
5.0045	+0.0384	-0.192
6.0080	+0.0270	-0.1619
6.9972	+0.0200	-0.1400
8.0095	+0.0154	-0.1229
9.0098	+0.0122	-0.1096
10.019	+0.0099	-0.0988
1.2497	+0.3906	-0.4875
1.5004	+0.3076	-0.4615
10.988	+0.0082	-0.0902
11.992	+0.0069	-0.0829
12.996	+0.0059	-0.0765
14.008	+0.0051	-0.0710
14.990	+0.0044	-0.0664
15.969	+0.0039	-0.0624
16.999	+0.0034	-0.0585
17.980	+0.0031	-0.0554
18.976	+0.0028	-0.0526
19.970	+0.0025	-0.0499
0.10011	+0.990	-0.0991
0.30001	+0.918	-0.2754
0.50004	+0.800	-0.400
2.5002	+0.138	-0.345
3.4989	+0.0756	-0.2644

ω	P_1	Q_1
0	-1	0
0.20012	-0.8135	+0.548
0.39997	-0.386	+0.844
0.60007	+0.0434	+0.857
0.80020	+0.3422	+0.702
1.0000	+0.500	+0.500
1.9999	+0.440	-0.0799
3.0003	+0.260	-0.180
4.0009	+0.1625	-0.180
5.0045	+0.1094	-0.1626
6.0080	+0.0780	-0.1445
6.9972	+0.0585	-0.1288
8.0095	+0.0451	-0.1153
9.0098	+0.0359	-0.1043
10.019	+0.0292	-0.0950
0.10011	-0.951	+0.2936
0.30001	-0.615	+0.735
0.50004	-0.160	+0.880
0.70021	+0.2122	+0.791
0.90040	+0.4375	+0.602
1.2002	+0.557	+0.314
1.4002	+0.5565	+0.166
1.6003	+0.527	+0.0554
1.8003	+0.485	-0.0241
10.988	+0.0244	-0.0872
11.992	+0.0205	-0.0805
12.996	+0.0175	-0.0747
14.008	+0.0151	-0.0696
14.990	+0.0132	-0.0653
15.969	+0.0117	-0.0614
16.999	+0.0103	-0.0578
17.980	+0.0092	-0.0547
18.976	+0.0083	-0.0520
19.970	+0.0075	-0.0494
2.5002	+0.338	-0.1545
3.4989	+0.2036	-0.1844
1.1003	+0.5385	+0.4025
1.3001	+0.5625	+0.2354
0.05007	-0.988	+0.1494
1.5004	+0.544	+0.1063
0.45012	-0.2712	+0.871
0.55013	-0.0543	+0.874
0.64982	+0.1320	+0.828

ω	P_2	Q_2
0	+1	0
0.10011	+0.8735	-0.476
0.20012	+0.540	-0.819
0.30001	+0.1085	-0.9525
0.39997	-0.3022	-0.878
0.50004	-0.608	-0.656
0.60007	-0.776	-0.3646
0.70021	-0.816	-0.0715
0.80020	-0.760	+0.1801
0.90040	-0.644	+0.372
1.0000	-0.500	+0.500
1.9999	+0.328	+0.304
3.0003	+0.316	+0.0120
4.0009	+0.228	-0.0823
5.0045	+0.1634	-0.1081
6.0080	+0.1206	-0.1115
6.9972	+0.0922	-0.1073
8.0095	+0.0721	-0.1007
9.0098	+0.0579	-0.0938
10.019	+0.0474	-0.0873
0.15005	+0.727	-0.671
0.24995	+0.3292	-0.9125
0.34987	-0.1055	-0.938
0.45012	-0.472	-0.781
0.55013	-0.709	-0.514
0.64982	-0.810	-0.216
0.74991	-0.7975	+0.0605
0.85006	-0.707	+0.2836
0.95007	-0.574	+0.4425
1.2002	-0.2084	+0.605
1.4002	+0.0237	+0.580
1.6003	+0.1813	+0.498
1.8003	+0.277	+0.3994
10.988	+0.0397	-0.0814
11.992	+0.0336	-0.0760
12.996	+0.0287	-0.0711
14.008	+0.0248	-0.0667
14.990	+0.0218	-0.0629
15.969	+0.0192	-0.0595
16.999	+0.0170	-0.0562
17.980	+0.0152	-0.0534
18.976	+0.0137	-0.0508

ω	P_2	Q_2
0	+1	0
19.970	+0.0124	-0.0484
2.5002	+0.350	+0.1206
3.4989	+0.2705	-0.0489
0.05007	-0.968	-0.2474
1.1003	-0.350	+0.575
1.3001	-0.0831	+0.604
2.2496	+0.352	+0.202
2.7500	+0.3365	+0.0589
0.27513	+0.2184	-0.939
0.32492	-0.0000	-0.951
0.37488	-0.206	-0.9125
1.0501	-0.425	+0.5445
1.1504	-0.2774	+0.594
1.5004	+0.1115	+0.543

ω	P_3	Q_3
0.000	-1.0	0
0.10011	-0.762	+0.640
0.20012	-0.1835	+0.963
0.30001	+0.434	+0.855
0.39997	+0.825	-0.427
0.50084	+0.889	-0.0930
0.60007	+0.687	-0.514
0.70021	+0.3462	-0.742
0.80020	-0.0091	-0.781
0.90040	-0.3028	-0.680
1.0000	-0.500	-0.500
1.1003	-0.605	-0.2932
1.2002	-0.633	-0.0956
1.3001	-0.605	+0.0747
1.4002	-0.5415	+0.2106
1.5004	-0.4585	+0.312
1.6003	-0.368	+0.3812
1.6999	-0.278	+0.4235
1.8003	-0.1924	+0.446
1.9007	-0.1146	+0.451
1.9999	-0.0465	+0.445
3.0003	+0.2458	+0.1993
4.0009	+0.240	+0.0347
5.0045	+0.1924	-0.0371
6.0080	+0.1502	-0.0664
6.9972	+0.1185	-0.0772
8.0095	+0.0946	-0.0799
9.0098	+0.0771	-0.0789
10.019	+0.0637	-0.0762
0.15005	-0.499	+0.855
0.24995	+0.1390	+0.960
0.34987	+0.6675	+0.6675
0.45012	+0.8975	+0.1642
0.55013	+0.814	-0.324
0.64982	+0.527	-0.6525
0.74991	+0.1655	-0.7825
0.85006	-0.166	-0.744
0.95007	-0.413	-0.596
2.2496	+0.0860	+0.3965
2.5002	+0.171	+0.330

ω	P_3	Q_3
0.000	-1.0	0
2.7500	+0.220	+0.2614
3.4989	+0.2556	+0.1014
10.988	+0.0538	-0.0729
11.992	+0.0457	-0.0694
12.996	+0.0393	-0.0659
14.008	+0.0341	-0.625
14.990	+0.030	-0.0594
15.969	+0.0265	-0.0566
16.999	+0.0235	-0.0537
17.980	+0.0211	-0.0514
18.976	+0.0190	-0.0491
19.970	+0.0172	-0.0470
0.05007	-0.9385	+0.3428
0.47483	+0.9025	+0.0347
1.1504	-0.627	-0.1918
1.2497	-0.6248	-0.0073
3.2506	+0.2556	+0.1457
3.7495	+0.2494	+0.0647
4.4983	+0.2168	-0.0086
5.5026	+0.170	-0.0550
0.22505	-0.0207	+0.9756
0.17513	-0.3444	+0.923
0.27513	+0.2928	+0.919
0.72521	+0.255	-0.768
0.77521	+0.076	-0.786
0.82483	-0.0889	-0.766

ω	P_4	Q_4
0.0000	+1	0
0.10011	+0.620	-0.7785
0.15005	+0.226	-0.963
0.20012	-0.2014	-0.960
0.24995	-0.5745	-0.7815
0.30001	-0.832	-0.475
0.34987	-0.938	-0.1061
0.39997	-0.8915	+0.2592
0.45012	-0.7175	+0.563
0.50004	-0.459	+0.767
0.55013	-0.162	+0.861
0.60007	+0.1299	+0.848
0.64982	+0.3824	+0.746
0,70021	+0.579	+0.579
0.74991	+0.705	+0.378
0.80020	+0.7635	+0.1625
0.85006	+0.760	-0.0439
0.90040	+0.706	-0.2298
0.95007	+0.616	-0.382
1.0000	+0.500	-0.500
1.1003	+0.2346	-0.630
1.2002	-0.0201	-0.640
1.3001	-0.2276	-0.566
1.4002	-0.375	-0.444
1.5004	-0.464	-0.3035
1.6003	-0.504	-0.1640
1.6999	-0.505	-0.0371
1.8003	-0.480	+0.0722
1.9007	-0.437	+0.1611
1.9999	-0.3834	+0.2295
2.2496	-0.2368	+0.3298
2.5002	-0.1037	+0.3566
2.7500	+0.0009	+0.3418
3.0003	+0.0770	+0.3066
3.4989	+0.1633	+0.221
4.0009	+0/1955	+0.1436
5.0045	+0.1919	+0.0398
6.0080	+0.1635	-0.0142
6.9972	+0.1354	-0.0409
8.0095	+0.1113	-0.0542
9.0098	+0.0926	-0.0601
10.019	+0.0773	-0.0621
10.988	+0.0661	-0.0620

ω	P_4	Q_4
0.0000	+1	0
11.992	+0.0566	-0.0609
12.996	+0.0489	-0.0591
14.008	+0.0426	-0.0570
14.990	+0.9376	-0.0550
15.969	+0.0334	-0.0529
16.999	+0.0296	-0.0507
17.980	+0.0266	-0.0487
18.976	+0.0240	-0.0468
19.970	+0.0218	-0.0450
0.02502	+0.975	-0.223
0.05007	+0.900	-0.435
0.07490	+0.780	-0.621
0.12485	+0.434	-0.892
0.17513	+0.0103	-0.985
0.22505	-0.399	-0.890
0.27513	-0.7215	-0.6395
0.32492	-0.9045	-0.2940
0.37488	-0.932	+0.0808
0.42516	-0.8165	+0.422
0.47483	-0.597	+0.6775
0.77521	+0.7425	+0.2698
0.87492	+0.740	-0.1391
4.4983	+0.2001	+0.0842
5.5026	+0.1787	+0.0084
0.52501	-0.313	+0.8275
0,57503	-0.0136	+0.8669
0.62487	+0.2622	+0.8065
1.0501	+0.370	-0.582
1.1504	+0.1027	-0.648
1.2497	-0.130	-0.6115

ω	P_5	Q_5	ω	P_5	Q_5
0	-1	0	0	-1	0
0.02502	-0.962	+0.2717	3.4989	+0.0218	+0.274
0.05007	-0.852	+0.523	4.0009	+0.1050	+0.2186
0.07490	-0.678	+0.731	5.0045	+0.1619	+0.1104
0.10011	-0.454	+0.886	6.0080	+0.1593	+0.0396
0.12485	-0.2014	+0.971	6.9972	+0.1415	-0.0013
0.15005	+0.0666	+0.987	8.0095	+0.1213	-0.0252
0.17513	+0.3252	+0.930	9.0098	+0.1034	-0.0383
0.20012	+0.555	+0.808	10.019	+0.0883	-0.0456
0.22505	+0.7385	+0.637	10.988	+0.0762	-0.0491
0.24995	+0.875	+0.420	11.992	+0.0659	-0.0507
0.27513	+0.947	+0.1805	12.996	+0.0574	-0.0509
0.30001	+0.956	-0.0618	14.008	+0.0503	-0.0504
0.32492	+0.9045	-0.2940	14.990	+0.0446	-0.0495
0.34987	+0.800	-0.502	15.969	+0.0397	-0.0483
0.37488	+0.650	-0.674	16.999	+0.0354	-0.0469
0.39997	+0.4675	-0.8025	17.980	+0.0319	-0.0445
0.42516	+0.2634	-0.881	18.976	+0.0288	-0.0440
0.45012	+0.0546	-0.910	19.970	+0.0262	-0.0426
0.47483	-0.1476	-0.891	0.00989	-0.994	+0.1086
0.50004	-0.3384	-0.8275	0.03987	-0.9045	+0.424
0.55013	-0.640	-0.597	0.05999	-0.7895	+0.612
0.60007	-0.809	-0.2844	0.09013	-0.5475	+0.8315
0.64982	-0.837	+0.0466	0.11011	-0.3545	+0.9285
0.70021	-0.742	+0.3460	0.13995	-0.0409	+0.990
0.74991	-0.560	+0.5715	0.15988	+0.1701	+0.9725
0.80020	-0.3258	+0.710	0.18986	+0.465	+0.8655
0.85006	-0.0792	+0.7575	0.21013	+0.636	+0.744
0.90040	+0.1547	+0.727	0.24008	+0.829	+0.508
0.95007	+0.3502	+0.635	0.25986	+0.911	+0.3276
1.0000	+0.500	+0.500	0.28990	+0.960	+0.0363
1.1003	+0.650	+0.1733	0.30987	+0.943	-0.1558
1.2002	+0.626	-0.1352	0.34010	+0.8465	-0.424
1.3001	+0.488	-0.3552	0.36002	+0.743	-0.577
1.4002	+0.298	-0.498	0.38988	+0.544	-0.757
1.5004	+0.1013	-0.545	0.41013	+0.387	-0.841
1.6003	-0.0738	-0.525	0.44001	+0.1392	-0.904
1.6999	-0.2132	-0.460	0.45995	-0.0270	-0.9085
1.8003	-0.3152	-0.3694	0.48989	-0.264	-0.858
1.9007	-0.380	-0.2686	0.52501	-0.5035	-0.7275
1.9999	-0.4135	-0.1691	0.57503	-0.742	-0.448
2.2496	-0.4035	+0.0450	0.62487	-0.840	-0.1181
2.5002	-0.321	+0.1870	0.67493	-0.804	+0.2032
2.7500	-0.2188	+0.2626	0.72521	-0.6595	+0.470
3.0003	-0.1224	+0.2916	0.77521	-0.4465	+0.652

ω	P_5	Q_5
0	-1	0
0.82483	-0.2038	+0.743
0.87492	+0.0396	+0.7515
0.92493	+0.2564	+0.688
0.97472	+0.430	+0.572
1.0501	+0.600	+0.341
1.1504	+0.656	+0.0115
1.2497	+0.568	-0.2606
1.3498	+0.3972	-0.443
1.4496	+0.200	-0.532
1.5497	+0.0110	-0.542
1.6501	-0.1482	-0.497
1.7496	-0.268	-0.417
1.8495	-0.351	-0.3206
1.9500	-0.400	-0.2186
2.0997	-0.423	-0.0756
4.4983	+0.1457	+0.161
5.5026	+0.1641	+0.0708
3.2506	-0.0414	+0.291
3.7495	+0.0698	+0.248

ω	P_6	Q_6	ω	P_6	Q_6
0	+1	0	0	+1	0
0.02502	+0.948	-0.3195	3.4989	-0.1263	+0.2440
0.05007	+0.795	-0.605	4.0009	-0.0103	+0.2422
0.07490	+0.562	-0.823	5.0045	+0.1070	+0.1643
0.10011	+0.2688	-0.958	6.0080	+0.138	+0.0891
0.12485	-0.0436	-0.992	6.9972	+0.1362	+0.0384
0.15005	-0.3534	-0.924	8.0095	+0.1238	+0.0056
0.17513	-0.6215	-0.764	9.0098	+0.1092	-0.0147
0.20012	-0.8235	-0.5325	10.019	+0.0955	-0.0272
0.22505	-0.9415	-0.2546	10.988	+0.0838	-0.0345
0.24995	-0.969	+0.0412	11.992	+0.0734	-0.0391
0.27513	-0.906	+0.3296	12.996	+0.0645	-0.0416
0.30001	-0.765	+0.578	14.008	+0.0569	-0.0427
0.32492	-0.559	+0.769	14.990	+0.0507	-0.0432
0.34987	-0.3125	+0.8915	15.969	+0.0454	-0.0430
0.37488	-0.0466	+0.935	16.999	+0.0406	-0.0424
0.39997	+0.2152	+0.904	17.980	+0.0367	-0.0417
0.42516	+0.4515	+0.8015	18.976	+0.0333	-0.0408
0.45012	+0.6455	+0.645	19.970	+0.0303	-0.0398
0.47483	+0.784	+0.449	0.00989	+0.992	-0.1282
0.50004	+0.865	+0.2258	0.03987	+0.868	-0.4945
0.55013	+0.847	-0.221	0.05999	+0.711	-0.7015
0.60007	+0.6315	-0.580	0.09013	+0.390	-0.916
0.64982	+0.2978	-0.784	0.11011	+0.1439	-0.984
0.70021	-0.0714	-0.816	0.13995	-0.2324	-0.9625
0.74991	-0.3914	-0.6975	0.15988	-0.465	-0.871
0.80020	-0.621	-0.473	0.18986	-0.750	-0.635
0.85006	-0.735	-0.2005	0.21013	-0.8815	-0.425
0.90040	-0.739	+0.0777	0.24008	-0.970	-0.0763
0.95007	-0.655	+0.3114	0.25986	-0.9555	+0.1576
1.0000	-0.500	+0.500	0.28990	-0.830	+0.4825
1.1003	-0.1107	+0.663	0.30987	-0.6895	+0.6615
1.2002	+0.2460	+0.591	0.34010	-0.4125	+0.852
1.3001	+0.478	+0.3784	0.36002	-0.205	+0.9175
1.4002	+0.568	+0.1203	0.38988	+0.1120	+0.925
1.5004	+0.5425	-0.1165	0.41013	+0.3146	+0.869
1.6003	+0.4395	-0.2965	0.44001	+0.5725	+0.713
1.6999	+0.2986	-0.409	0.45995	+0.707	+0.571
1.8003	+0.1472	-0.4625	0.48989	+0.840	+0.3176
1.9007	+0.0061	-0.4656	0.52501	+0.885	-0.0016
1.9999	-0.1129	-0.4325	0.57503	+0.760	-0.4165
2.2496	-0.3038	-0.2696	0.62487	+0.474	-0.703
2.5002	-0.3614	-0.086	0.67493	+0.1123	-0.8215
2.7500	-0.3364	+0.0606	0.72521	-0.2414	+0.7725
3.0003	-0.273	+0.1597	0.77521	-0.520	+0.5945

ω	P_6	Q_6
0	+1	0
0.82483	-0.6915	+0.342
0.87492	-0.750	+0.0606
0.92493	-0.706	+0.202
0.97472	-0.583	+0.4155
1.0501	-0.311	+0.615
1.1504	+0.0799	+0.651
1.2497	+0.379	+0.497
1.3498	+0.540	+0.251
1.4496	+0.568	-0.0020
1.5497	+0.498	-0.2132
1.6501	+0.3716	-0.361
1.7496	+0.223	-0.4425
1.8495	+0.076	-0.469
1.9500	-0.0559	-0.4525
2.0997	-0.2082	-0.3762
2.2998	-0.3248	-0.232
2.4004	-0.3512	-0.1567
2.6006	-0.3582	-0.0215
2.7009	-0.3454	+0.0356
3.2506	-0.1980	+0.2174
3.7495	-0.0629	+0.2498
4.4983	+0.0637	+0.2076
5.5026	+0.1289	+0.124

ω	P_7	Q_7
0	-1	0
0.00989	-0.989	+0.1478
0.02007	-0.955	+0.2965
0.02997	-0.9007	+0.4345
0.03987	-0.825	+0.5625
0.05007	-0.731	+0.6815
0.05999	-0.6215	+0.781
0.06993	-0.499	+0.864
0.07987	-0.3654	+0.927
0.09013	-0.220	+0.971
0.10011	-0.0737	+0.992
0.11011	+0.0737	+0.991
0.12013	+0.2192	+0.968
0.12988	+0.3392	+0.9315
0.13995	+0.488	+0.862
0.15005	+0.609	+0.780
0.15988	+0.714	+0.6835
0.17004	+0.805	+0.569
0.17993	+0.8765	+0.447
0.18986	+0.9305	+0.316
0.20012	+0.965	+0.1744
0.21013	+0.9785	+0.0342
0.21986	+0.970	-0.1022
0.22995	+0.944	-0.240
0.24008	+0.8985	-0.372
0.24995	+0.836	-0.4925
0.25986	+0.7575	-0.6025
0.27013	+0.6615	-0.703
0.28015	+0.556	-0;787
0.28990	+0.4435	-0.852
0.30001	+0.320	-0.903
0.30987	+0.1945	-0.935
0.32010	+0.0623	-0.951
0.33007	-0.0663	-0.9475
0.34010	-0.1929	-0.927
0.34987	-0.3112	-0.8915
0.36002	-0.4275	-0.838
0.36991	-0.5315	-0.773
0.37986	-0.626	-0.695
0.38988	-0.708	-0.605
0.39997	-0.780	-0.506
0.41013	-0.835	-0.398
0.42002	-0.8755	-0.2886
0.42998	-0.902	-0.1753
0.44001	-0.9135	-0.0599
0.45012	-0.910	+0.0557
0.45995	-0.893	+0.1656
0.46985	-0.8635	+0.2720
0.47984	-0.820	+0.375

ω	P_7	Q_7
0	-1	0
0.48989	-0.766	+0.4695
0.50004	-0.700	+0.557
0.50989	-0.6275	+0.6325
0.51983	-0.546	+0.699
0.52985	-0.4585	+0.755
0.53995	-0.365	+0.801
0.55013	-0.2672	+0.834
0.56003	-0.1703	+0.856
0.57000	-0.0720	+0.866
0.58007	+0.0264	+0.865
0.58983	+0.1199	+0.852
0.60007	+0.2146	+0.830
0.61000	+0.3026	+0.790
0.62003	+0.386	+0.757
0.63014	+0.464	+0.707
0.63994	+0.533	+0.652
0.64982	+0.596	+0.5905
0.65980	+0.651	+0.5225
0.66986	+0.6985	+0.4495
0.68002	+0.738	+0.372
0.68985	+0.7685	+0.295
0.70021	+0.791	+0.212
0.70979	+0.804	+0.1345
0.71990	+0.810	+0.0531
0.73010	+0.8075	-0.0282
0.73996	+0.7975	-0.1050
0.74991	+0.7795	-0.180
0.75996	+0.755	-0.2525
0.77010	+0.7235	-0.322
0.77988	+0.688	-0.3854
0.79022	+0.645	-0.4475
0.80020	+0.5975	-0.502
0.80978	+0.5495	-0.5495
0.81995	+0.4945	-0.595
0.83022	+0.436	-0.634
0.84009	+0.3774	-0.6665
0.85006	+0.316	-0.693
0.86014	+0.2532	-0.715
0.86980	+0.1920	-0.729
0.88007	+0.1273	-0.740
0.88992	+0.0652	-0.7435
0.89988	+0.0032	-0.7433
0.90993	-0.058	-0.7375
0.92008	-0.1183	-0.7265
0.92980	-0.1740	-0.7105
0.94016	-0.2312	-0.691
0.95007	-0.2836	-0.6675
0.96008	-0.3332	-0.640

ω	P_7	Q_7
0	-1	0
0.97020	-0.3806	-0.609
0.97984	-0.422	-0.576
0.99016	-0.464	-0.538
1.00000	-0.500	-0.500
1.1197	-0.664	+0.0407
1.1403	-0.6465	+0.1287
1.1599	-0.619	+0.2072
1.1799	-0.582	+0.281
1.2002	-0.537	+0.3486
1.2203	-0.486	+0.408
1.2401	-0.431	+0.458
1.2602	-0.3698	+0.500
1.2799	-0.308	+0.533
1.3001	-0.2434	+0.559
1.3198	-0.1792	+0.577
1.3400	-0.1142	+0.5875
1.3597	-0.0517	+0.590
1.3798	+0.0102	+0.5868
1.4002	+0.0708	+0.577
1.4202	+0.1271	+0.5615
1.4397	+0.1787	+0.542
1.4596	+0.228	+0.517
1.4798	+0.274	+0.4885
1.5004	+0.3162	+0.456
1.5204	+0.3538	+0.421
1.5399	+0.3852	+0.3852
1.5597	+0.413	+0.3474
1.5798	+0.438	+0.307
1.6003	+0.459	+0.265
1.6202	+0.475	+0.224
1.6404	+0.4875	+0.1823
1.6599	+0.496	+0.1421
1.6797	+0.501	+0.1020
1.6999	+0.503	+0.0618
1.7205	+0.5025	+0.0219
1.7402	+0.498	-0.0152
1.7603	+0.4915	-0.0517
1.7796	+0.4825	-0.0852
1.8003	+0.4705	-0.1195
1.8202	+0.4575	-0.1509
1.8405	+0.442	-0.1810
1.8598	+0.4265	-0.2080
1.8807	+0.4065	-0.2348
1.9007	+0.387	-0.2586
1.9196	+0.3674	-0.2794
1.9402	+0.346	-0.3006

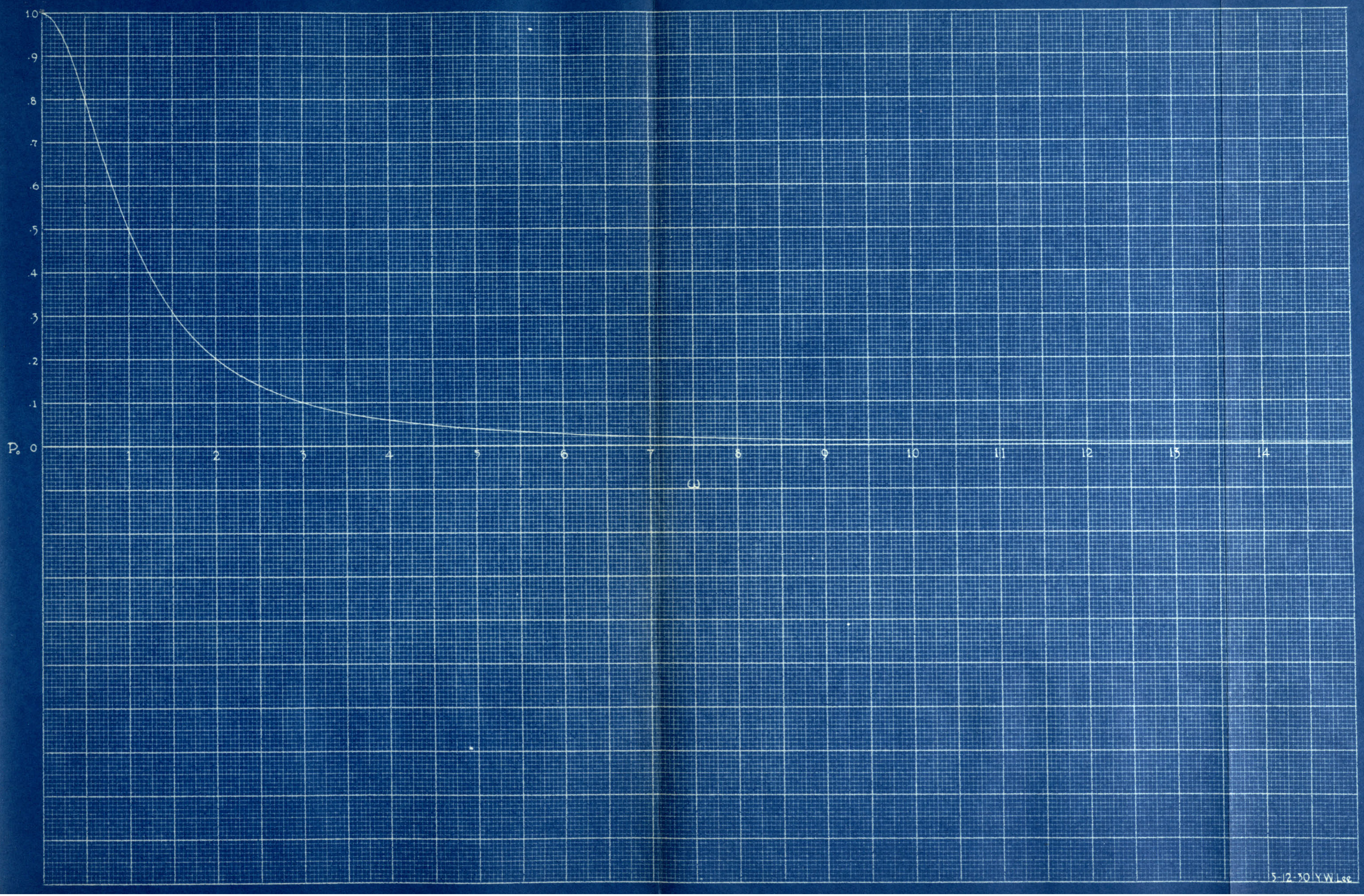
ω	P_7	Q_7
0	-1	0
1.9598	+0.3242	-0.3186
1.9999	+0.2782	-0.350
2.0997	+0.1611	-0.3986
2.1994	+0.0487	-0.4115
2.2998	-0.0521	-0.3958
2.4004	-0.1363	-0.3596
2.5002	-0.2024	-0.3118
2.6006	-0.2516	-0.2562
2.7009	-0.2854	-0.1980
2.8006	-0.3052	-0.1410
2.8987	-0.314	-0.0871
3.0003	-0.314	-0.0358
3.2008	-0.2938	+0.0505
3.4014	-0.257	+0.116
3.6018	-0.213	+0.162
3.7983	-0.1679	+0.1915
4.0009	-0.1230	+0.2088
4.1976	-0.0831	+0.2164
4.4015	-0.0461	+0.2166
4.5993	-0.0148	+0.2120
4.8007	+0.0125	+0.2034
4.9969	+0.0349	+0.1931
5.2011	+0.0544	+0.1809
5.4043	+0.0704	+0.1678
5.6045	+0.0831	+0.1548
5.7994	+0.0931	+0.142
5.9972	+0.1013	+0.1296
6.4971	+0.1143	+0.1002
6.9972	+0.1200	+0.0750
7.4947	+0.1208	+0.0538
8.0095	+0.1186	+0.0357
8.4913	+0.1149	+0.0218
9.0098	+0.1098	+0.0096
9.5144	+0.1045	+0.0000
10.019	+0.0990	-0.0078
10.988	+0.0886	-0.0188
11.992	+0.0788	-0.0264
12.996	+0.0701	-0.0312
14.008	+0.0624	-0.0342
14.990	+0.056	-0.0360
15.969	+0.0504	-0.0370
16.999	+0.0453	-0.0373
17.980	+0.0411	-0.0373
18.976	+0.0373	-0.0371
19.970	+0.341	-0.0366
1.0200	-0.5625	-0.4165

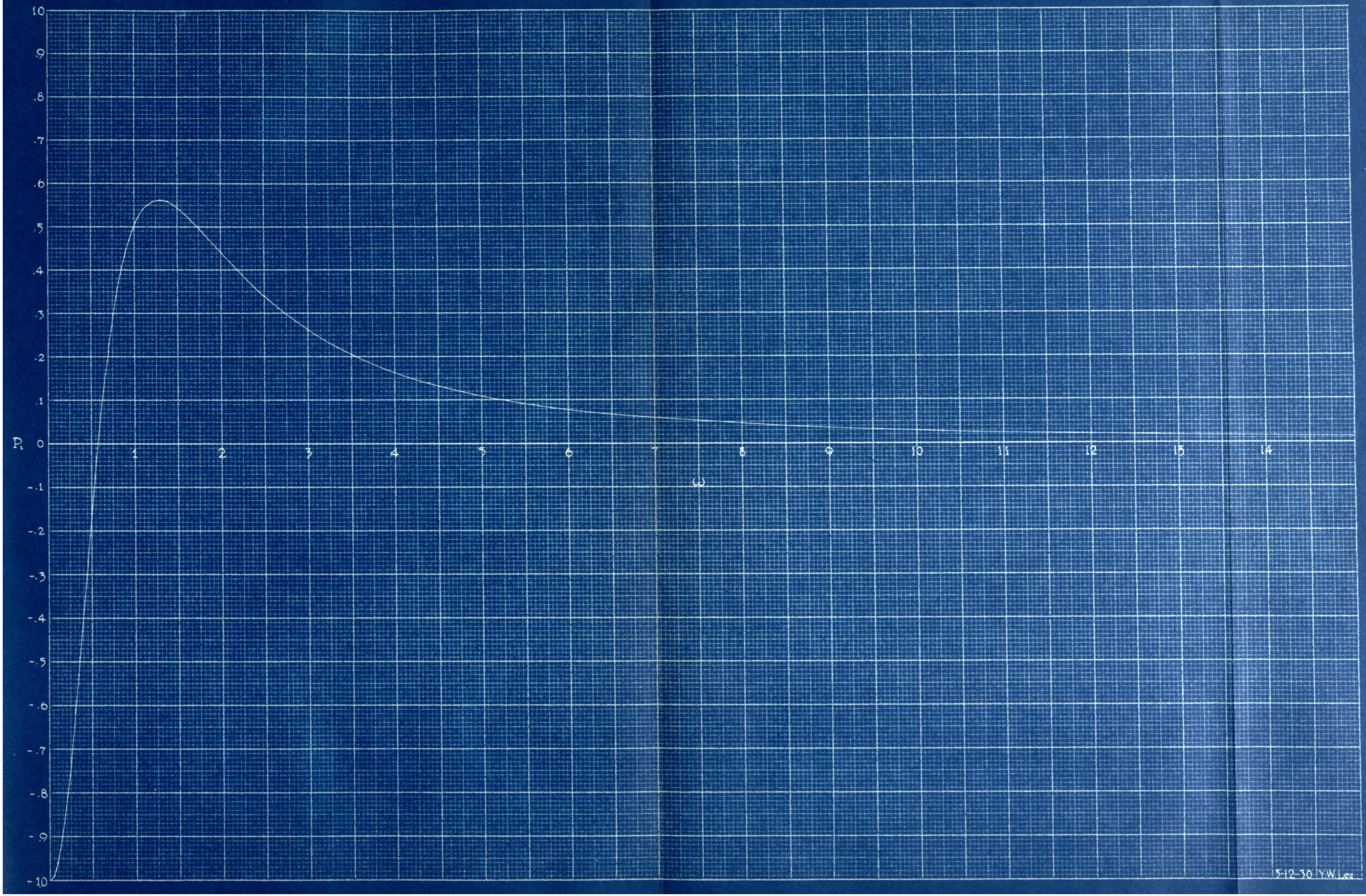
ω	P_7	Q_7
0	-1	0
1.0398	-0.6105	-0.328
1.0599	-0.645	-0.2346
1.0799	-0.6645	-0.1413
1.1003	-0.671	-0.0469

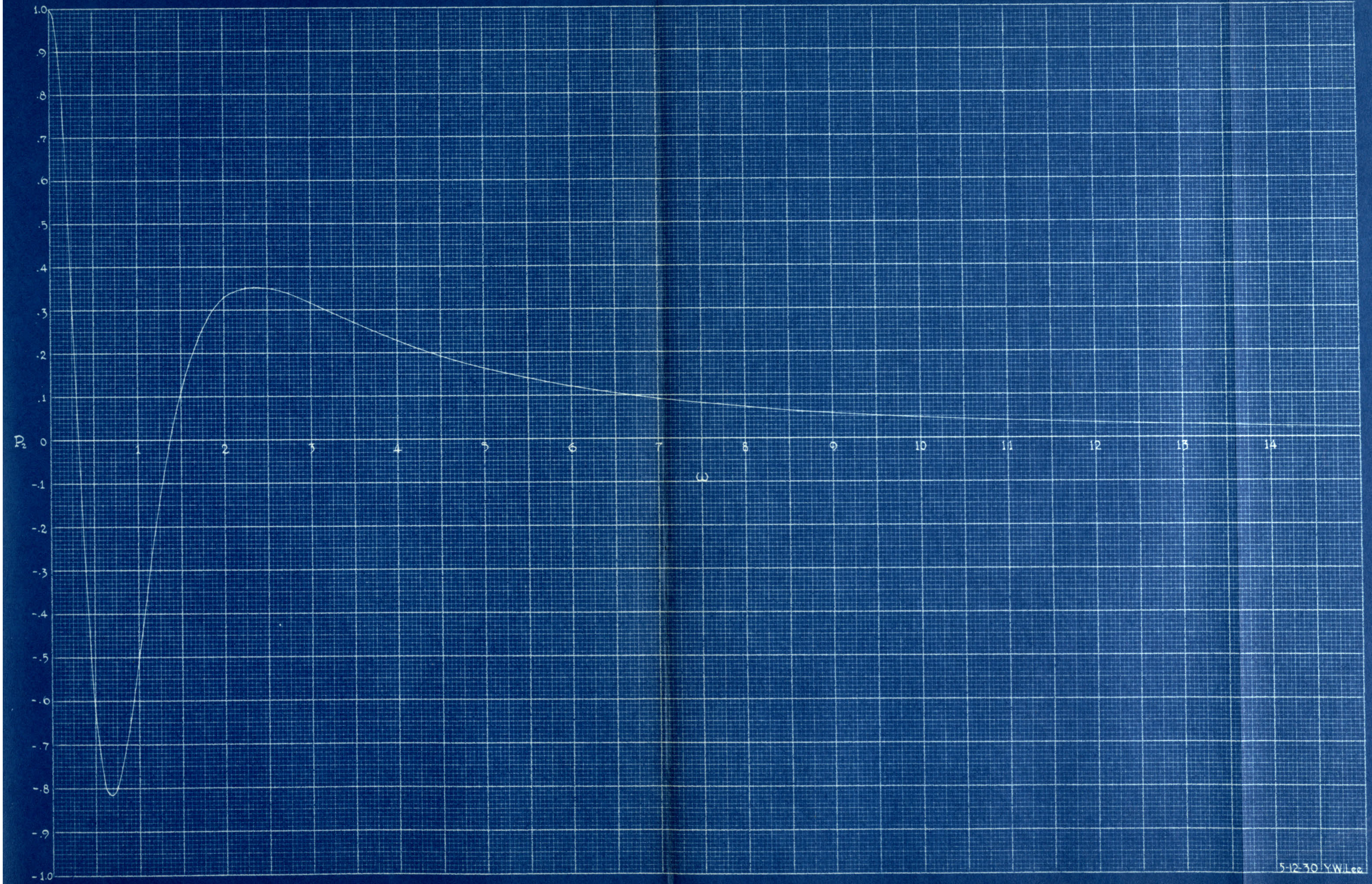
ω	P_8	Q_8	ω	P_8	Q_8
0	+1	-0	0	+1	-0
0.00989	+0.986	-0.1673	0.45012	+0.562	-0.719
0.02007	+0.942	-0.3346	0.45995	+0.456	-0.786
0.02997	+0.873	-0.4876	0.46985	+0.3414	-0.8375
0.03987	+0.779	-0.627	0.47984	+0.2216	-0.874
0.05007	+0.659	-0.751	0.48989	+0.0986	-0.893
0.05999	+0.524	-0.850	0.50004	-0.0255	-0.894
0.06993	+0.374	-0.925	0.50989	-0.1437	-0.879
0.07987	+0.2136	-0.974	0.51983	-0.2584	-0.849
0.09013	+0.0426	-0.995	0.52985	-0.3678	-0.804
0.10011	-0.1244	-0.987	0.53995	-0.469	-0.745
0.11011	-0.2874	-0.951	0.55013	-0.5615	-0.6725
0.12013	-0.4425	-0.889	0.56003	-0.641	-0.5925
0.12988	-0.580	-0.8035	0.57000	-0.708	-0.5035
0.13995	-0.706	-0.695	0.58007	-0.764	-0.407
0.15005	-0.811	-0.567	0.58983	-0.804	-0.3076
0.15988	-0.891	-0.427	0.60007	-0.834	-0.2014
0.17004	-0.948	-0.271	0.61000	-0.849	-0.0964
0.17993	-0.977	-0.1132	0.62003	-0.01047	+0.0089
0.18986	-0.982	+0.0472	0.63014	-0.8375	+0.1131
0.20012	-0.958	+0.2104	0.63994	-0.815	+0.2105
0.21013	-0.910	+0.3622	0.64982	-0.782	+0.3046
0.21986	-0.839	+0.500	0.65980	-0.736	+0.3932
0.22995	-0.745	+0.628	0.66986	-0.6815	+0.4755
0.24008	-0.6315	+0.739	0.68002	-0.6175	+0.550
0.24995	-0.506	+0.8275	0.68985	-0.549	+0.614
0.25086	-0.3685	+0.896	0.70021	-0.470	+0.671
0.27013	-0.2174	+0.940	0.70979	-0.3924	+0.715
0.28015	-0.0664	+0.961	0.71990	-0.3074	+0.7515
0.28990	+0.0810	+0.956	0.73010	-0.219	+0.7775
0.30001	+0.230	+0.930	0.73996	-0.1329	+0.7925
0.30987	+0.3684	+0.881	0.74991	-0.0456	+0.799
0.32010	+0.501	+0.810	0.75996	+0.0412	+0.795
0.33007	+0.6175	+0.722	0.77010	+0.1267	+0.782
0.34010	+0.7175	+0.6175	0.77988	+0.2062	+0.761
0.34987	+0.799	+0.503	0.79022	+0.2862	+0.730
0.36002	+0.864	+0.374	0.80020	+0.3582	+0.693
0.36991	+0.906	+0.2412	0.80978	+0.423	+0.652
0.37986	+0.929	+0.1043	0.81995	+0.487	+0.602
0.38988	+0.931	-0.0341	0.83022	+0.543	+0.545
0.39997	+0.9125	-0.1710	0.84009	+0.592	+0.487
0.41013	+0.874	-0.3026	0.85006	+0.633	+0.423
0.42002	+0.8185	-0.4235	0.86014	+0.669	+0.3574
0.42998	+0.748	-0.534	0.86980	+0.696	+0.2914
0.44001	+0.661	-0.633	0.88007	+0.717	+0.220

ω	P_8	Q_8	ω	P_8	Q_8
0	+1	-0	0	+1	-0
0.88992	+0.7315	+0.1510	1.7796	+0.3236	+0.368
0.89988	+0.739	+0.0814	1.8003	+0.350	+0.3365
0.90993	+0.7396	+0.0116	1.8202	+0.3726	+0.305
0.92008	+0.734	-0.0576	1.8405	+0.392	+0.2725
0.92980	+0.722	-0.1220	1.8598	+0.4075	+0.2408
0.94016	+0.704	-0.1883	1.8807	+0.422	+0.206
0.95007	+0.681	-0.2488	1.9007	+0.4325	+0.1725
0.96008	+0.653	-0.3066	1.9196	+0.440	+0.1411
0.97020	+0.621	-0.362	1.9402	+0.446	+0.1073
0.97984	+0.584	-0.411	1.9598	+0.448	+0.0755
0.99016	+0.5425	-0.458	1.9999	+0.447	+0.0127
1.00000	+0.500	-0.500	2.0997	+0.411	-0.1263
1.1197	-0.1153	-0.656	2.1994	+0.342	-0.2338
1.1403	-0.2120	-0.6245	2.2998	+0.2538	-0.308
1.1599	-0.2964	-0.582	2.4004	+0.1595	-0.350
1.1799	-0.3726	-0.528	2.5002	+0.0684	-0.365
1.2002	-0.440	-0.465	2.6006	-0.0153	-0.3588
1.2203	-0.495	-0.396	2.7009	-0.0876	-0.336
1.2401	-0.538	-0.324	2.8006	-0.1471	-0.3022
1.2602	-0.571	-0.2466	2.8987	-0.1938	-0.2622
1.2799	-0.5915	-0.1698	3.0003	-0.239	-0.2172
1.3001	-0.603	-0.0916	3.2008	-0.2702	-0.1258
1.3198	-0.6035	-0.0164	3.4014	-0.279	-0.0418
1.3400	-0.595	+0.0577	3.6018	-0.266	+0.0290
1.3597	-0.579	+0.1266	3.7983	-0.2405	+0.0839
1.3798	-0.554	+0.1925	4.0009	-0.2068	+0.1264
1.4002	-0.5225	+0.254	4.1976	-0.1718	+0.1559
1.4202	-0.486	+0.309	4.4015	-0.1352	+0.1755
1.4397	-0.445	+0.3565	4.5993	-0.1016	+0.1867
1.4596	-0.400	+0.399	4.8007	-0.0699	+0.1916
1.4798	-0.3512	+0.4365	4.9969	-0.0421	+0.1918
1.5004	-0.299	+0.467	5.2011	-0.0165	+0.1880
1.5204	-0.247	+0.4915	5.4043	+0.0057	+0.1819
1.5399	-0.1952	+0.508	5.6045	+0.0245	+0.1740
1.5597	-0.1428	+0.521	5.7994	+0.0402	+0.165
1.5798	-0.0898	+0.5275	5.9972	+0.0538	+0.1555
1.6003	-0.0370	+0.529	6.4971	+0.0789	+0.1300
1.6202	+0.0127	+0.525	6.9972	+0.0942	+0.1056
1.6404	+0.0613	+0.517	7.4947	+0.1025	+0.0836
1.6599	+0.1061	+0.505	8.0095	+0.1061	+0.0638
1.6797	+0.1492	+0.489	8.4913	+0.1067	+0.0479
1.6999	+0.1905	+0.470	9.0098	+0.1051	+0.0335
1.7205	+0.2296	+0.447	9.5144	+0.1022	+0.0217
1.7402	+0.2640	+0.4225	10.019	+0.0986	+0.0119
1.7603	+0.2962	+0.3956	10.988	+0.0906	-0.0025

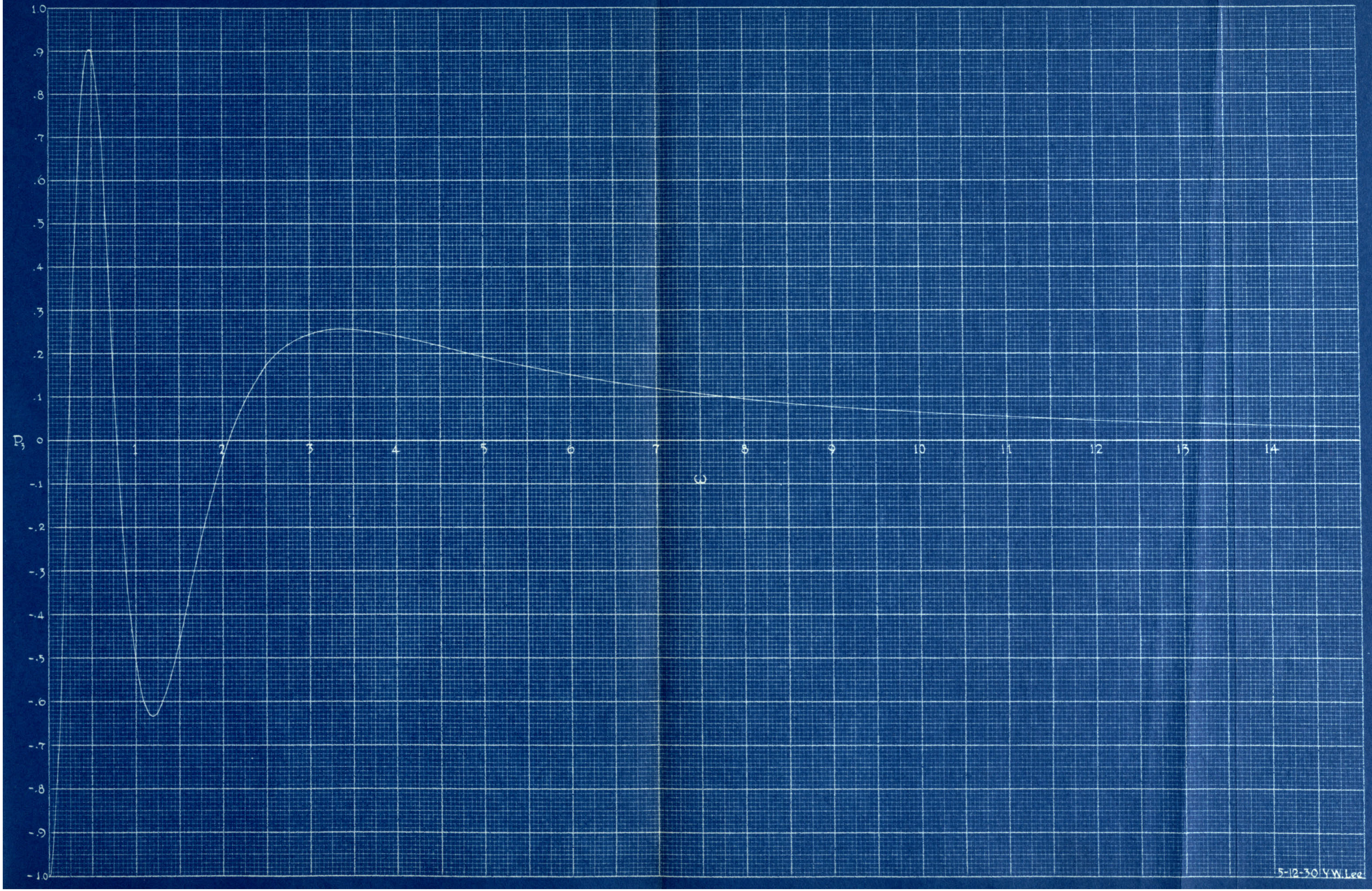
ω	P_8	Q_8
0	+1	-0
11.992	+0.0821	-0.0130
12.996	+0.0740	-0.0201
14.008	+0.0666	-0.0251
14.990	+0.0603	-0.0283
15.969	+0.0547	-0.0304
16.999	+0.0494	-0.0318
17.980	+0.0450	-0.0326
18.976	+0.041	-0.0329
19.970	+0.0376	-0.0330
1.0200	+0.405	-0.571
1.0398	+0.304	-0.6225
1.0599	+0.1969	-0.657
1.0799	+0.0900	-0.673
1.1003	-0.0172	-0.6725


1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
P_o 0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
ω
5-12-30 Y.W. Lee

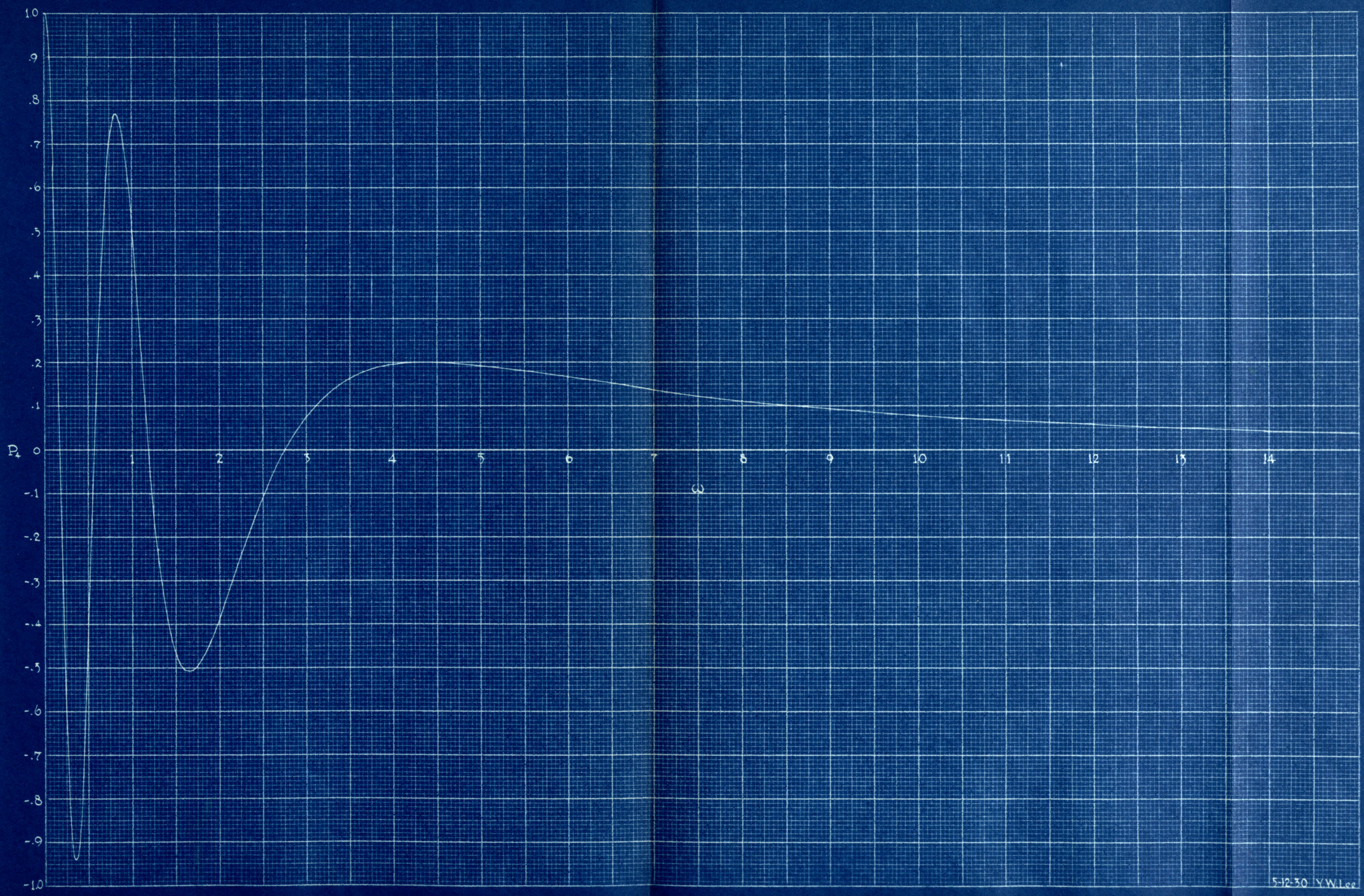

1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
P1 0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
1 2 3 4 5 6 7 8 9 10 11 12 13 14
ω
15-12-30 Y.W.Lee



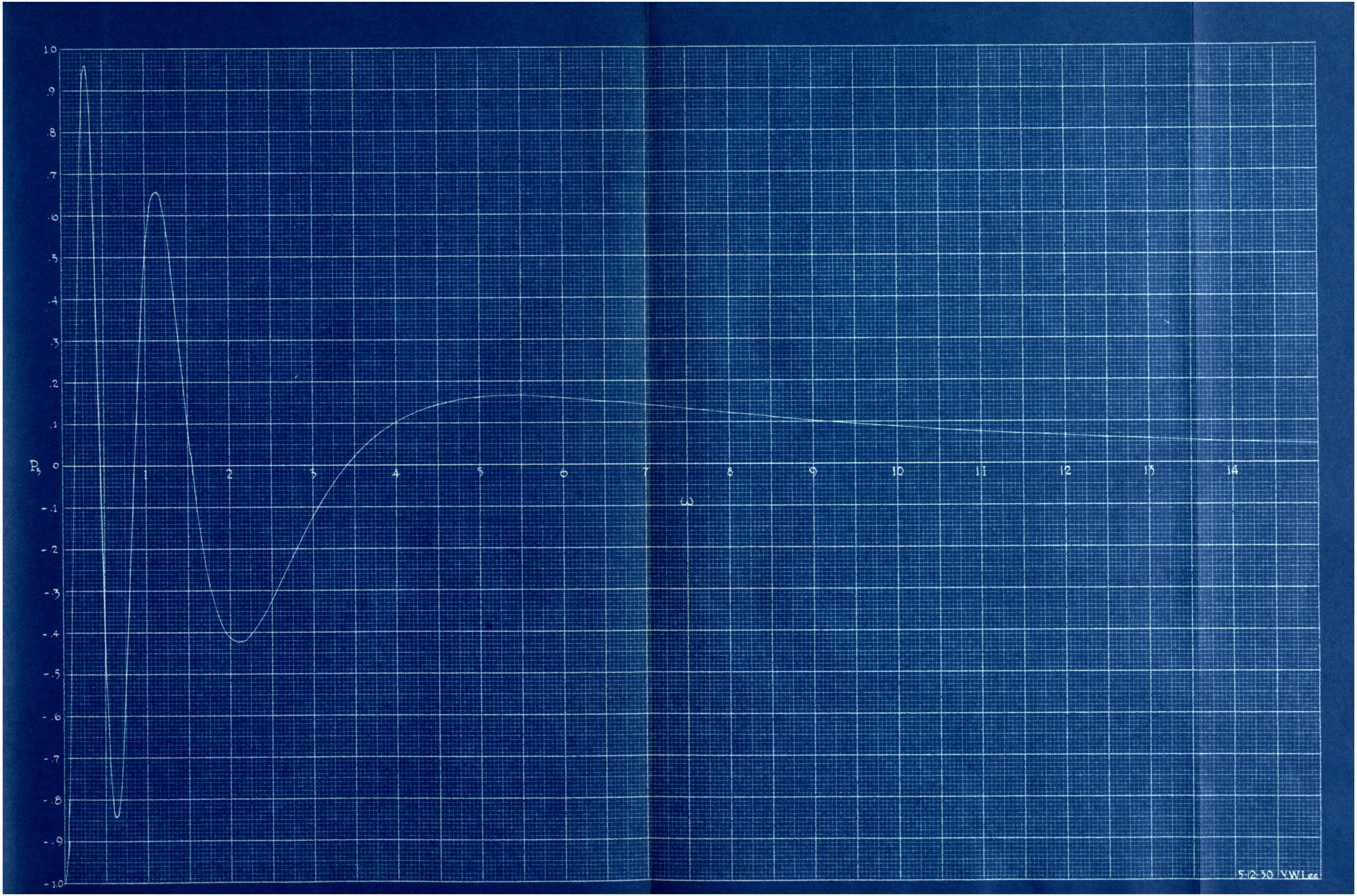
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
P_2 0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
1 2 3 4 5 6 7 8 9 10 11 12 13 14
ω
5-12-30 Y.W. Lee



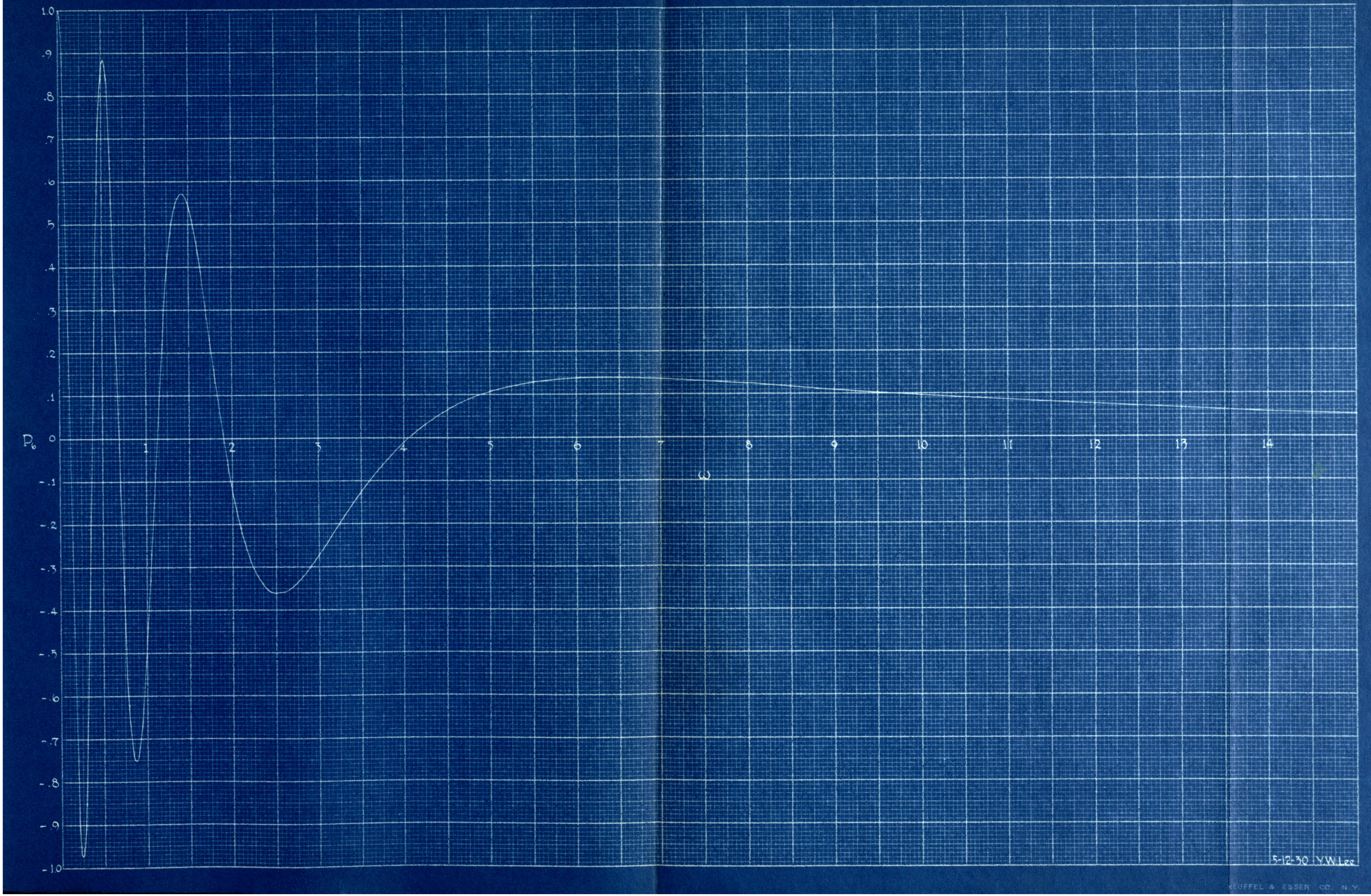
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
P3 0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
ω
5-12-30 Y.W. Lee



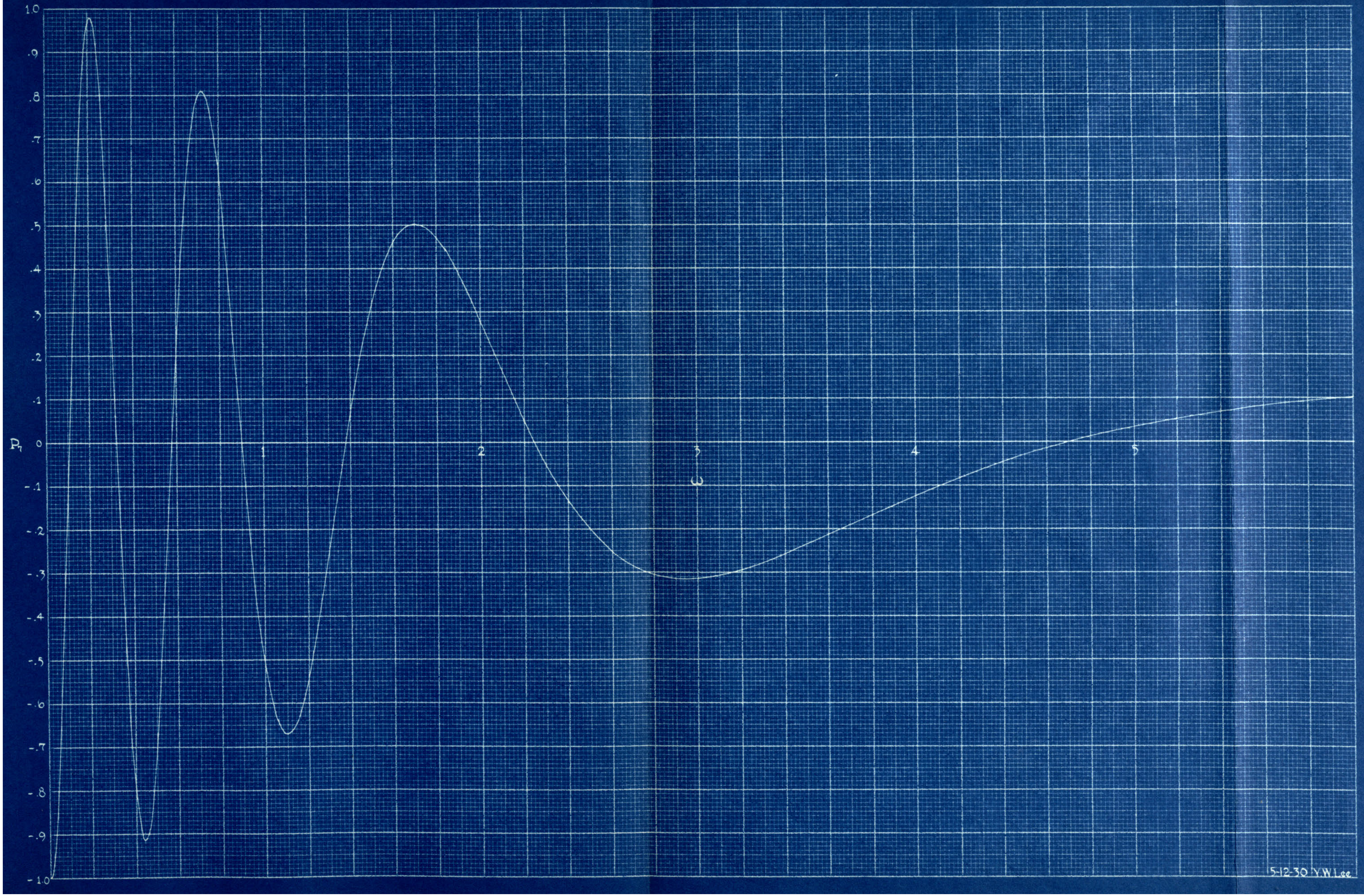
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
P_4
1
2
3
4
5
6
7
8
9
10
11
12
13
14
ω
5-12-30 Y.W. Lee



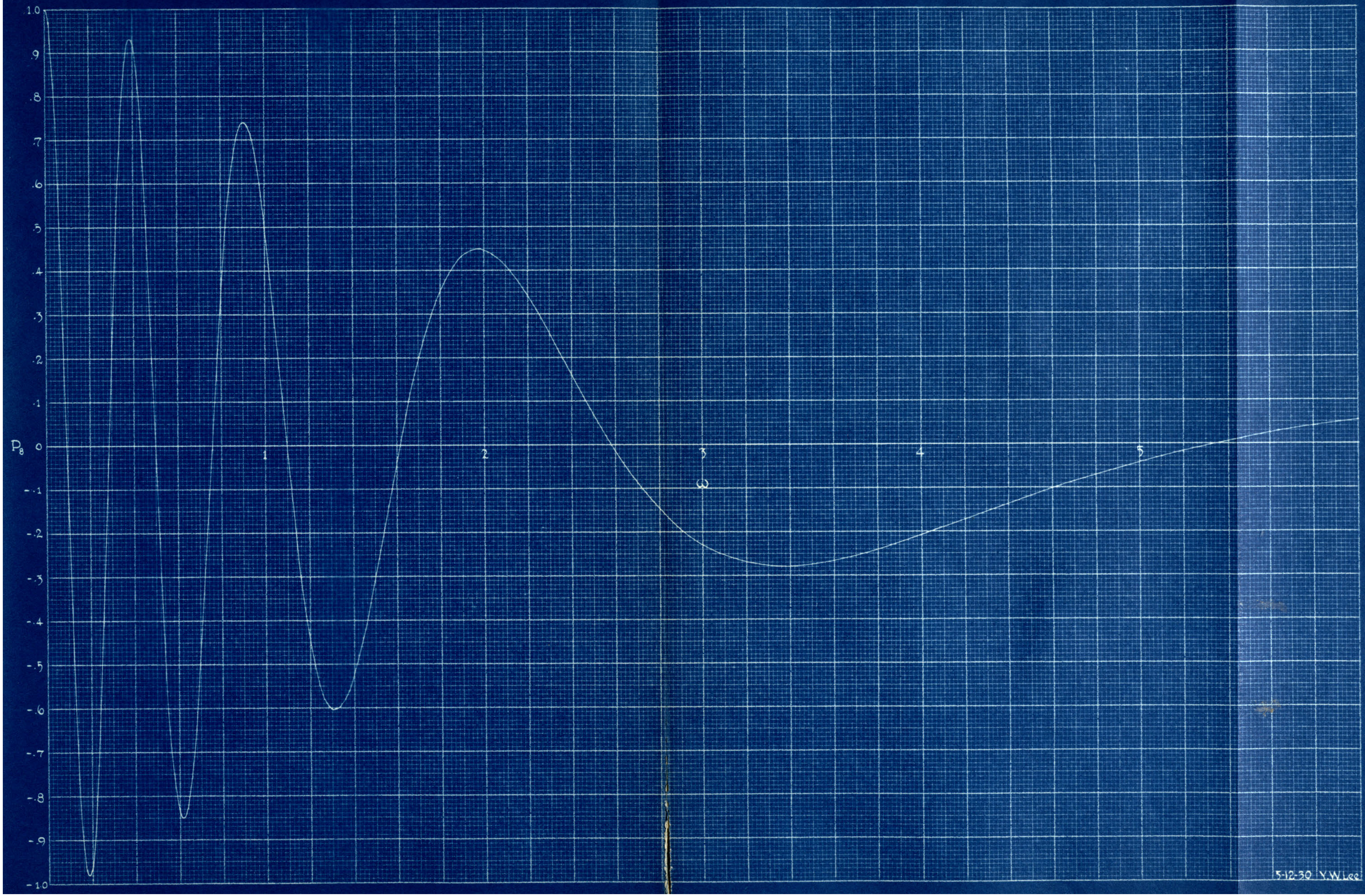
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
P_5
1
2
3
4
5
6
7
8
9
10
11
12
13
14
ω
5-12-30 Y.W.Lee



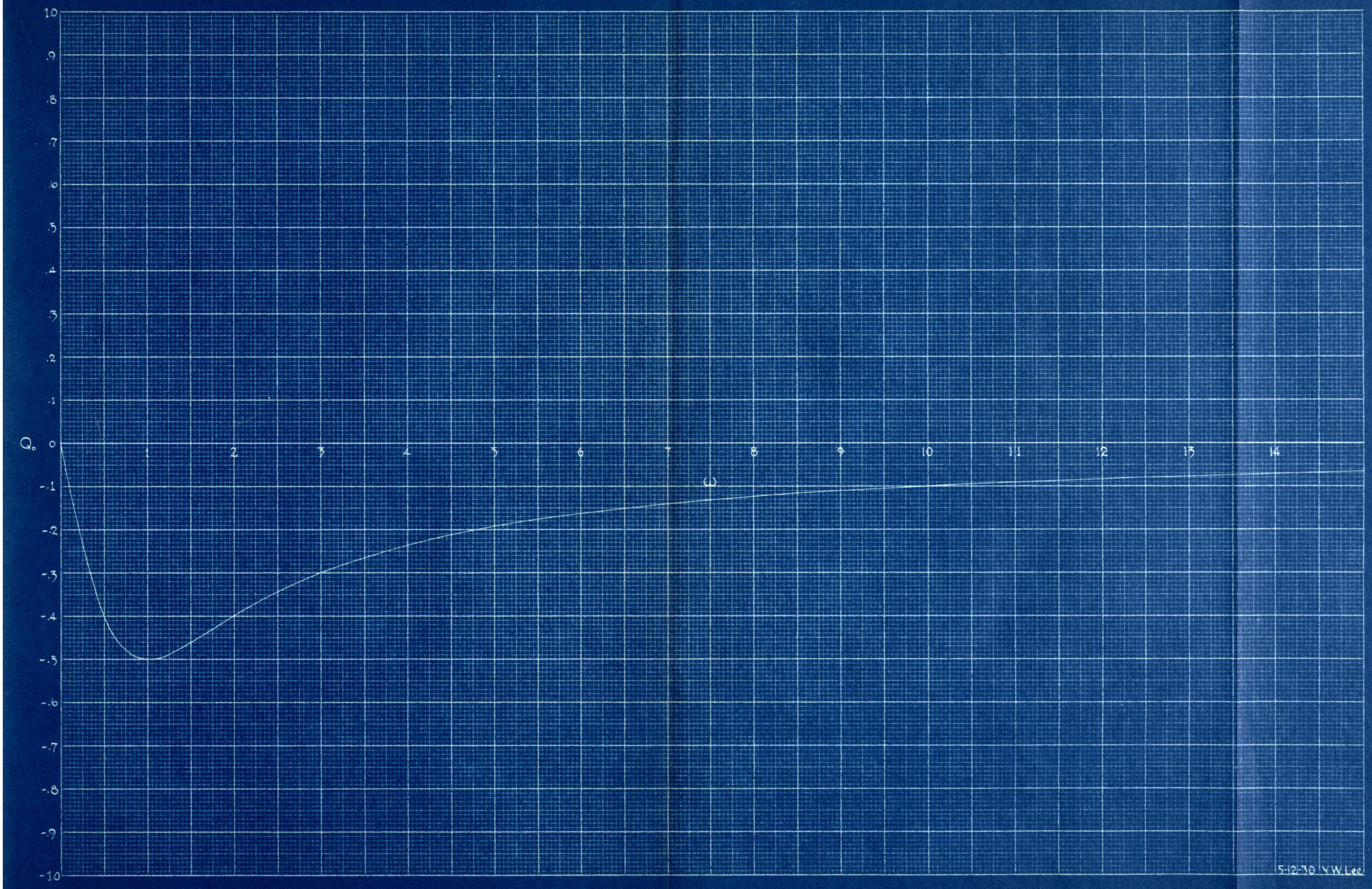
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
P_6
1
2
3
4
5
6
7
8
9
10
11
12
13
14
ω
5-12-30 Y.W. Lee
KEUFFEL & ESSER CO. N.Y.



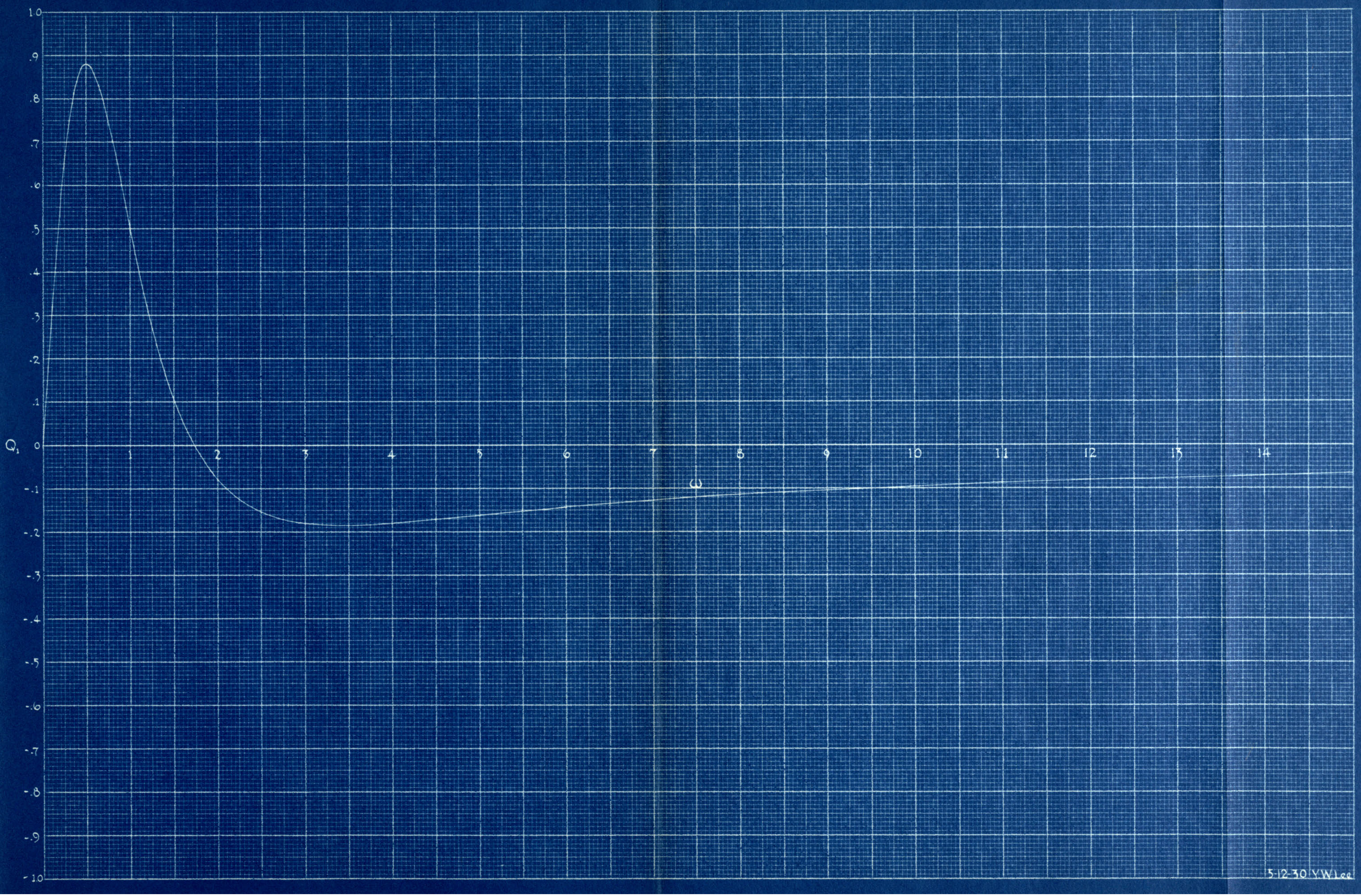
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
P7 0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
1
2
3
4
5
ω
5-12-30 Y.W.Lee



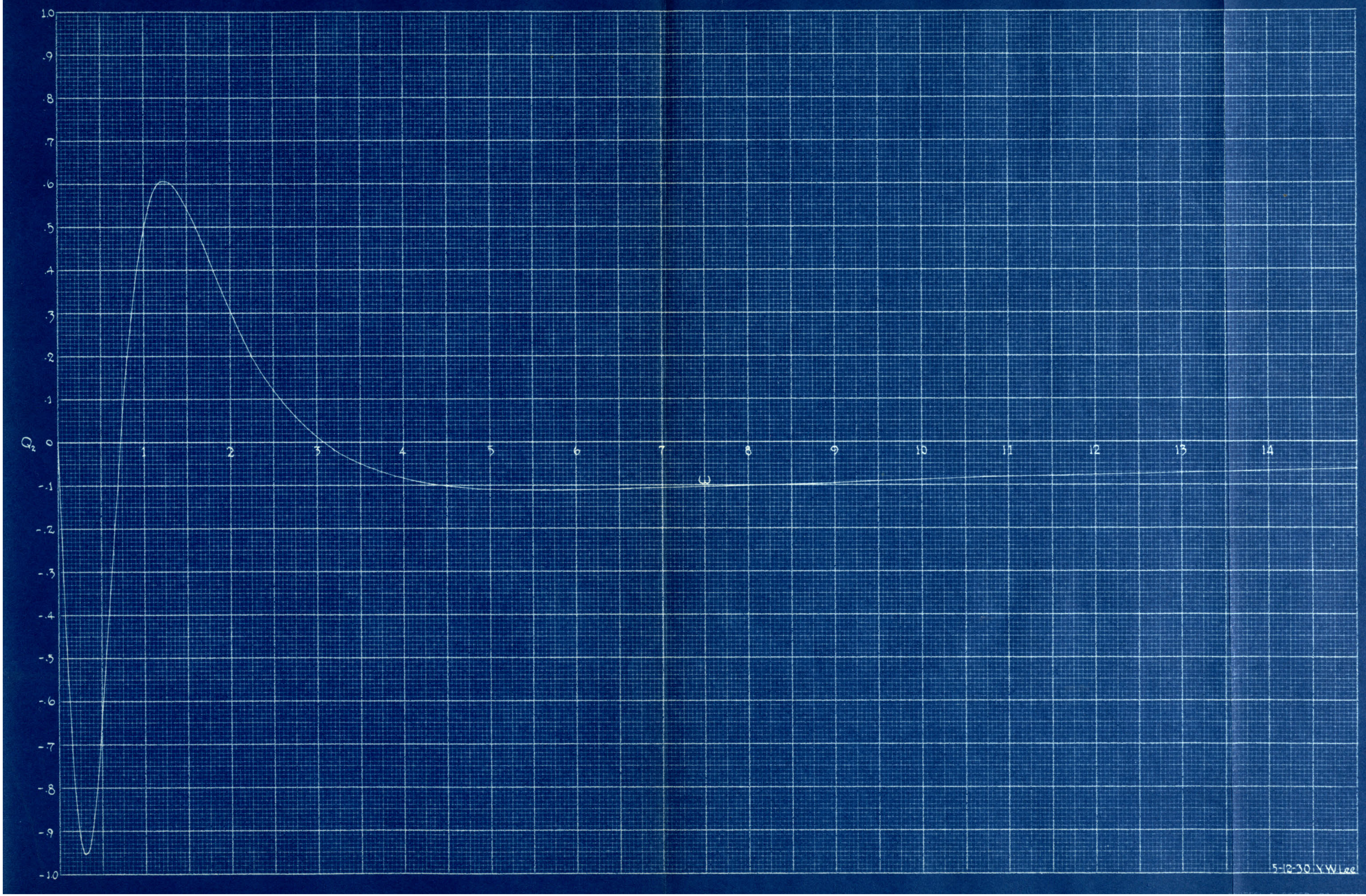
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
P_8 0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
1
2
3
4
5
ω
5-12-30 Y.W.Lee



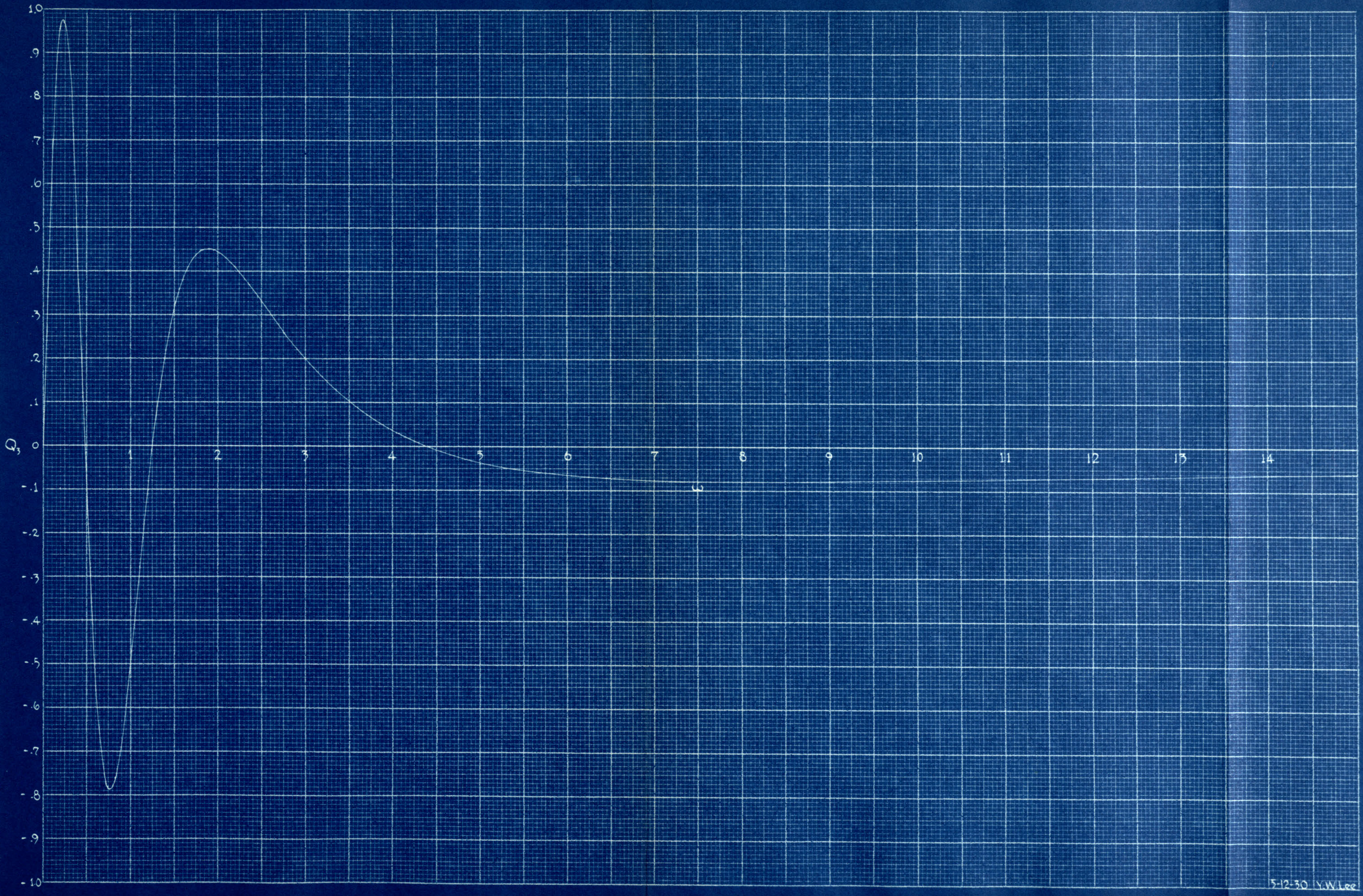
Q₀
ω
5-12-30 Y.W. Lee

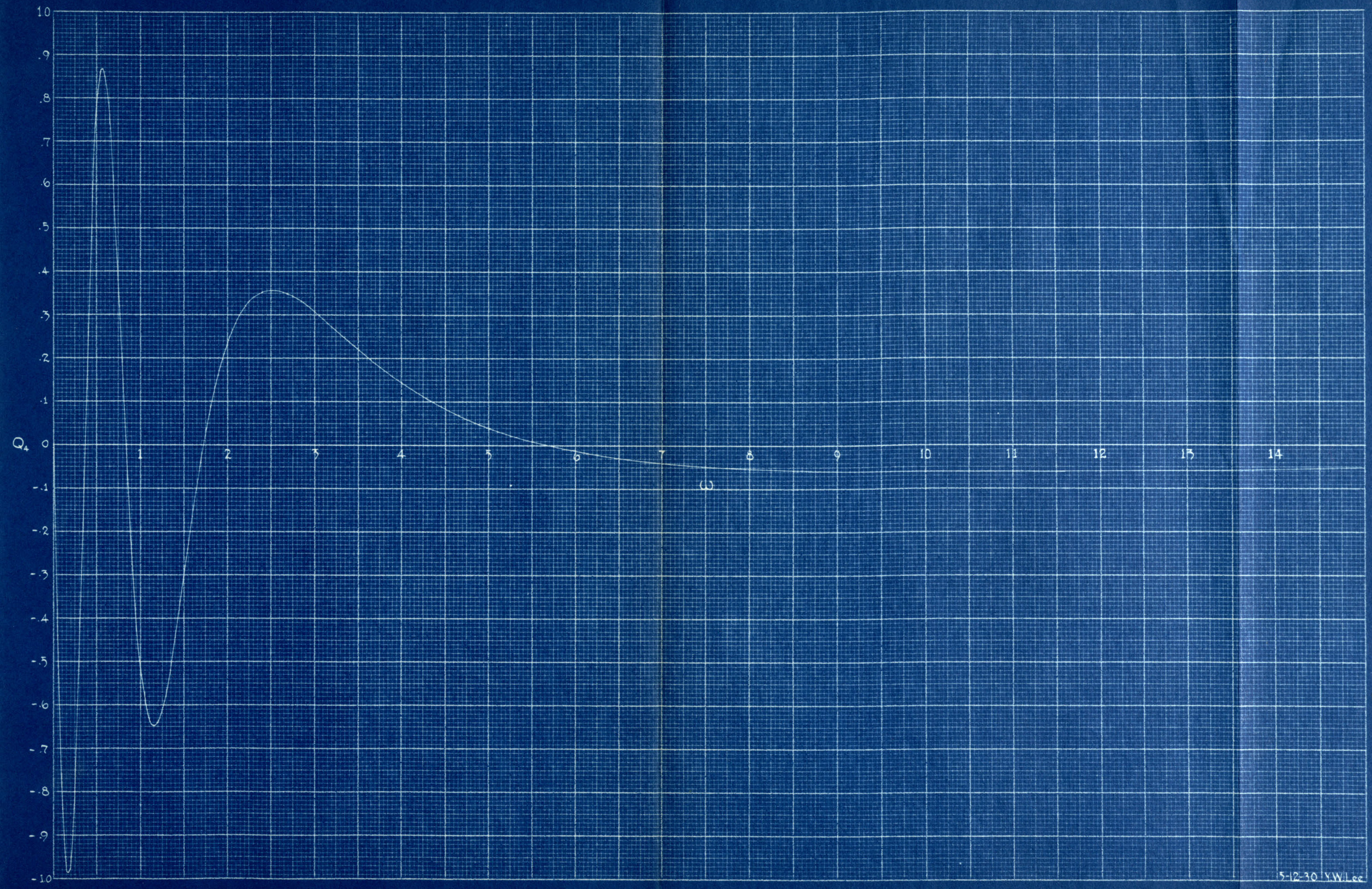


Q₁
ω
5-12-30 Y.W. Lee

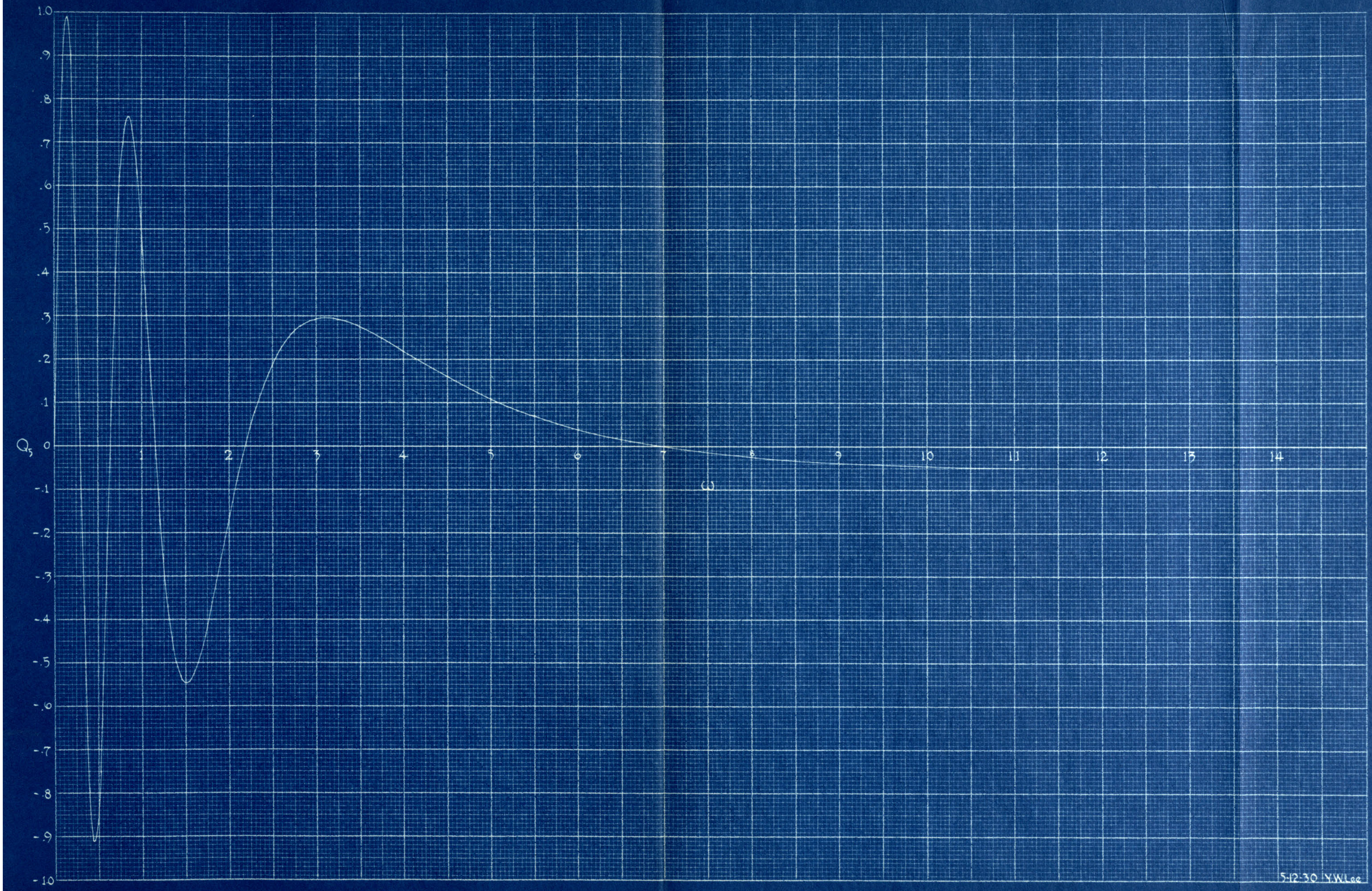


1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
Q2 0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
ω
5-12-30 Y.W.Lee

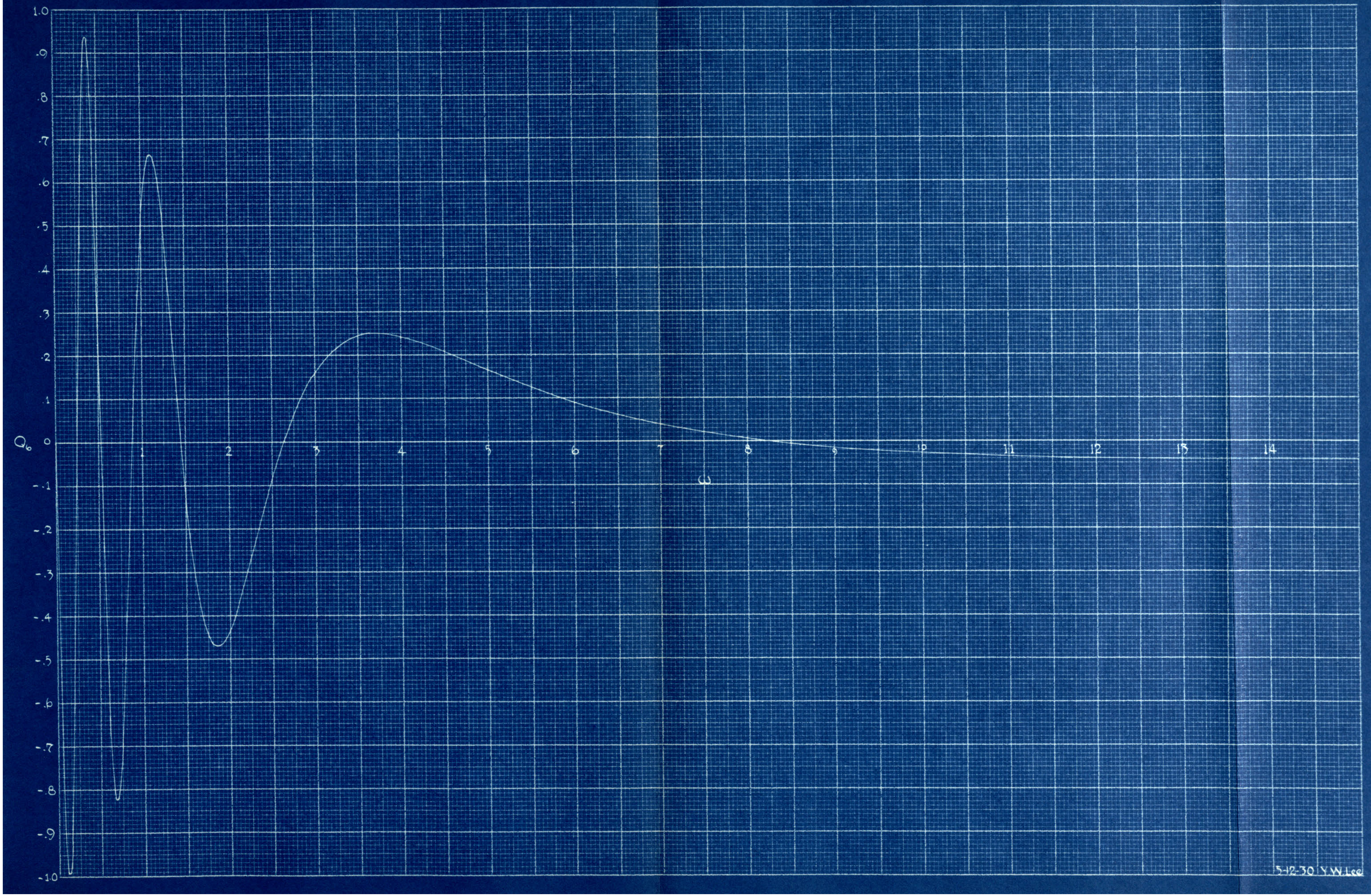

1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
Q_3
0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
ω
5-12-30 Y.W.Lee



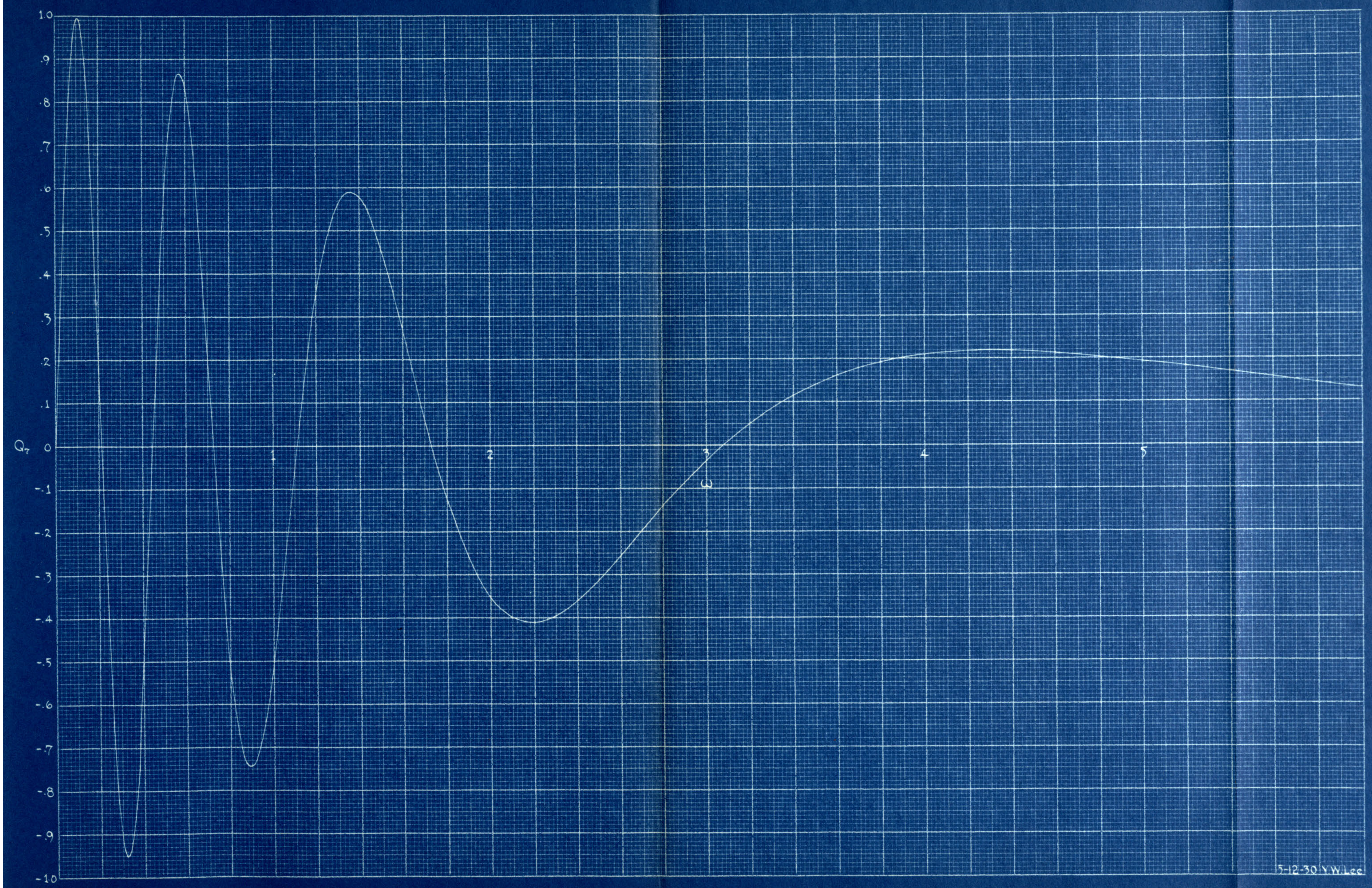
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
Q_4
1
2
3
4
5
6
7
8
9
10
11
12
13
14
ω
15-12-30 Y.W.Lee



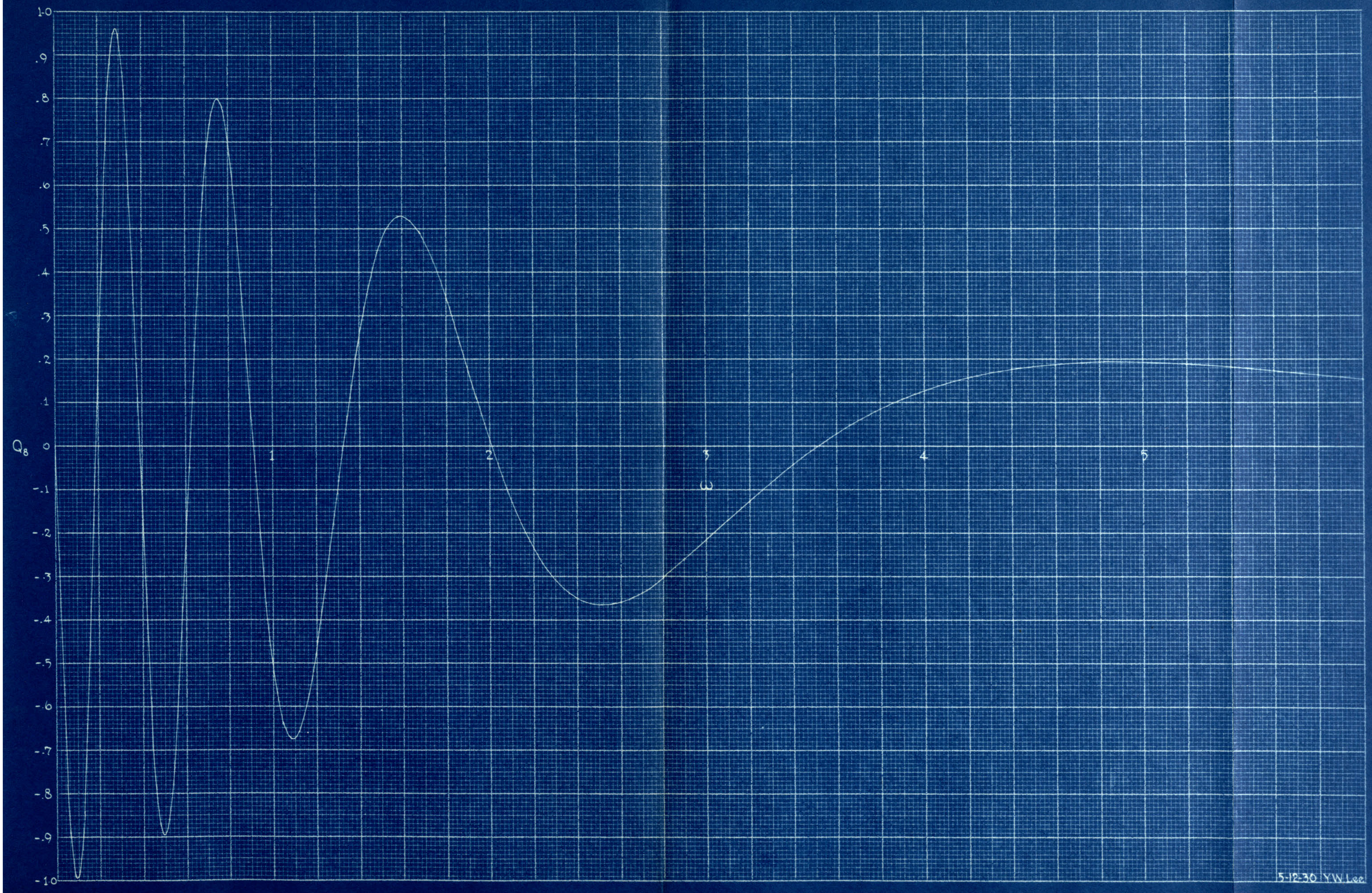
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
Q_5
1
2
3
4
5
6
7
8
9
10
11
12
13
14
ω
5-12-30 Y.W.Lee



1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
Q_6
1
2
3
4
5
6
7
8
9
10
11
12
13
14
ω
5-12-30 Y.W.Lee



1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
Q7 0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
1
2
3
4
5
ω
5-12-30 Y.W.Lee



1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
Q_8
0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
1
2
3
4
5
ω
5-12-30 Y.W. Lee

b.

TABLE 2

ω	$0.854P_o$	$-0.238P_1$	$-0.0956P_2$	$+0.854Q_0$	$-0.238Q_1$	$-0.0956Q_2$
0	+0.854	+0.238	-0.0956	0	0	0
0.10	+0.846	+0.226	-0.0835	-0.0847	-0.0698	+0.0456
0.20	+0.8215	+0.1933	-0.0517	-0.1646	-0.1303	+0.0785
0.30	+0.784	+0.1462	-0.0104	-0.2354	-0.1748	+0.0911
0.40	+0.736	+0.0918	+0.0289	-0.2948	-0.2006	+0.0840
0.50	+0.683	+0.0381	+0.0582	-0.342	-0.2092	+0.0628
0.60	+0.628	-0.0103	+0.0742	-0.3778	-0.2038	+0.0349
0.70		-0.0505	+0.0780		-0.1880	+0.0068
0.80	+0.521	-0.0814	+0.0727	-0.4175	-0.1669	-0.0172
0.90		-0.1040	+0.0616		-0.1430	-0.0356
1.00	+0.4275	-0.1189	+0.0478	-0.4275	-0.1189	-0.0478
1.10		-0.1280	+0.0335		-0.0957	-0.0550
1.20		-0.1325	+0.0199		-0.0746	-0.0579
1.30		-0.1339	+0.0080		-0.0560	-0.0578
1.40		-0.1325	-0.0023		-0.0395	-0.0555
1.50	+0.2628	-0.1293	-0.0107	-0.3942	-0.0253	
1.60		-0.1253	-0.0174		-0.0132	-0.0477
1.80		-0.1152	-0.0265		+0.0057	-0.0382
2.00	+0.1710	-0.1046	-0.0314	-0.342	+0.0190	-0.0291
2.25			-0.0337			-0.0193
2.50	+0.1180	-0.0804	-0.0335	-0.2948	+0.0367	-0.0115
2.75			-0.0322			-0.0056
3.00	+0.0854	-0.0618	-0.0302	-0.2565	+0.0428	-0.0011
3.50	+0.0646	-0.0484	-0.0259	-0.226	+0.0439	+0.0047
4.00	+0.0503	-0.0386	-0.0218	-0.201	+0.0428	+0.0079
5.00	+0.0328	-0.0260	-0.0156	-0.1641	+0.0387	+0.0103
6.00	+0.0230	-0.0186	-0.0115	-0.1383	+0.0344	+0.0107
7.00	+0.0171	-0.0139	-0.0088	-0.1197	+0.0306	+0.0103
8.00	+0.0131	-0.0107	-0.0069	-0.1050	+0.0274	+0.0096
9.00	+0.0104	-0.0085	-0.0055	-0.0936	+0.0248	+0.0090
10.00	+0.0084	-0.0069	-0.0045	-0.0844	+0.0226	+0.0084
1.25	+0.334			-0.417		

TABLE 3

ω	$0.854P_0 - 0.238P_1$	$0.854P_0 - 0.238P_1 - 0.0956P_2$	ω	$0.854P_0 - 0.238P_1$	$0.854P_0 - 0.238P_1 - 0.0956P_2$
0	+1.092	+0.996	1.9999	+0.066	+0.0350
0.10	+1.072	+0.988	2.2496		
0.20	+1.0148	+0.9631	2.5002	+0.038	+0.004
0.30	+0.930	+0.920	2.7500		
0.40	+0.828	+0.857	3.0003	+0.0236	-0.0066
0.50	+0.721	+0.779	3.4989	+0.9162	-0.0097
0.60	+0.618	+0.692	4.0009	+0.0116	-0.0102
0.70			5.0045	+0.0068	-0.0088
0.80	+0.440	+0.513	6.0080	+0.0045	-0.0070
0.90			6.9972	+0.0032	-0.0056
1.00	+0.309	+0.356	8.0095	+0.0024	-0.00450
1.10			9.0098	+0.0019	-0.0036
1.20			10.019	+0.0015	-0.0030
1.40			1.2497		
1.50	+0.134	+0.123			

TABLE 4

ω	$0.854Q_0 - 0.238Q_1$	$0.854Q_0 - 0.238Q_1 - 0.0956Q_2$	ω	$0.854Q_0 - 0.238Q_1$	$0.854Q_0 - 0.238Q_1 - 0.0956Q_2$
0	0		2.00	-0.323	-0.3521
0.10	-0.1545	-0.1089			
0.20	-0.2949	-0.2164	2.50	-0.2581	-0.2696
0.30	-0.4102	-0.3191			
0.40	-0.4954	-0.4114	3.00	-0.2137	-0.2148
0.50	-0.5512	-0.4884	3.50	-0.1821	-0.1774
0.60	-0.5816	-0.5467	4.00	-0.158	-0.150
			5.00	-0.1254	-0.1151
0.80	-0.5844	-0.6016	6.00	-0.104	-0.093
1.00	-0.5464	-0.5942			
1.50	-0.4195				

TABLE 5

ω	$0.484Q_0$	$-0.610Q_1$	$0.1253Q_3$	$0.484P_0$	$-0.610P_1$	$0.1253P_3$
0	0	0	0	+0.484	+0.610	-0.1253
0.10	-0.0480	-0.1790	+0.0802	+0.479	+0.580	-0.0955
0.20	-0.0932	-0.3345	+0.1207	+0.465	+0.496	-0.9230
0.30	-0.1333	-0.448	+0.1071	+0.4445	+0.375	+0.0544
0.40	-0.1670	-0.515	+0.0535	+0.4175	+0.2356	+0.1034
0.50	-0.1937	-0.537	-0.0117	+0.3872	+0.0976	+0.1114
0.60	-0.214	-0.5225	-0.0644	+0.356	-0.0265	+0.0861
0.70		-0.4825	-0.0930	+	-0.1295	+0.0434
0.80	-0.2365	-0.428	-0.0978	+0.2954	-0.2086	-0.0011
0.90		-0.3672	-0.0852		-0.267	-0.038
1.00	-0.242	-0.305	-0.0627	+0.242	-0.305	-0.0627
1.10		-0.2456	-0.0367		-0.3284	-0.0758
1.20		-0.1915	-0.0120		-0.340	-0.0793
1.30		-0.1435	+0.0094		-0.3432	-0.0758
1.40		-0.1013	+0.0264		-0.3396	-0.0679
1.50	-0.2234	-0.0648	+0.0391	+0.1490	-0.332	-0.0575
1.60		-0.0338	+0.0478		-0.3214	-0.0462
1.70			+0.0531			-0.0348
1.80		+0.0147	+0.0559		-0.296	-0.0241
1.90			+0.0565			-0.0144
2.00	+0.0968	-0.2684	-0.0058	-0.1937	+0.0488	+0.0558
2.25			+0.0108			+0.0497
2.50	+0.0668	-0.2062	+0.0214	-0.167	+0.0943	+0.0413
2.75			+0.0276			+0.0328
3.00	+0.0484	-0.1586	+0.0308	-0.1453	+0.1099	+0.0250
3.50	+0.0366	-0.1242	+0.0320	-0.1280	+0.1125	+0.0127
4.00	+0.0285	-0.0991	+0.0301	-0.1139	+0.1099	+0.0044
5.00	+0.0186	-0.0668	+0.0241	-0.0930	+0.0991	-0.0046
6.00	+0.0131	-0.0476	+0.0188	-0.0784	+0.0881	-0.0083
7.00	+0.0097	-0.0357	+0.0149	-0.0678	+0.0785	-0.0097
8.00	+0.0074	-0.0275	+0.0119	-0.0595	+0.0703	-0.0100
9.00	+0.0059	-0.0219	+0.0097	-0.0530	+0.0636	-0.0099
10.00	+0.0048	-0.0178	+0.0080	-0.0478	+0.0579	-0.0096

TABLE 6

ω	$0.484P_0 - 0.610P_1$	$0.484P_0 - 0.610P_1 + 0.1253P_3$	ω	$0.484P_0 - 0.610P_1$	$0.484P_0 - 0.610P_1 + 0.1253P_3$
0	+1.094	+0.969	2.00	-0.1716	-0.1774
0.10	+1.059	+0.963			
0.20	+0.961	+0.938	2.50	-0.1394	-0.1180
0.30	+0.820	+0.874			
0.40	+0.6531	+0.7565	3.00	-0.1102	-0.0794
0.50	+0.4848	+0.5962	3.50	-0.0876	-0.0556
0.60	+0.3295	+0.4156	4.00	-0.0706	-0.0405
0.70			5.00	-0.0482	-0.0240
0.80	+0.0868	+0.0857	6.00	-0.0345	-0.0157
0.90			7.00	-0.0260	-0.0111
1.00	-0.063	-0.126	8.00	-0.0201	-0.0082
1.10			9.00	-0.0160	-0.0063
1.20			10.00	-0.0131	-0.0051
1.50	-0.183	-0.240			

TABLE 7

ω	$0.484Q_0 - 0.610Q_1$	$0.484Q_0 - 0.610Q_1 + 0.1253Q_3$	ω	$0.484Q_0 - 0.610Q_1 +$	$0.484Q_0 - 0.510Q_1 + 0.1253Q_3$
0	0	0	2.00	-0.1449	-0.0891
0.10	-0.2270	-0.1468			
0.20	-0.4277	-0.3070	2.50	-0.0727	-0.0314
0.30	-0.5813	-0.4742			
0.40	-0.6820	-0.6285	3.00	-0.0354	-0.0104
0.50	-0.7307	-0.7424	3.50	-0.0155	-0.0028
0.60	-0.7365	-0.8009	4.00	-0.0040	+0.0004
0.70			5.00	+0.0061	+0.0015
0.80	-0.6645	-0.7623	6.00	+0.0097	+0.0014
0.90			7.00	+0.0107	+0.0010
1.00	-0.547	-0.610	8.00	+0.0108	+0.0008
1.10			9.00	+0.0106	+0.0007
1.20			10.00	+0.0101	+0.0005
1.50	-0.2882	-0.2491			

ABSTRACT

Synthesis of Electric Networks by Means of the Fourier Transforms of Laguerre's Functions

by

Yuk-Wing Lee

This thesis presents a new method for the design of electric networks with assigned transfer admittances. This problem occurs frequently in electrical communication engineering where the design of transmission networks such as wave filters, balancing networks, artificial lines, and phase correction networks, is of prime importance.

It is known in the Operational Calculus that the Fourier transform of a function vanishing to the left of the origin is a function analytic over the right half of the complex plane. Since an admittance function is analytic over the right half of the complex plane, it seems possible to expand this function in terms of the Fourier transform of a function which vanishes to the left of the origin, and if we have a set of transforms corresponding to a known set of networks, a proper linear combination of these will yield any desired admittance function.

Dr. Norbert Wiener, Associate Professor of Mathematics of the Massachusetts Institute of Technology, conceived the idea and suggested the use, for this purpose, of Laguerre's functions which are defined for

positive values of the argument. The expansion of an admittance Y (ω) in terms of the Fourier transforms of the Laguerre's functions G_n (ω) is:

$$Y(\omega) = \sum_{n=0}^{\infty} G_n(\omega) \int_{-\infty}^{\infty} Y(\omega)\, \bar{G}_n(\omega)\, d\omega .$$

Here ω is 2π times the frequency.

The Fourier transforms of Laguerre's functions are found to represent the transfer admittances of a set of networks built up in a simple and definite manner, of positive resistances, inductances, and capacitances. By proper choice of the polarity of the terminals of the network, both positive and negative coefficients may be represented.

This scheme of connections leads in the first instance to a short-circuit transfer admittance function. In order to introduce a load in the short-circuit, a circuit theorem is established. The theorem is: <u>If an admittance Y be inserted into any mesh of a network, the transfer admittance of that mesh is the product of the transfer admittance prior to the addition and Y, divided by the sum of the driving point admittance of the same mesh prior to the addition and Y</u>.

The coefficients of the expansion are completely determined either by the real or the imaginary part of the transfer admittance function, so that it is not necessary to know both to determine the network. A change of variables reduces the evaluation of the coefficients to a very simple process. The procedure consists of plotting the given function on paper with a tangent scale on the abscissa and making a harmonic analysis of it. Of course analytical means may be used when possible.

The real and imaginary parts of an admittance function are Hilbert transforms of each other, that is, they are related by the reciprocal formulas:

$$P(\omega) = \frac{1}{\pi}\int_0^\infty \frac{Q(\omega+t) - Q(\omega-t)}{t}\,dt$$

and

$$Q(\omega) = -\frac{1}{\pi}\int_0^\infty \frac{P(\omega+t) - P(\omega-t)}{t}\,dt\,.$$

Likewise, the logarithm of the absolute value of the admittance function and its phase are Hilbert transforms of each other (due to Dr. N. Wiener), that is, if

$$|Y| = (P^2 + Q^2)^{\frac{1}{2}}$$

and

$$\phi = \tan^{-1}\frac{Q}{P}\,,$$

then $$\ln|Y| = \frac{1}{\pi}\int_0^\infty \frac{\phi(\omega+t) - \phi(\omega-t)}{t}\,dt$$

and

$$\phi = -\frac{1}{\pi}\int_0^\infty \frac{\ln|Y(\omega+t)| - \ln|Y(\omega-t)|}{t}\,dt\,.$$

Because of these interrelations, a network can be designed when any one of the four values of the admittance function, namely, the real part, the imaginary part, the absolute value, and the phase, is given. Owing to the fact that a zero of a function is an infinity of its logarithm, the determination of phase by amplitude and vice versa is not unique. It is this that permits the existance of phase correction networks.

When either the absolute value or the phase of the admittance function is given, it is necessary to solve the Hilbert transform. If this cannot be done analytically, Mr. T.S. Gray's photo-electric integrating machine developed under the supervision of Dr. V. Bush at the Massachusetts Institute of Technology may be used.

Two illustrative problems are given. The first is the design of a network having the real part of the transfer admittance function assigned. In the second problem, the imaginary part of the transfer admittance is specified. Both problems are carried out analytically, and the required functions are very closely approximated by only three terms of the expansions.