

The 1-dimensional λ -self shrinkers in \mathbb{R}^2 and the nodal sets of biharmonic Steklov problems

by

Jui-En Chang

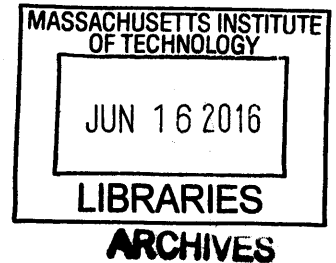
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Abstract

This thesis contains two of my projects. Chapter 1 and 2 describe the behavior of 1-dimensional λ -self shrinkers, which are also known as λ -curves in other literature. Chapter 3 and 4 focus on the estimation of the asymptotic behavior of the nodal set of biharmonic Steklov problems.

Chapter 1 gives the background of mean curvature flow and the importance of self-shrinkers as solitons of the flow equation. We also introduce the background of the λ -hypersurface and explain how this is related to the self shrinkers. In chapter 2, we examine the solutions of 1-dimensional λ -self shrinkers and show that for certain $\lambda < 0$, there are some closed, embedded solutions other than circles. For negative λ near zero, there are embedded solutions with 2-symmetry. For negative λ with large absolute value, there are embedded solutions with m -symmetry, where m is greater than 2.

Chapter 3 focuses on the background of spectral geometry. Several eigenvalue problems are introduced. We have a brief survey of some of the important problems such as the asymptotic distribution of the eigenvalues, the shape optimization problem and the bound of nodal sets. This project focuses on establishing a lower bound of the measure of the nodal set. In chapter 4, we use layer potential to establish that the boundary biharmonic Steklov operators are elliptic pseudo-differential operators. Thus we are able to establish lower bounds on both the measure of boundary nodal sets and interior nodal sets for biharmonic Steklov eigenfunctions.

Thesis Supervisor: William P. Minicozzi II
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Chapter 1

Background of mean curvature flow

1.1 Mean curvature flow

We start with the variation of a surface in Euclidean space. Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a closed hypersurface. Let $F : \Sigma \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^{n+1}$ be a smooth map and $F(x, 0) = x$ for all $x \in \Sigma$. Therefore, Σ_t , the image of $F(\Sigma, t)$, is a one-parameter family of hypersurfaces. When the surface changes with respect to t , the area, which is a function of the surface, also changes with respect to t . We can differentiate the area with respect to time at $t = 0$ and get the following first variation formula:

$$\left. \frac{d}{dt} \right|_{t=0} \text{Area}(\Sigma_t) = \int_{\Sigma_t} \langle \partial_t F(x, 0), HN \rangle d\sigma(x), \quad (1.1.1)$$

where N is the unit normal vector on Σ and H is the mean curvature of Σ . Note that if $\{e_i\}$ is a local orthonormal frame of Σ , the mean curvature H is defined as $\sum_i \langle \nabla_{e_i} N, e_i \rangle$, which changes sign if the unit normal vector points in the opposite direction.

From the first variation formula, in order to have the most rapid descent of the area at any time, the variation vector field $\partial_t F$ should be proportional to $-HN$. A family of hypersurfaces Σ_t in \mathbb{R}^{n+1} satisfies the mean curvature flow(MCF) if

$$\partial_t x = -HN, \quad (1.1.2)$$

where $N(t)$ is the unit normal vector and $H(t)$ is the mean curvature of the hypersurface Σ_t at x . This is the negative gradient flow of the area functional.

Example 1.1.1. *The following are some examples of MCF:*

1. *If Σ is a minimal surface in \mathbb{R}^{n+1} , i.e. the mean curvature $H \equiv 0$. The flow is given by $\Sigma_t = \Sigma$. The surface does not move at all. This is the most trivial example.*
2. *The generalized cylinder and the sphere: $\Sigma_t = \mathbb{S}^k(\sqrt{-2kt}) \times \mathbb{R}^{n-k}$ in \mathbb{R}^{n+1} with $1 \leq k \leq n$ and t ranges from $-\infty$ to 0 . The surface shrinks to $\{0\} \times \mathbb{R}^{n-k}$ at time $t = 0$ and becomes extinct.*
3. *The curves $\Sigma_t = (s, -\log \cos s + t)$ in \mathbb{R}^2 , $s \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $t \in \mathbb{R}$. This curve moves by translating upward in \mathbb{R}^2 . We can generalize this example to higher dimension and find rotationally symmetric solution which move by translating along the x_{n+1} direction in \mathbb{R}^{n+1} .*

For the examples given above, the shape of the surface is the same along the flow. Unfortunately, in the general case, the behavior is complicated. Usually, we cannot express the solution explicitly and singularities may occur due to the nonlinearity of the differential equation.

1.1.1 Properties of mean curvature flow

For the one dimensional curve in \mathbb{R}^2 , the mean curvature flow is also called the curve shortening flow. Grayson[28] proves the following:

Theorem 1.1.2 (Grayson[28]). *Let $C(\cdot, 0) : \mathbb{S}^1 \rightarrow \mathbb{R}^2$ be a smooth embedded curve in the plane. Then $C : \mathbb{S}^1 \times [0, T)$ exists, satisfying*

$$\partial_t C = kN, \tag{1.1.3}$$

where k is the curvature of C , and N its unit inward normal vector. $C(\cdot, t)$ is smooth for all t , it converges to a point as $t \rightarrow T$, and its limiting shape as $t \rightarrow T$ is a round

circle, with convergence in the C^∞ norm.

The result is established by showing that embedded curves will first become convex at some time. According to Gage and Hamilton[26], they then will become round and extinct at a point. This completely characterizes the behavior of the 1-dimensional closed embedded case.

For the higher dimensional case, the behavior of the flow is more complicated. Since the flow equation is a nonlinear heat equation, one of the most important tools for the study of MCF is the parabolic maximum principle. There are some properties which are preserved under the flow. The reader may refer to [21]:

1. If two surfaces are disjoint, they remain disjoint under the flow.
2. If the original surface is embedded, the surface remains embedded.
3. If the original surface is mean convex, i.e. $H \geq 0$, the surface remains mean convex.
4. If the original surface is convex, it remains convex.

For the last case, the behavior of the flow is similar to the one dimensional case. Huisken[34] generalizes the result of Gage and Hamilton[26] to the higher dimension.

Theorem 1.1.3 (Huisken[34]). *If Σ_0^n , $n \geq 2$, is smooth, convex and compact without boundary, then the mean curvature flow Σ_t exists on a maximal finite time interval $[0, T)$ and Σ_t converges to a single point as $t \rightarrow T$. The surfaces Σ_t converge to a round sphere after appropriate rescaling.*

For general non-convex hypersurfaces, even though the surface is embedded, the singularities may occur before the flow becomes extinct. The topology may change when a singularity occurs. Therefore, the study of the singularity is important in MCF.

1.1.2 Huisken's monotonicity formula

Huisken's monotonicity formula is a powerful tool to study what happens before the singularity. Define Φ on $\mathbb{R}^{n+1} \times (-\infty, 0)$ by

$$\Phi(x, t) = (-4\pi t)^{-\frac{n}{2}} e^{\frac{|x|^2}{4t}}. \quad (1.1.4)$$

Φ is the backward heat kernel defined on \mathbb{R}^n and extended to \mathbb{R}^{n+1} . Huisken proves the following theorem

Theorem 1.1.4 (Huisken[33]). *If Σ_t satisfies the mean curvature flow, then we have the formula*

$$\frac{d}{dt} \int_{\Sigma_t} \Phi(x, t) d\sigma_t(x) = - \int_{\Sigma_t} \Phi(x, t) \left| H N + \frac{x^\perp}{2t} \right|^2 d\sigma_t(x). \quad (1.1.5)$$

where $d\sigma_t(x)$ is the surface element on Σ_t .

Note that if Σ_t is flowing by MCF, for all constant $\mu > 0$, the parabolic scaling $\mu\Sigma_{\mu^{-2}t}$ is also a solution of MCF. This quantity $\int_{\Sigma_t} \Phi(x, t) d\sigma_t(x)$ is invariant under the parabolic scaling, that is, $\int_{\Sigma_t} \Phi(x, t) d\sigma_t(x) = \int_{\mu\Sigma_{\mu^{-2}t}} \Phi(x, \mu^{-2}t) d\sigma_t(x)$.

The first corollary we can get from this theorem is that the quantity $\int_{\Sigma_t} \Phi(x, t) d\sigma_t(x)$ is non-increasing in time. As the time goes on, the surface will have less and less $\int_{\Sigma_t} \Phi(x, t) d\sigma_t(x)$ value. Therefore, the bound of the initial integration gives a restriction of the first singularity.

Without loss of generality, we can translate the first singularity to the origin of $\mathbb{R}^{n+1} \times \mathbb{R}$. According to Huisken's monotonicity formula, all the parabolic scaling with $\mu > 1$ will give us a mean curvature flow with less $\int_{\Sigma_t} \Phi(x, t) d\sigma_t(x)$ value and with the first singularity occurring at the origin of the product space. Together with Brakke's weak compactness theorem for MCF, there exists a sequence of $\mu_i \rightarrow \infty$ such that $\mu_i\Sigma_{\mu_i^{-2}t}$ will converge to a limiting flow Σ_t^∞ . This flow is called a tangent flow at the origin and the integration of the backward heat kernel is constant on the flow. This gives us information about the singularity.

Remark 1.1.5. *If we take a different sequence of $\hat{\mu}_i \rightarrow \infty$, the problem whether they will converges to the same tangent flow is still open. This is the uniqueness of the tangent flow problem.*

1.2 Self-shrinkers

For a tangent flow at a singularity at the origin, the integration of the backward heat kernel is constant. From Huisken's formula, we can conclude that

$$HN + \frac{x^\perp}{2t} = 0 \tag{1.2.1}$$

everywhere. At any time, note that $t < 0$, the flow $\partial_t x = -HN = \frac{x^\perp}{2t}$ is moving by scaling inward with respect to the origin and reparametrization on the surface.

If we consider the flow which moves by scaling with respect to the origin and has a singularity at the origin of the product space, it should be of the form $f(t)\Sigma$. For this type of solution, the space and the time variable in the mean curvature flow equation can be separated and solved independently. Put this into the flow equation, we have

$$f'(t)x^\perp(t_0) = \frac{\partial x}{\partial t} = -HN = -H(t_0)f(t)^{-1}N(t_0). \tag{1.2.2}$$

Therefore, the scaling function satisfies

$$\frac{f'(t)}{f'(t_0)} = \frac{f(t_0)}{f(t)}. \tag{1.2.3}$$

If we want to find solutions with a singularity at $t = 0$, the solution will be $f = \sqrt{Ct}$ for some C . When $C > 0$, the solution is defined for $t > 0$. It expands as time goes on. We are more interested in the case $C < 0$. In this case, the solution is defined for $t < 0$ and can be expressed as

$$\Sigma_t = \sqrt{-t}\Sigma_{-1}. \tag{1.2.4}$$

Since the scaling factor in time is fixed for all solutions, we only need the infor-

maison of Σ_{-1} to describe the flow. The hypersurface Σ_{-1} satisfies

$$H = \frac{\langle x, N \rangle}{2}. \quad (1.2.5)$$

This equation is called the self-shrinker equation, which express the $t = -1$ surface of a solution which shrinks with respect to the origin and becomes extinct at time $t = 0$. If we consider the equation satisfied by the tangent flow, we will see the tangent flow at a singularity is a self-shrinker. Therefore, self-shrinkers are models of the singularities.

In the previous examples of mean curvature flow, generalized cylinders and the sphere flow by scaling with respect to the origin. The plane \mathbb{R}^n which passes through the origin is minimal and thus it is fixed in the mean curvature flow. It is also a cone which is invariant under the scaling with respect to the origin. Therefore, the minimality and invariance under scaling make it satisfies the self-shrinker equation.

1.2.1 Classification of self-shrinkers

For 1-dimensional self-shrinkers in \mathbb{R}^2 , Abresch and Langer[2] completely characterized the closed solutions in the following theorem.

Theorem 1.2.1 ([2]). *Let $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}^2$ be a unit speed closed curve representing a homothetic solution of the curve shortening flow. Then γ is an n -covered circle γ_n , or γ is a member of the family of transcendental curves $\{\gamma_{n,m}\}$ having the following description: if n and m are positive integers satisfying $\frac{1}{2} < \frac{n}{m} < \frac{\sqrt{2}}{2}$, there is (up to congruence) a unique unit speed curve $\gamma_{n,m} : \mathbb{S}^1 \rightarrow \mathbb{R}^2$ having rotation index n and closing up in m periods of its curvature function $k > 0$, a solution to the equations*

$$B'' + 2\omega^2(e^B - 1) = 0, \quad B = \frac{2 \log k}{\omega}, \quad (1.2.6)$$

for some constant ω .

If (r, θ) are polar coordinates with origin at the center of mass of $\gamma_{n,m}$, then k and r are related by $k = Ce^{\frac{1}{2}\omega^2 r^2}$ for some constant C .

Since the rotation index n is always greater than 1, the curves $\gamma_{n,m}$ are not embedded. As a result, circles are the only closed, embedded solutions.

It is worth mentioning that Halldorssen[30] classifies all the curves which move by scaling(shrinking or expanding), rotation or a combination of scaling and rotation. The same author[31] also classifies the self-similar solution for the curve shortening flow in the Minkowski plane $\mathbb{R}^{1,1}$.

For higher dimensional cases, in \mathbb{R}^3 , Angenent[3] discovered a well-known solution, “the Angenent’s doughnuts”, which is an embedded rotationally symmetric solution with the topology of a torus. This example can be generalized to any higher dimensions as an embedding of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ into \mathbb{R}^{n+1} with rotational symmetry in the first n coordinate. Møller[44] constructs more closed embedded solutions. For noncompact examples, the reader can refer to Kleen and Møller[36]. These examples have different topology and make the complete classification of self-shrinkers almost impossible.

In what follows, we list some of the most important literature concerning the classification of the self-shrinkers. First, Huisken[34] classifies the self-shrinkers under the condition of nonnegative mean curvature, bounded second fundamental form and polynomial volume growth.

Theorem 1.2.2 (Huisken[34]). *If Σ is a smooth self shrinker in \mathbb{R}^{n+1} , with nonnegative mean curvature $H \geq 0$, then Σ is one of the following:*

1. \mathbb{S}^n ,
2. $\mathbb{S}^{n-m} \times \mathbb{R}^m$,
3. $\gamma \times \mathbb{R}^{n-1}$,

where γ is one of the homothetically shrinking curves in \mathbb{R}^2 found by Abresch and Langer in Theorem 1.2.1.

We include the proof of the compact case here. This illustrates the most basic idea in the classification. First, we introduce the drift Laplacian \mathcal{L} , which is defined as

$$\mathcal{L}f = e^{\frac{|x|^2}{4}} \operatorname{div}(e^{-\frac{|x|^2}{4}} \nabla f) = \Delta f - \frac{1}{2} \langle x^T, \nabla f \rangle. \quad (1.2.7)$$

This operator is self-adjoint in the Gaussian weighted space. For functions u, v which are controlled at infinity such that we can do the integration by parts, we have

$$\int_{\Sigma} u(\mathcal{L}v)e^{-\frac{|x|^2}{4}} d\sigma = - \int_{\Sigma} \langle \nabla u, \nabla v \rangle e^{-\frac{|x|^2}{4}} d\sigma = \int_{\Sigma} (\mathcal{L}u)v e^{-\frac{|x|^2}{4}} d\sigma \quad (1.2.8)$$

The following proof is adapted from [34], [33].

Proof. On the compact surface Σ , let e_1, e_2, \dots, e_n be an orthonormal frame on Σ . Let $A = (h_{ij})$ be the second fundamental form. We can differentiate the equation (1.2.5) and get the following

$$\begin{aligned} \mathcal{L}H &= \frac{1}{2}H - H|A|^2, \\ h_{ij}\nabla_i\nabla_j H &= \frac{1}{2}|A|^2 - H\text{tr}(A^3) + \frac{1}{2}\langle x, e_l \rangle h_{ij}\nabla_l h_{ij}. \end{aligned} \quad (1.2.9)$$

From Simon's identity, we have

$$\mathcal{L}|A|^2 = -2|A|^4 + |A|^2 + 2|\nabla A|^2. \quad (1.2.10)$$

Since $H \geq 0$, we can compute the drift Laplacian of $\frac{|A|^2}{H^2}$.

$$\mathcal{L}\left(\frac{|A|^2}{H^2}\right) = 2\left|\nabla\frac{A}{H}\right|^2 - 2\left\langle\frac{\nabla H}{H}, \nabla\frac{|A|^2}{H^2}\right\rangle. \quad (1.2.11)$$

Move all the terms with $\frac{|A|^2}{H^2}$ to the same side, multiply the equation by $H^2e^{-\frac{|x|^2}{4}}$ and integrate. We have

$$\begin{aligned} \int_{\Sigma} 2\left|\nabla\frac{A}{H}\right|^2 H^2 e^{-\frac{|x|^2}{4}} d\sigma &= \int_{\Sigma} \left[\mathcal{L}\left(\frac{|A|^2}{H^2}\right) + 2\left\langle\frac{\nabla H}{H}, \nabla\frac{|A|^2}{H^2}\right\rangle \right] H^2 e^{-\frac{|x|^2}{4}} d\sigma \\ &= \int_{\Sigma} \mathbf{div}\left[H^2 e^{-\frac{|x|^2}{4}} \nabla\left(\frac{|A|^2}{H^2}\right) \right] d\sigma = 0. \end{aligned} \quad (1.2.12)$$

Therefore, we can conclude $\nabla\frac{A}{H} \equiv 0$, the tensor $\frac{A}{H}$ is parallel. We have $|h_{ij}\nabla_l H - \nabla_l h_{ij}H|^2 \equiv 0$. From the Codazzi equation and the curvature tensor vanishes identically in \mathbb{R}^{n+1} , $|h_{ij}\nabla_l H - h_{il}\nabla_j H|^2 \equiv 0$. Now, choose the orthonormal frame such

that e_1 is parallel to ∇H . In this case, $\nabla_j H = 0$ for all $j > 1$. Therefore,

$$0 = |h_{ij}\nabla_l H - h_{il}\nabla_j H|^2 = 2|\nabla H|^2(|A|^2 - \sum_{i=1}^n h_{1i}^2). \quad (1.2.13)$$

We have either $|\nabla H| = 0$ or $|A|^2 = \sum_{i=1}^n h_{1i}^2$. For the case that $|\nabla H| = 0$, the mean curvature is constant and therefore it can only be a sphere in the compact case. If $|A|^2 = \sum_{i=1}^n h_{1i}^2$, it is only possible when $h_{ij} = 0$ except $(i, j) = (1, 1)$. We can deduce that $|A|^2 = H^2$. Integrating equation (1.2.9) on Σ , we have

$$\begin{aligned} \int_{\Sigma} H^3 d\sigma &= \frac{1}{2} \int_{\Sigma} H + \langle x^T, \nabla H \rangle d\sigma \\ &= \frac{1}{2} \int_{\Sigma} H d\sigma - \frac{n}{2} \int_{\Sigma} H d\sigma + \int_{\Sigma} \frac{\langle x, N \rangle}{2} H^2 d\sigma \\ &= \frac{1}{2} \int_{\Sigma} H d\sigma - \frac{n}{2} \int_{\Sigma} H d\sigma + \int_{\Sigma} H^3 d\sigma. \end{aligned} \quad (1.2.14)$$

Therefore, we have $\frac{n-1}{2} \int_{\Sigma} H d\sigma = 0$, which is impossible for compact Σ when $n > 1$ and $H \geq 0$. \square

For the noncompact case, there are more technical details involved to do the integration by parts. In that case, it is possible for the mean curvature H not to be a constant. This corresponds to the case 3 in the classification. In Huisken's proof for noncompact manifold, some bounds of the growth rate of the second fundamental form and the volume are assumed. Later, Colding and Minicozzi[21] remove the requirement of the second fundamental form bound for the classification result. They also show that the generalized cylinders are generic solutions in the sense that one can perturb a flow to make all the singularities of this type.

If a self-shrinker is also an entire graph over \mathbb{R}^n , Ecker and Huisken prove the following:

Proposition 1.2.3 (Ecker and Huisken[24]). *If Σ is an entire graph of at most polynomial growth satisfying the self-shrinker equation (1.2.5), then Σ is a plane.*

Later, Wang[53] removes the assumption of growth rate.

For the genus 0 surfaces in \mathbb{R}^3 , Simon Brendle establishes the following theorem for the compact case.

Theorem 1.2.4 (Brendle[8]). *Let Σ be a compact embedded self-shrinker in \mathbb{R}^3 of genus 0. Then Σ is a round sphere.*

And the following for the non-compact case

Theorem 1.2.5 (Brendle[8]). *Suppose that Σ is a properly embedded self-shrinker in \mathbb{R}^3 with the property that any two loops in Σ have vanishing intersection number mod 2. Then Σ is a round sphere or a cylinder or a plane.*

1.3 λ -hypersurfaces

Now, we consider the λ -hypersurface equation. Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a hypersurface satisfying

$$H = \frac{\langle x, N \rangle}{2} + \lambda, \quad (1.3.1)$$

where N is the normal vector on the surface, H is the mean curvature and λ is a constant. Our goal is to describe the behavior of the 1-dimension solutions in \mathbb{R}^2 .

1.3.1 Gaussian isoperimetric problem

The equation (1.3.1) is first studied by McGonagle and Ross[43] and is named as λ -hypersurface in the work of Cheng and Wei[17]. The equation arises in the Gaussian isoperimetric problem: In \mathbb{R}^{n+1} , the weighted Gaussian volume element is given by $dV_\mu = \exp(-\frac{|x|^2}{4})dV$, where dV is the volume element induced by the Euclidean metric. For the case that Σ is closed and $\Sigma = \partial\Omega$ for some bounded region $\Omega \subset \mathbb{R}^{n+1}$, let the r -neighborhood of Ω , $\Omega_r = \{x \in \mathbb{R}^{n+1} | \text{dist}(x, \Omega) < r\}$. The boundary measure is defined by

$$P_\mu = \liminf_{r \rightarrow 0} \frac{V_\mu(\Omega_r) - V_\mu(\Omega)}{r}. \quad (1.3.2)$$

This measures the relative rate of change of Gaussian volume for small change from Ω to Ω_r . When Σ is smooth, the boundary measure $P_\mu(\Omega) = A_\mu(\Sigma) = \int_\Sigma d\sigma_\mu$, where $d\sigma_\mu$

is the weighted Gaussian area element defined by $d\sigma_\mu = \exp(-\frac{|x|^2}{4})d\sigma$. The Gaussian isoperimetric problem asks: Among all regions with the same weighted volume V_0 , which one has the least weighted boundary area? The answer to this problem is given by Borel[7], Sudakov and Tsirel'son[51]: The half space minimizes the weighted boundary area.

The problem above can be considered locally as follows. Let $F : \Sigma \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^{n+1}$ be a smooth, normal variation which fixed the weighted volume, $\partial_t F(x, 0) = uN$ when $t = 0$. We have

$$\begin{aligned} \frac{d}{dt} V_\mu(\Omega)|_{t=0} &= \int_{\Sigma} u e^{-\frac{|x|^2}{4}} d\sigma, \\ \frac{d}{dt} A_\mu(\Sigma)|_{t=0} &= \int_{\Sigma} u \left(H - \frac{\langle x, N \rangle}{2} \right) e^{-\frac{|x|^2}{4}} d\sigma. \end{aligned} \tag{1.3.3}$$

Let Σ be a surface minimizing the weighted boundary area among all surfaces enclosing the same weighted volume. Because of the minimality, the surface is a critical point for all variations that fix the enclosed weighted volume. We can deduce the equation (1.3.1). This equation is defined on Σ locally and it can be studied even if Σ does not enclose a region. The solutions can be thought of as the critical points to the weighted area functional. In McGonagle and Ross' work[43] they show the hyperplane is the only stable smooth solution to the Gaussian isoperimetric problem in terms of the second derivative of the weighted Gaussian area functional.

Remark 1.3.1. *In the special case $\lambda = 0$, the equation becomes the original shrinker equation (1.2.5). This comes from the fact that the self-shrinkers are also the critical points of the weighted area functional under any variation, not only the volume preserving ones, in Gaussian space. Therefore, we call equation (1.3.1) the λ -self shrinker equation in my work.*

Remark 1.3.2. *λ -hypersurface also arises in other studies. For example, it plays an important role in the study of weighted volume-preserving mean curvature flow by Cheng and Wei[17].*

Example 1.3.3. *The following are λ -hypersurfaces in \mathbb{R}^{n+1} :*

1. The hyperplane \mathbb{R}^n which is λ away from the origin in the normal direction.
2. The cylinder $\mathbb{S}^k(\sqrt{\lambda^2 + 2k} - \lambda) \times \mathbb{R}^{n-k}$.
3. The sphere $\mathbb{S}^n(\sqrt{\lambda^2 + 2n} - \lambda)$.

The examples above admit properties such as polynomial volume growth and constant mean curvature. They also admit good symmetry. It is important to investigate under which assumption we can deduce that a λ -hypersurface is one of the above. Some rigidity results can be found in [16], [18], [29] and [43].

1.3.2 Classification of λ -hypersurfaces

The problem of classification of λ -hypersurfaces can be regarded as a generalization of the case of self-shrinkers. The $H \geq 0$ case in self-shrinkers is now replaced by $H - \lambda \geq 0$ and discussed in the work of Cheng and Wei[17]. However, this is not enough to guarantee the round solution. Further quantities arise in the differentiation, and we also need the condition about the following quantities:

$$\begin{aligned}
 |A|^2 &= \sum_{i,j} h_{ij}^2, \\
 \text{tr}A^3 &= \sum_{i,j,k} h_{ij}h_{jk}h_{ki},
 \end{aligned}
 \tag{1.3.4}$$

where h_{ij} is the second fundamental form corresponding to an orthonormal frame. Now, the result of the classification is

Theorem 1.3.4 (Cheng, Wei[17]). *Let Σ be an n -dimensional embedded λ -hypersurface in \mathbb{R}^{n+1} , either compact or complete with polynomial area growth. If $H - \lambda \geq 0$ and $\lambda(2\text{tr}A^3(H - \lambda) - |A|^2) \geq 0$, then Σ is one of the standard round solutions in example 1.3.3.*

The proof is similar to the self-shrinker case. In the λ -hypersurface case, the drift

Laplacian of the curvature terms are given by

$$\begin{aligned}\mathcal{L}H &= \frac{1}{2}H + |A|^2(\lambda - H), \\ \mathcal{L}|A|^2 &= |A|^2 - 2|A|^4 + 2|\nabla A|^2 + 2\lambda\text{tr}A^3, \\ \mathcal{L}\frac{1}{(H-\lambda)^2} &= 6\frac{|\nabla(H-\lambda)|^2}{(H-\lambda)^4} + \frac{2|A|^2(H-\lambda) - H}{(H-\lambda)^3}.\end{aligned}\tag{1.3.5}$$

The proof is established by replacing $\frac{|A|^2}{H^2}$ in the proof of the self-shrinker case by $\frac{|A|^2}{(H-\lambda)^2}$. There will be an extra term involving $\lambda(2\text{tr}A^3(H-\lambda) - |A|^2)$, therefore, we need a further condition to control the sign of this term.

Remark 1.3.5. *We cannot remove the condition $\lambda(2\text{tr}A^3(H-\lambda) - |A|^2) \geq 0$. My work, which will be introduced in the next section, shows that when $\lambda < 0$ there are 1-dimensional solutions in \mathbb{R}^2 which are not the standard circle.*

1.4 My results

My work focuses on the behavior of the equation (1.3.1) in \mathbb{R}^2 . In what follows, to simplify the equation, we scale the curve by a factor of $\sqrt{2}$ to make the constant $\frac{1}{2}$ become 1 and use the 1-dimensional curvature k in place of mean curvature H . The equation becomes

$$k = -\langle x, N \rangle + \lambda.\tag{1.4.1}$$

Generally, the behavior of the solution is described by the following theorem. We state the theorem in a way similar to the theorem given by Abresch and Langer so that the reader can make a comparison.

Theorem 1.4.1. *The curvature function $k > 0$ of the λ -curves satisfies*

$$B'' = 2(1 + \lambda e^{\frac{B}{2}} - e^B), \quad B = 2 \log k.\tag{1.4.2}$$

Also, let r be the distance to the origin. Then k and r are related by $k = Ce^{\frac{1}{2}r^2}$ for some constant C .

If n and m are positive integers satisfying

$$\min\left\{\frac{1}{2}, \frac{\sqrt{2}}{2} \sqrt{\frac{\lambda}{\sqrt{\lambda^2+4}} + 1}\right\} < \frac{n}{m} < \max\left\{\frac{1}{2}, \frac{\sqrt{2}}{2} \sqrt{\frac{\lambda}{\sqrt{\lambda^2+4}} + 1}\right\}, \quad (1.4.3)$$

then there is a closed curve $\gamma_{n,m} : \mathbb{S}^1 \rightarrow \mathbb{R}^2$ having rotation index n and closing up in m periods of its curvature function $k > 0$, which is a solution to the equation (1.4.1).

Remark 1.4.2. In the theorem above, we give a sufficient condition of existence of solutions. The condition may not be a necessary condition. Therefore, in each of the following results, further work is needed.

Remark 1.4.3. For the case $k < 0$, we can choose the normal vector N to be the opposite. In that case, it would correspond to a solution with the curvature replaced with $-k$ and the λ replaced with $-\lambda$. More details will be given in section 2.1.

1.4.1 The case for $\lambda < 0$

From this theorem, for $\lambda < \frac{-7}{2\sqrt{2}}$, there are solutions with $n = 1$ and therefore embedded.

Theorem 1.4.4. For $\lambda < \frac{-7}{2\sqrt{2}}$, there exists an embedded solution. The embedded solution admit m -symmetry for some $m > 2$.

Remark 1.4.5. From the behavior of the differential equation, we are able to extend the range for λ . Actually, there exist $\delta > 0$ such that there is such embedded solution for $\lambda \leq \frac{-7}{2\sqrt{2}} + \delta$.

For the embedded solution with 2-symmetry, it is subtle because $\frac{1}{2}$ is either the lower bound or the upper bound of $\frac{n}{m}$, so the theorem 1.4.1 cannot guarantee the existence of an embedded solution with change of angle exactly π in a period. Further detail for the behavior of the differential equation when energy is near infinity is needed to establish the existence of solution with 2-symmetry.

Theorem 1.4.6. For $\frac{-2}{\sqrt{3}} < \lambda < 0$, there exists an embedded solution with 2-symmetry.

Therefore, for certain negative λ , there exists embedded solutions other than the circle.

Unlike the result of Abresch and Langer[2] that for the $\lambda = 0$ case, the circle is the only closed embedded solution, we surprisingly find other embedded solutions. This affects the understanding of the rigidity problem about the classification of λ -hypersurfaces. If we product the curve with \mathbb{R}^{n-1} , we obtain a λ -hypersurface in \mathbb{R}^{n+1} which is topologically $S^1 \times \mathbb{R}^{n-1}$ with non-vanishing mean curvature and polynomial area growth. However, this is not the standard round cylinder as in example 1.3.3. This is the λ -hypersurface analogue of the case 3 in theorem 1.2.2.

We can also compare the result with the isoperimetric problem in Euclidean space, where the critical surface to the area functional should admit constant mean curvature. Thus the only 1-dimensional solutions of the isoperimetric problem in \mathbb{R}^2 are circles. However, the embedded solutions, which are the critical surface in the Gaussian isoperimetric problem, can be other than circles.

1.4.2 The case for $\lambda \geq 0$

For positive λ , the behavior is similar to the self-shrinking curves. Since the theorem 1.4.1 gives only a sufficient condition, we need to compare the change of angle with the self-shrinkers and use Abresch and Langer's result to rule out the possibility of embedded solutions.

Theorem 1.4.7. *When $\lambda > 0$, there are no embedded solutions to the equation (1.4.1) with $k > 0$.*

Remark 1.4.8. *It is worth mentioning that Guang[29] establishes the same result as in theorem 1.4.7 with a different proof. He considers the part of the curve where the curvature decreases from the maximum to the minimum.*

Chapter 2

1-dimensional λ -self shrinkers

In this chapter, we are going to establish the result of 1-dimensional λ -self shrinkers. This chapter will be structured as follows:

In section 1, starting from the defining equation, we derive an ODE system for the 1-dimensional λ -self shrinkers. The approach used here is similar to that in the work of Halldorsson[30]. This is a powerful tool to study the curves in \mathbb{R}^2 , which describe the curve by its tangent component and normal component.

In section 2, we define the energy η for a solution and analyze the behavior of the solution for the extreme cases: The energy is near the minimum and the energy is near infinity. We first get the change of angle at the both extreme case and use continuity to establish the theorem 1.4.1. As mentioned before, we need more detail when the energy is near the minimum and near infinity to establish theorem 1.4.6 and 1.4.4.

In section 3, we fix the relative energy and find the relation between $\Delta\theta$ and λ . We can compare the change of angle with the case of self-shrinker in the work of Abresch and Langer[2] and establish theorem 1.4.7.

In section 4, we use Matlab to get numerical solutions which approximately solve the equation. Some pictures of the curves are provided for better understandings of the behavior for each of the different cases in the main theorems. We also give some conjectures about the behavior of the solution here.

2.1 Setting up the ODE system

For a curve $x(s) \in \mathbb{R}^2$ parametrized by arc length s , we have

$$\begin{aligned}\frac{d}{ds}x &= T, \\ \frac{d}{ds}T &= kN,\end{aligned}\tag{2.1.1}$$

where T and N are the tangent vector and the normal vector of the curve, respectively. Note that for any curve in \mathbb{R}^2 , we have two possible choices of N : either rotate T clockwise by $\frac{\pi}{2}$ or $-\frac{\pi}{2}$. If we let $N^- = -N$, $k^- = -k$, we have $kN = k^-N^-$. Therefore, we have

$$k^- = -k = \langle x, N \rangle - \lambda = -\langle x, N^- \rangle - \lambda.\tag{2.1.2}$$

This tells us that selecting the opposite normal vector will change the sign of k and result in a solution corresponding to $-\lambda$.

Using the method as in Halldorsson's work[30], we decompose the position vector x into the tangent part and the normal part. The curve can be reconstructed by these data up to a rotation. Let $\tau = \langle x, T \rangle$, $\nu = \langle x, N \rangle$. We can obtain the ODE system

$$\begin{cases} \frac{d\tau}{ds} = 1 + k\nu = 1 - \nu^2 + \lambda\nu, \\ \frac{d\nu}{ds} = -k\tau = \nu\tau - \lambda\tau. \end{cases}\tag{2.1.3}$$

The equilibrium is the point where $\frac{d}{ds}\tau = \frac{d}{ds}\nu = 0$. They are given by $(0, \nu_{\pm}^{\circ})$, where

$$\nu_{\pm}^{\circ} = \frac{\lambda \pm \sqrt{\lambda^2 + 4}}{2}\tag{2.1.4}$$

are the positive and the negative solutions of the equation $\nu^2 - \lambda\nu - 1 = 0$, respectively.

At the equilibrium, the curvature is a nonzero constant. It corresponds to the circle centered at the origin. For $(0, \nu_+^{\circ})$, it is a circle of radius ν_+° with the normal pointed outward and $k < 0$. For $(0, \nu_-^{\circ})$, it is a circle of radius $-\nu_-^{\circ} = |\nu_-^{\circ}|$ with the normal pointed inward and $k > 0$. Also, note that $(\tau, \nu) = (s, \lambda)$ is a solution which

corresponds to a line with the minimum distance to the origin equal to λ . From now on, without loss of generality, we only consider solutions with $k \geq 0$. They are the solutions with the trajectory contained in the half plane $\{\nu \leq \lambda\}$ of $\tau - \nu$ space. For the solutions with $k < 0$, choose the opposite normal vector and study them as the solutions corresponding to $-\lambda$ with positive k .

2.1.1 Periodicity of the solution

For a solution to the system, the function

$$F(\tau, \nu) = (\lambda - \nu) \exp\left(-\frac{\nu^2 + \tau^2}{2}\right) \quad (2.1.5)$$

is positive in the $\{\nu \leq \lambda\}$ half plane. Differentiating it with respect to s , we have

$$\begin{aligned} \frac{d}{ds}F &= \left(-\frac{d}{ds}\nu - (\lambda - \nu)\left(\nu\frac{d}{ds}\nu + \tau\frac{d}{ds}\tau\right)\right) \exp\left(-\frac{\nu^2 + \tau^2}{2}\right) \\ &= \left(-(\nu - \lambda)\tau + (\nu - \lambda)\left(\nu(\nu - \lambda)\tau + \tau(1 - \nu(\nu - \lambda))\right)\right) \exp\left(-\frac{\nu^2 + \tau^2}{2}\right) \\ &= 0. \end{aligned} \quad (2.1.6)$$

The trajectory of each solution lies in a level set of F . Since each level set of F is a simple closed curve except the level set $\{F = 0\}$, which corresponds to the line mentioned before, we have a uniform lower bound of the speed of $(\tau(s), \nu(s))$ curve away from 0 on each level set. Therefore, the solution $(\tau(s), \nu(s))$ should be periodic in s .

Remark 2.1.1. *Note that if $x(s)$ is periodic, then τ, ν are periodic. But the converse may not be true. Starting from a periodic solution of $(\tau(s), \nu(s))$, the resulting $x(s)$ is periodic only when the change of angle in a period can be expressed as $\frac{2n\pi}{m}$, where n, m are positive integers. In this case, the period of $x(s)$ is m times the period of $(\tau(s), \nu(s))$, and it will be a closed solution.*

2.1.2 Change of angle in a period

Now, since k is more directly related to the geometric behavior than ν , we use (τ, k) as the variable instead of (τ, ν) . Plugging $\nu = \lambda - k$ into the previous ODE system, it becomes

$$\begin{cases} \frac{d\tau}{ds} = 1 + \lambda k - k^2, \\ \frac{dk}{ds} = k\tau, \end{cases} \quad (2.1.7)$$

and $\{\nu \leq \lambda\}$ becomes $\{k \geq 0\}$ in $\tau - k$ plane. Note that after the change of variable, we still have the equilibrium at $(0, k_{\pm}^{\circ})$, where $k_{\pm}^{\circ} = \nu_{\pm}^{\circ}$ because they satisfy exactly the same equation. However, $(0, k_{\pm}^{\circ})$ correspond to $(0, \nu_{\mp}^{\circ})$, respectively. The $\nu = \lambda$ line in $\tau - \nu$ space now becomes $k = 0$ line in $\tau - k$ space. Also, the function F becomes

$$F = ke^{-\frac{|x|^2}{2}}. \quad (2.1.8)$$

From now on, we will work in the $\{k > 0\}$ half space. For simplicity, write $k^{\circ} = k_{+}^{\circ}$ and we will not use k_{-}° anymore. Let $B = 2 \log k$. We have $\frac{dB}{ds} = 2\tau$ and

$$\frac{d^2 B}{ds^2} = 2 \frac{d\tau}{ds} = 2 + 2k(\lambda - k) = 2 + 2\lambda e^{\frac{B}{2}} - 2e^B. \quad (2.1.9)$$

Remark 2.1.2. *This equation can also be derived from the λ -hypersurface analogue of equation (1.2.9). If we apply the Laplacian to the scaled equation*

$$H = \langle x, N \rangle - \lambda, \quad (2.1.10)$$

the result would be

$$\Delta H = H + \langle x^T, \nabla H \rangle - H|A|^2 - \lambda H^2. \quad (2.1.11)$$

In 1-dimensional case, together with the knowledge of $x^T = \tau T$ we can get the same second order equation. We derive the equation from the 1-dimensional frame τ, ν to emphasise the geometric structure of a curve.

Multiplying both sides by $\frac{dB}{ds}$ and integrating with respect to s , we get

$$\frac{1}{2}\left(\frac{dB}{ds}\right)^2 + 2e^B - 4\lambda e^{\frac{B}{2}} - 2B = -4\log F - 2\lambda^2. \quad (2.1.12)$$

If we define $F_\lambda = F \cdot \exp \frac{\lambda^2}{2}$, $V(B) = e^B - 2\lambda e^{\frac{B}{2}} - B$, the equation becomes

$$\frac{1}{2}\left(\frac{dB}{ds}\right)^2 + 2V(B) = -4\log F_\lambda. \quad (2.1.13)$$

The minimum of $V(B)$ is attained when $\frac{dV}{dB}(B) = 0$. $e^B - \lambda e^{\frac{B}{2}} - 1 = 0$. $e^{\frac{B}{2}} = k^\circ$. This corresponds to the equilibrium at $(0, k^\circ)$ and $\min V(B) = -\lambda k^\circ - 2\log k^\circ + 1$. Now, since the value $(\frac{dB}{ds})^2$ is always nonnegative, the maximum value $V(B)$ can attain is $-2\log F_\lambda$.

Now, define the energy η of the curve by $\eta = -2\log F_\lambda$. The range of the energy is from $\min V(B)$ to infinity. For any η in this range, we can find the solutions $B_\eta^- < B_\eta^+$ of $V(B) = \eta$. Considering the differential equation of B , we get

$$\begin{aligned} \frac{1}{2}\left(\frac{dB}{ds}\right)^2 + 2V(B) &= 2\eta, \\ \frac{dB}{ds} &= \pm 2\sqrt{\eta - V(B)}. \end{aligned} \quad (2.1.14)$$

Therefore, the length of the curve in a period is given by

$$\oint ds = 2 \int_{B_\eta^-}^{B_\eta^+} \left(\frac{dB}{ds}\right)^{-1} dB = \int_{B_\eta^-}^{B_\eta^+} \frac{1}{\sqrt{\eta - V(B)}} dB, \quad (2.1.15)$$

and the change of the angle in a period is given by

$$\Delta\theta = \oint k ds = \int_{B_\eta^-}^{B_\eta^+} \frac{e^{\frac{B}{2}}}{\sqrt{\eta - V(B)}} dB. \quad (2.1.16)$$

In order to simplify the calculation, we can switch the variable back to k . Let $k_\eta^\pm = e^{\frac{B_\eta^\pm}{2}}$, respectively. $V(k) = k^2 - 2\lambda k - 2\log k$ after the change of variable. The change

of angle in a period of curvature, $\Delta\theta$, is given by

$$\Delta\theta = \int_{k_\eta^-}^{k_\eta^+} \frac{2dk}{\sqrt{\eta - V(k)}}. \quad (2.1.17)$$

2.2 The behavior of the solutions

Now, we will focus on the behavior of $\Delta\theta$ when the energy η varies from $\min V$ to ∞ .

2.2.1 The behavior of the solution when η is near $\min V$

The following focuses on the behavior of $\Delta\theta$. When the energy is near the minimum, the behavior is closed to a simple harmonic oscillator.

Lemma 2.2.1. *For any potential function $V \in C^2$, at a local minimum k° with positive second derivative, let k_η^- be the largest solution of $V(k) = \eta$ which is below k° and let k_η^+ be the smallest solution of $V(k) = \eta$ which is above k° . We have*

$$\lim_{\eta \rightarrow V(k^\circ)^+} \int_{k_\eta^-}^{k_\eta^+} \frac{dk}{\sqrt{\eta - V(k)}} = \sqrt{\frac{2}{V''(k^\circ)}}\pi, \quad (2.2.1)$$

where ' denotes $\frac{d}{dk}$

Proof. First, note that for the case in which the potential is quadratic, $V(k) = V(k^\circ) + \frac{V''(k^\circ)(k-k^\circ)^2}{2}$, a simple calculation shows that

$$\int_{k_\eta^-}^{k_\eta^+} \frac{dk}{\sqrt{\eta - V(k)}} = \sqrt{\frac{2}{V''(k^\circ)}}\pi \quad (2.2.2)$$

for any $\eta > V(k^\circ)$ and is independent of η .

For arbitrary potential function $V \in C^2$ and $\epsilon > 0$, there is $\delta > 0$ such that for all $V(k^\circ) < \eta < V(k^\circ) + \delta$, we have $|V''(k) - V''(k^\circ)| < \epsilon$ for $k \in [k_\eta^-, k_\eta^+]$. Let V_\pm be the quadratic functions which pass through (k_η^-, η) , (k_η^+, η) with $V_\pm'' = V''(k^\circ) \mp \epsilon$.

We have $V_- < V < V_+$ in (k_η^-, u_η^+) . Therefore,

$$\begin{aligned} \sqrt{\frac{2}{V''(k^\circ) + \epsilon}} \pi &= \int_{k_\eta^-}^{k_\eta^+} \frac{dk}{\sqrt{\eta - V_-(k)}} \leq \int_{k_\eta^-}^{k_\eta^+} \frac{dk}{\sqrt{\eta - V(k)}} \\ &\leq \int_{k_\eta^-}^{k_\eta^+} \frac{dk}{\sqrt{\eta - V_+(k)}} = \sqrt{\frac{2}{V''(k^\circ) - \epsilon}} \pi. \end{aligned} \quad (2.2.3)$$

Letting ϵ goes to 0 yields the desired result. \square

Proposition 2.2.2. *When $\eta \rightarrow \min V^+$, $\Delta\theta$ approaches $\pi\sqrt{2}\sqrt{\frac{\lambda}{\sqrt{\lambda^2+4}} + 1}$. Moreover, $\Delta\theta$ is decreasing in a neighborhood of $\min V$.*

Proof. Let $\eta \rightarrow \min V^+$. The derivatives of $V(k)$ with respect to k at the minimum point are

$$\begin{aligned} V^{(2)}(k^\circ) &= 2 + 2(k^\circ)^{-2}, \\ V^{(3)}(k^\circ) &= -4(k^\circ)^{-3}, \\ V^{(4)}(k^\circ) &= 12(k^\circ)^{-4}. \end{aligned} \quad (2.2.4)$$

Therefore, from the lemma above and recall that $k^\circ = \frac{\lambda + \sqrt{\lambda^2 + 4}}{2}$, we have

$$\begin{aligned} \lim_{\eta \rightarrow \min V^+} \Delta\theta &= \lim_{\eta \rightarrow \min V^+} \int_{k_\eta^-}^{k_\eta^+} \frac{2dk}{\sqrt{\eta - V(k)}} \\ &= 2\pi \cdot \sqrt{\frac{2(k^\circ)^2}{2(k^\circ)^2 + 2}} = \pi\sqrt{2}\sqrt{\frac{\lambda}{\sqrt{\lambda^2 + 4}} + 1}. \end{aligned} \quad (2.2.5)$$

From the result of Chicone[19], since

$$\begin{aligned} 5(V^{(3)})^2 - 4V^{(2)}V^{(4)} &= 80(k^\circ)^{-6} - 96((k^\circ)^{-6} + (k^\circ)^{-4}) \\ &= -16(k^\circ)^{-6} - 96(k^\circ)^{-4} < 0, \end{aligned} \quad (2.2.6)$$

the function $\Delta\theta$ is decreasing near $\min V$ with respect to η . \square

Remark 2.2.3. *For the case of self shrinkers, we have $\lambda = 0$. The proposition above gives $\Delta\theta \rightarrow \sqrt{2}\pi$, as the result in Abresch and Langer[2]. This function is strictly*

increasing with respect to λ . When λ approaches ∞ , $\Delta\theta$ approaches 2π . When λ approaches $-\infty$, $\Delta\theta$ approaches 0.

2.2.2 The behavior of the solution when η is near infinity

Now, we turn our attention to the behavior of $\Delta\theta$ when the energy approaches infinity. An upperbound of $\Delta\theta$ is given by the following proposition.

Proposition 2.2.4. *For any $L > 1$,*

$$\Delta\theta \leq \pi + 2\left(\lambda - 1 + \sqrt{\frac{L}{L-1}}\right) \frac{1}{\sqrt{\eta}} + o\left(\frac{1}{\sqrt{\eta}}\right) \quad (2.2.7)$$

as η goes to infinity.

Proof. In order to get an upper bound of $\Delta\theta$, separate the integration into two terms.

$$\Delta\theta = \int_{k_{\eta}^-}^{k_{\eta}^+} \frac{2dk}{\sqrt{\eta - V(k)}} = \int_{k_{\eta}^-}^1 \frac{2dk}{\sqrt{\eta - V(k)}} + \int_1^{k_{\eta}^+} \frac{2dk}{\sqrt{\eta - V(k)}}. \quad (2.2.8)$$

When $1 \leq k \leq k_{\eta}^+$, we want to construct a quadratic function which is larger than $V(k)$ in this interval such that the integration can be computed explicitly. Let \bar{k}_{η}^+ be the positive solution of $\eta = k^2 - 2\lambda k$. Note that $\bar{k}_{\eta}^+ < k_{\eta}^+$. The function $k^2 - 2\lambda k$ is quadratic and passes $(1, 1 - 2\lambda)$, we want to modify this function so that it passes (k_{η}^+, η) . Using scaling, let $\bar{V}(k) = \left(\frac{\bar{k}_{\eta}^+ - 1}{k_{\eta}^+ - 1}(k - 1) + 1\right)^2 - 2\lambda\left(\frac{\bar{k}_{\eta}^+ - 1}{k_{\eta}^+ - 1}(k - 1) + 1\right)$. At $k = k_{\eta}^+$, $V(k) = \bar{V}(k) = \eta$. At $k = 1$, $V(k) = \bar{V}(k) = 1 - 2\lambda$. The second derivative of $\bar{V}(k) - V(k)$ is

$$\left(\bar{V}(k) - V(k)\right)'' = 2\left(\frac{\bar{k}_{\eta}^+ - 1}{k_{\eta}^+ - 1}\right)^2 - \left(2 + 2\frac{1}{k^2}\right) < 0. \quad (2.2.9)$$

We can conclude

$$\bar{V}(k) - V(k) \geq 0 \quad (2.2.10)$$

for any $1 < k < k_\eta^+$. Therefore, we have

$$\begin{aligned}
\int_1^{k_\eta^+} \frac{2dk}{\sqrt{\eta - V(k)}} &\leq \int_1^{k_\eta^+} \frac{2dk}{\sqrt{\eta - \bar{V}(k)}} \\
&= \frac{k_\eta^+ - 1}{\bar{k}_\eta^+ - 1} \int_1^{\bar{k}_\eta^+} \frac{2dv}{\sqrt{\eta - v^2 + 2\lambda v}} \\
&= 2 \frac{k_\eta^+ - 1}{\bar{k}_\eta^+ - 1} \left(\frac{\pi}{2} - \sin^{-1} \frac{1 - \lambda}{\sqrt{\eta + \lambda^2}} \right).
\end{aligned} \tag{2.2.11}$$

We need an upper bound for $\frac{k_\eta^+ - 1}{\bar{k}_\eta^+ - 1}$. Starting from $\bar{k}_\eta^+ = \lambda + \sqrt{\lambda^2 + \eta}$, $V(\bar{k}_\eta^+) = \eta - 2 \log \bar{k}_\eta^+$, $V(k_\eta^+) = \eta$ and $V'(k) \geq 2\bar{k}_\eta^+ - 2\lambda - 2\frac{1}{\bar{k}_\eta^+}$ for $\bar{k}_\eta^+ < k < k_\eta^+$, we have

$$k_\eta^+ - \bar{k}_\eta^+ \leq \frac{2 \log \bar{k}_\eta^+}{2\bar{k}_\eta^+ - 2\lambda - 2\frac{1}{\bar{k}_\eta^+}} = \frac{\log \bar{k}_\eta^+}{\bar{k}_\eta^+ - \lambda - \frac{1}{\bar{k}_\eta^+}} \leq \frac{C \log \eta}{\frac{\sqrt{\eta}}{2}} = O(\eta^{-\frac{1}{2}} \log \eta) \tag{2.2.12}$$

for η large enough. Hence,

$$\frac{k_\eta^+ - 1}{\bar{k}_\eta^+ - 1} = 1 + \frac{k_\eta^+ - \bar{k}_\eta^+}{\bar{k}_\eta^+ - 1} = 1 + O(\eta^{-1} \log \eta). \tag{2.2.13}$$

Therefore,

$$\int_1^{k_\eta^+} \frac{2dk}{\sqrt{\eta - V(k)}} \leq 2 \frac{k_\eta^+ - 1}{\bar{k}_\eta^+ - 1} \left(\frac{\pi}{2} - \sin^{-1} \frac{1 - \lambda}{\sqrt{\eta + \lambda^2}} \right) = \pi + 2 \frac{\lambda - 1}{\sqrt{\eta}} + o\left(\frac{1}{\sqrt{\eta}}\right). \tag{2.2.14}$$

Now, we are going to estimate the other term. For all $L > 1$, let $k_{\eta,L}^- = \exp(-\frac{\eta}{2L} + \frac{1}{2} + |\lambda|)$. Note that when η is large enough, $k_{\eta,L}^- < 1$ and $V(k_{\eta,L}^-) < \frac{\eta}{L}$.

$$\int_{k_\eta^-}^1 \frac{2dk}{\sqrt{\eta - V(k)}} = \int_{k_\eta^-}^{k_{\eta,L}^-} \frac{2dk}{\sqrt{\eta - V(k)}} + \int_{k_{\eta,L}^-}^1 \frac{2dk}{\sqrt{\eta - V(k)}}. \tag{2.2.15}$$

For the first term, since $V(k_\eta^-) = \eta$, $V(k_{\eta,L}^-) < \frac{\eta}{L}$, $V'' > 0$, we have

$$V(k) < \frac{\eta}{L} + \left(\frac{L-1}{L}\right) \eta \frac{k - k_{\eta,L}^-}{k_\eta^- - k_{\eta,L}^-} \tag{2.2.16}$$

for $k_{\eta}^{-} < k < k_{\eta,L}^{-}$ and

$$\begin{aligned} \int_{k_{\eta}^{-}}^{k_{\eta,L}^{-}} \frac{2dk}{\sqrt{\eta - V(k)}} &\leq \int_{k_{\eta}^{-}}^{k_{\eta,L}^{-}} \frac{2dk}{\sqrt{\frac{L-1}{L}\eta\left(\frac{k_{\eta}^{-}-k}{k_{\eta}^{-}-k_{\eta,L}^{-}}\right)}} = (k_{\eta,L}^{-} - k_{\eta}^{-}) \int_0^1 \frac{2dv}{\sqrt{\frac{L-1}{L}\eta v}} \\ &\leq k_{\eta,L}^{-} \int_0^1 \frac{2dv}{\sqrt{\frac{L-1}{L}\eta v}} = k_{\eta,L}^{-} \sqrt{\frac{L}{L-1}} \frac{4}{\sqrt{\eta}}. \end{aligned} \quad (2.2.17)$$

The second term can be bounded by the following,

$$\int_{k_{\eta,L}^{-}}^1 \frac{2dk}{\sqrt{\eta - V(k)}} \leq \int_{k_{\eta,L}^{-}}^1 \frac{2dk}{\sqrt{\frac{L-1}{L}\eta}} \leq \sqrt{\frac{L}{L-1}} \frac{2}{\sqrt{\eta}}. \quad (2.2.18)$$

Therefore, we get

$$\int_{k_{\eta}^{-}}^1 \frac{2dk}{\sqrt{\eta - V(k)}} \leq \sqrt{\frac{L}{L-1}} \frac{2}{\sqrt{\eta}} + o\left(\frac{1}{\sqrt{\eta}}\right). \quad (2.2.19)$$

Combine the estimation of both terms, we have

$$\Delta\theta \leq \pi + 2\left(\lambda - 1 + \sqrt{\frac{L}{L-1}}\right) \frac{1}{\sqrt{\eta}} + o\left(\frac{1}{\sqrt{\eta}}\right) \quad (2.2.20)$$

□

After establishing an upper bound of $\Delta\theta$, a lower bound is given by the following:

Proposition 2.2.5. *We have*

$$\Delta\theta \geq \pi + 2 \sin^{-1} \frac{\lambda - k_{\eta}^{-}}{\sqrt{\eta + 2 \log k_{\eta}^{+} + \lambda^2}} = \pi + 2 \sin^{-1} \frac{\lambda - k_{\eta}^{-}}{k_{\eta}^{+} - \lambda}. \quad (2.2.21)$$

Proof. To get a lower bound of $\Delta\theta$, use $\log k \leq \log k_\eta^+$ when $k_\eta^- \leq k \leq k_\eta^+$.

$$\begin{aligned}
\Delta\theta &= \int_{k_\eta^-}^{k_\eta^+} \frac{2dk}{\sqrt{\eta - V(k)}} \\
&\geq \int_{k_\eta^-}^{k_\eta^+} \frac{2dk}{\sqrt{\eta - k^2 + 2\lambda k + 2\log k_\eta^+}} \\
&= \int_{k_\eta^-}^{k_\eta^+} \frac{2dk}{\sqrt{(\eta + 2\log k_\eta^+ + \lambda^2) - (k - \lambda)^2}} \\
&= \pi + 2\sin^{-1} \frac{\lambda - k_\eta^-}{\sqrt{\eta + 2\log k_\eta^+ + \lambda^2}} \\
&= \pi + 2\sin^{-1} \frac{\lambda - k_\eta^-}{k_\eta^+ - \lambda}.
\end{aligned} \tag{2.2.22}$$

□

Note that in this proposition, we do not require the energy to go to infinity. From this lower bound, we can get a partial proof for the theorem 1.4.7 for the case $\lambda \geq \frac{1}{\sqrt{3}}$.

Proof. Separate into 2 cases: $k_\eta^- < \frac{1}{\sqrt{3}}$ and $k_\eta^- \geq \frac{1}{\sqrt{3}}$.

For the case $k_\eta^- < \frac{1}{\sqrt{3}}$, applying proposition 2.2.5 yields the result $\Delta\theta > \pi$.

For the case $k_\eta^- \geq \frac{1}{\sqrt{3}}$, we have $V''(u) > 8$ in the interval (k_η^-, k_η^+) . Let \bar{V} be the quadratic curve passing through (k_η^-, η) and (k_η^+, η) with second derivative equals 8. From the maximum principle, we have $V > \bar{V}$ in (k_η^-, k_η^+) . Therefore,

$$\begin{aligned}
\Delta\theta &= \int_{k_\eta^-}^{k_\eta^+} \frac{2dk}{\sqrt{\eta - V(k)}} \\
&> \int_{k_\eta^-}^{k_\eta^+} \frac{2dk}{\sqrt{\eta - \bar{V}(k)}} = 2\sqrt{\frac{2}{V''}}\pi = \pi.
\end{aligned} \tag{2.2.23}$$

□

Combining both the upper bound and the lower bound, we can get the limit of $\Delta\theta$ when the energy η goes to infinity.

Proposition 2.2.6. *For any λ , when the energy η goes to infinity,*

$$\lim_{\eta \rightarrow \infty} \Delta\theta = \pi. \quad (2.2.24)$$

Proof. As η goes to infinity, k_η^+ goes to infinity and k_η^- goes to zero. Therefore, the upper bound and the lower bound established above both approach π as η goes to infinity. \square

2.2.3 Existence of closed solutions and embedded solutions

The proof of the theorem 1.4.1 is given by this proposition together with the behavior when η is near 0.

Proof of Theorem 1.4.4. For any $\lambda < \frac{-7}{2\sqrt{2}}$, when $\eta \rightarrow \min V^+$, the limit of $\Delta\theta$ is less than $\frac{2\pi}{3}$. When η is large enough, $\Delta\theta$ approaches π . From continuity, there exist η such that $\Delta\theta$ is exactly $\frac{2\pi}{m}$ for some integer $m > 2$.

$\Delta\theta$ is decreasing when η is near $\min V$. When $\lambda = \frac{-7}{2\sqrt{2}}$, $\min_\eta \Delta\theta < \frac{2\pi}{3}$. From the continuity, the range for λ can be extend a little higher than $\frac{-7}{2\sqrt{2}}$. \square

Furthermore, from the upper bound, we can get more information for the case $\lambda < 0$ and η large.

Corollary 2.2.7. *When $\lambda < 0$, $\Delta\theta < \pi$ for η large enough.*

Proof. Since

$$\Delta\theta \leq \pi + 2\left(\lambda - 1 + \sqrt{\frac{L}{L-1}}\right) \frac{1}{\sqrt{\eta}} + o\left(\frac{1}{\sqrt{\eta}}\right) \quad (2.2.25)$$

for arbitrary $L > 1$, choose L large enough so that $\lambda - 1 + \sqrt{\frac{L}{L-1}} < 0$. \square

With the knowledge of the behavior of $\Delta\theta$ for small energy and large energy, we can proof the theorem concerning the case $\lambda < 0$.

Proof of theorem 1.4.6. For $\frac{-2}{\sqrt{3}} < \lambda$,

$$\lim_{\eta \rightarrow \min V^+} \Delta\theta > \pi. \quad (2.2.26)$$

For $\lambda < 0$, $\Delta\theta < \pi$ when η is large enough. From continuity, there exists η such that $\Delta\theta$ is exactly π . \square

2.3 Relation between λ and $\Delta\theta$

For the case $\lambda > 0$, the behavior is similar to the original case for self shrinking curve in Abresch and Langer's paper. We want to compare the change of angle with the case $\lambda = 0$.

Recall k° is a function of λ . Translate the minimum point of $V_\lambda(k)$ to the origin, define $\hat{V}_\lambda(u) = V_\lambda(u + k^\circ) - \min V_\lambda$, where $\min V_\lambda = V_\lambda(k^\circ) = (k^\circ)^2 - 2\lambda k^\circ - 2 \log k^\circ$. Let $\bar{\eta} = \eta - \min V_\lambda$ be the energy relative to the minimum. We have the following theorem:

Theorem 2.3.1. *With the setting above, for a fixed $\bar{\eta}$, $\Delta\theta$ is increasing with respect to λ .*

Proof. Note that in this setting,

$$\Delta\theta = \int_{u_{\bar{\eta},\lambda}^-}^{u_{\bar{\eta},\lambda}^+} \frac{2du}{\sqrt{\bar{\eta} - \hat{V}_\lambda(u)}}, \quad (2.3.1)$$

where $u_{\bar{\eta},\lambda}^\pm$ are the positive and negative solution of $\bar{\eta} = \hat{V}_\lambda(u)$, respectively.

Now, for a fixed $\bar{\eta}$, we want to know the relation between λ and u when $\bar{\eta} = \hat{V}_\lambda(u)$.

Differentiate the equation with respect to λ , we have

$$\begin{aligned} 0 &= 2 \left((u + k^\circ) - \lambda - \frac{1}{u + k^\circ} \right) \left(\frac{du}{d\lambda} + \frac{dk^\circ}{d\lambda} \right) - 2(u + k^\circ) - 2 \left(k^\circ - \lambda - \frac{1}{k^\circ} \right) \frac{dk^\circ}{d\lambda} + 2k^\circ \\ &= 2 \frac{(u + k^\circ)^2 - \lambda(u + k^\circ) - 1}{u + k^\circ} \left(\frac{du}{d\lambda} + \frac{dk^\circ}{d\lambda} \right) - 2u \\ &= 2 \frac{u^2 + 2k^\circ u - \lambda u}{u + k^\circ} \left(\frac{du}{d\lambda} + \frac{dk^\circ}{d\lambda} \right) - 2u \\ &= 2u \left[\frac{u + 2k^\circ - \lambda}{u + k^\circ} \left(\frac{du}{d\lambda} + \frac{dk^\circ}{d\lambda} \right) - 1 \right]. \end{aligned} \quad (2.3.2)$$

Therefore,

$$\frac{du}{d\lambda} + \frac{dk^\circ}{d\lambda} = \frac{u + k^\circ}{u + 2k^\circ - \lambda}. \quad (2.3.3)$$

Since $k^\circ = \frac{\lambda + \sqrt{\lambda^2 + 4}}{2}$, we have

$$\frac{dk^\circ}{d\lambda} = \frac{1 + \frac{\lambda}{\sqrt{\lambda^2 + 4}}}{2} = \frac{\sqrt{\lambda^2 + 4} + \lambda}{2\sqrt{\lambda^2 + 4}} = \frac{k^\circ}{2k^\circ - \lambda}, \quad (2.3.4)$$

and

$$\frac{du}{d\lambda} = \frac{u + k^\circ}{u + 2k^\circ - \lambda} - \frac{k^\circ}{2k^\circ - \lambda} = \frac{(k^\circ - \lambda)u}{(u + 2k^\circ - \lambda)(2k^\circ - \lambda)}. \quad (2.3.5)$$

Since $k^\circ - \lambda$, $2k^\circ - \lambda$, $u + k^\circ$ are all positive, $\frac{du}{d\lambda}$ has the same sign as u , i.e. $\frac{\partial u_{\bar{\eta}, \lambda}^+}{\partial \lambda} > 0$, $\frac{\partial u_{\bar{\eta}, \lambda}^-}{\partial \lambda} < 0$.

Starting from $\hat{V}'_\lambda(u) = 2(u + k^\circ - \lambda - \frac{1}{u + k^\circ})$, for any $\tilde{\eta} < \bar{\eta}$, we want to know the change of the slope of \hat{V}'_λ at $u_{\tilde{\eta}, \lambda}^\pm$ with respect to λ . Differentiating the equation with respect to λ , we have

$$\begin{aligned} \frac{d}{d\lambda}(\hat{V}'_\lambda(u)) &= 2\left(1 + \frac{1}{(u + k^\circ)^2}\right)\left(\frac{du}{d\lambda} + \frac{dk^\circ}{d\lambda}\right) - 1 \\ &= 2\left(1 + \frac{1}{(u + k^\circ)^2}\right)\frac{u + k^\circ}{u + 2k^\circ - \lambda} - 1 \\ &= \frac{2(\lambda - k^\circ)u}{(u + k^\circ)(u + 2k^\circ - \lambda)}. \end{aligned} \quad (2.3.6)$$

Note that $\lambda - k^\circ < 0$ and therefore $\frac{d}{d\lambda}(\hat{V}'_\lambda(u))$ and u have the opposite sign.

Now, fix $\bar{\eta}$, $\lambda_1 < \lambda_2$. Since $\frac{du}{d\lambda}$ and u has the same sign, we have $u_{\bar{\eta}, \lambda_2}^- < u_{\bar{\eta}, \lambda_1}^- < 0 < u_{\bar{\eta}, \lambda_1}^+ < u_{\bar{\eta}, \lambda_2}^+$. Consider the function $\hat{V}_{\lambda_1}(u)$ and $\hat{V}_{\lambda_2}(u + u_{\bar{\eta}, \lambda_2}^+ - u_{\bar{\eta}, \lambda_1}^+)$, Both of them have the same value $\bar{\eta}$ at $u = u_{\bar{\eta}, \lambda_1}^+$. Now, for all fixed $\tilde{\eta} \in (0, \bar{\eta})$, $\frac{d}{d\lambda}(\hat{V}'_\lambda(u))$ and u have the opposite sign, we have

$$\frac{\partial u_{\tilde{\eta}, \lambda_1}^+}{\partial \tilde{\eta}} = \frac{1}{\hat{V}'_{\lambda_1}(u_{\tilde{\eta}, \lambda_1}^+)} < \frac{1}{\hat{V}'_{\lambda_2}(u_{\tilde{\eta}, \lambda_2}^+)} = \frac{\partial u_{\tilde{\eta}, \lambda_2}^+}{\partial \tilde{\eta}}. \quad (2.3.7)$$

Therefore, for any $\tilde{\eta} \in (0, \bar{\eta})$, $u_{\tilde{\eta}, \lambda_1}^+ > u_{\tilde{\eta}, \lambda_2}^+ - (u_{\tilde{\eta}, \lambda_2}^+ - u_{\tilde{\eta}, \lambda_1}^+)$, i.e. the graph of $(u_{\tilde{\eta}, \lambda_1}^+, \tilde{\eta})$ lies on the right of the graph of $(u_{\tilde{\eta}, \lambda_2}^+ - (u_{\tilde{\eta}, \lambda_2}^+ - u_{\tilde{\eta}, \lambda_1}^+), \tilde{\eta})$. Since $\hat{V}'_\lambda(u_{\tilde{\eta}, \lambda}^+) > 0$, it

implies that $\hat{V}_{\lambda_1}(u) < \hat{V}_{\lambda_2}(u + u_{\bar{\eta}, \lambda_2}^+ - u_{\bar{\eta}, \lambda_1}^+)$ for $u \in (0, u_{\bar{\eta}, \lambda_1}^+)$. We can employ the same argument for the negative part and obtain $\hat{V}_{\lambda_1}(u) < \hat{V}_{\lambda_2}(u + u_{\bar{\eta}, \lambda_2}^- - u_{\bar{\eta}, \lambda_1}^-)$ for $u \in (u_{\bar{\eta}, \lambda_1}^-, 0)$.

Therefore,

$$\begin{aligned}
\Delta\theta(\bar{\eta}, \lambda_1) &= \int_{u_{\bar{\eta}, \lambda_1}^-}^{u_{\bar{\eta}, \lambda_1}^+} \frac{2du}{\sqrt{\bar{\eta} - \hat{V}_{\lambda_1}(u)}} = \int_{u_{\bar{\eta}, \lambda_1}^-}^0 \frac{2du}{\sqrt{\bar{\eta} - \hat{V}_{\lambda_1}(u)}} + \int_0^{u_{\bar{\eta}, \lambda_1}^+} \frac{2du}{\sqrt{\bar{\eta} - \hat{V}_{\lambda_1}(u)}} \\
&< \int_{u_{\bar{\eta}, \lambda_1}^-}^0 \frac{2du}{\sqrt{\bar{\eta} - \hat{V}_{\lambda_2}(u + u_{\bar{\eta}, \lambda_2}^- - u_{\bar{\eta}, \lambda_1}^-)}} \\
&\quad + \int_0^{u_{\bar{\eta}, \lambda_1}^+} \frac{2du}{\sqrt{\bar{\eta} - \hat{V}_{\lambda_2}(u + u_{\bar{\eta}, \lambda_2}^+ - u_{\bar{\eta}, \lambda_1}^+)}} \\
&= \int_{u_{\bar{\eta}, \lambda_2}^-}^{u_{\bar{\eta}, \lambda_2}^- - u_{\bar{\eta}, \lambda_1}^-} \frac{2du}{\sqrt{\bar{\eta} - \hat{V}_{\lambda_2}(u)}} + \int_{u_{\bar{\eta}, \lambda_2}^+ - u_{\bar{\eta}, \lambda_1}^+}^{u_{\bar{\eta}, \lambda_2}^+} \frac{2du}{\sqrt{\bar{\eta} - \hat{V}_{\lambda_2}(u)}} \\
&< \int_{u_{\bar{\eta}, \lambda_2}^-}^{u_{\bar{\eta}, \lambda_2}^+} \frac{2du}{\sqrt{\bar{\eta} - \hat{V}_{\lambda_2}(u)}} = \Delta\theta(\bar{\eta}, \lambda_2).
\end{aligned} \tag{2.3.8}$$

□

Now, compare the case $\lambda > 0$ with the self shrinker case.

Proof of theorem 1.4.7. Use the result in Abresch and Langer[2] that when $\lambda = 0$, $\Delta\theta(\bar{\eta}, 0)$ is a decreasing function of $\bar{\eta}$, $\lim_{\bar{\eta} \rightarrow 0} \Delta\theta(\bar{\eta}, 0) = \sqrt{2}\pi$, $\lim_{\bar{\eta} \rightarrow \infty} \Delta\theta(\bar{\eta}, 0) = \pi$. Hence for all $\bar{\eta}$, $\Delta\theta(\bar{\eta}, 0) > \pi$. Plugging $\lambda_1 = 0$, $\lambda_2 = \lambda$ in the previous theorem, we get $\Delta\theta(\bar{\eta}, \lambda) > \Delta\theta(\bar{\eta}, 0) > \pi$ for $\lambda > 0$. □

2.3.1 Alternative proof of theorem 1.4.7 given by Guang

Here, we also include the proof given by Guang for a better understanding.

Alternative proof of theorem 1.4.7 in [29]. Define $\theta = \arg N$, this θ describe the polar angle of the normal vector and the change of angle in a period is $\Delta\theta$ as introduced

before. In this proof, we use θ as a variable instead of a function. We have

$$\frac{d\theta}{ds} = k. \quad (2.3.9)$$

Therefore,

$$\begin{aligned} k_\theta &= \frac{dk}{ds} \frac{ds}{d\theta} = \tau, \\ k_{\theta\theta} &= \frac{d\tau}{ds} \frac{ds}{d\theta} = \frac{1}{k} + \lambda - k. \end{aligned} \quad (2.3.10)$$

Multiply both side by $2k_\theta$ and integrate with respect to θ , we have

$$k_\theta^2 + k - 2\lambda k - 2\log k = \eta. \quad (2.3.11)$$

Note that this equation is equivalent to the equation (2.1.13).

If η is at the minimum, k must be constant and the solution is a circle. To study the solution which is not a round circle, we have η is greater than the minimum.

Now, rotate the solution such that $\theta = 0$ corresponds to a point with maximum k on the curve. Let $\bar{\theta}$ be the smallest positive θ which attains the minimum k . The function k is periodic with period $T = 2\bar{\theta}$. For an embedded solution, we have $T \leq \pi$.

Now, compute the third differentiation of k . We have

$$(k^2)_{\theta\theta\theta} + 4(k^2)_\theta = 4\frac{k_\theta}{k} + 6\lambda k_\theta. \quad (2.3.12)$$

Multiply by $\sin 2\theta$ and integrate from 0 to $\frac{T}{2}$.

$$\begin{aligned} 4 \int_0^{\frac{T}{2}} \sin 2\theta \frac{k_\theta}{k} d\theta &= \int_0^{\frac{T}{2}} \sin 2\theta ((k^2)_{\theta\theta\theta} + 4(k^2)_\theta - 6\lambda k_\theta) d\theta \\ &= (k^2)_{\theta\theta} \sin 2\theta \Big|_0^{\frac{T}{2}} + \int_0^{\frac{T}{2}} \sin 2\theta (4(k^2)_\theta - 6\lambda k_\theta) - 2 \cos 2\theta (k^2)_{\theta\theta} d\theta \\ &= (k^2)_{\theta\theta} \sin 2\theta \Big|_0^{\frac{T}{2}} - 2(k^2)_\theta \cos 2\theta \Big|_0^{\frac{T}{2}} - 6\lambda \int_0^{\frac{T}{2}} \sin 2\theta k_\theta d\theta. \end{aligned} \quad (2.3.13)$$

Using $T \leq \pi$, k attains minimum at $\frac{T}{2}$ and k is decreasing from 0 to $\frac{T}{2}$, we have

$$0 \geq 4 \int_0^{\frac{T}{2}} \sin 2\theta \frac{k_\theta}{k} d\theta = (k^2)_{\theta\theta} \left(\frac{T}{2}\right) \sin(T) - 6\lambda \int_0^{\frac{T}{2}} \sin 2\theta k_\theta d\theta \geq 0. \quad (2.3.14)$$

The equality must hold, therefore we have $k_\theta \equiv 0$, which is a contradiction. \square

Remark 2.3.2. *Guang's proof states explicitly why $\lambda \geq 0$ is needed. My proof, even it depends on the work of Abresch and Langer, describes the geometric behavior λ and gives us further information of the solutions.*

2.4 Simulation of the curves

By using Matlab program to solve the ODE system, we can obtain numerical solutions which approximately solve the ODE. The following are the curves of the numerical solutions. The curves behave as what expected from the theorem 1.4.7, theorem 1.4.6 and theorem 1.4.4. We put the pictures of the curves here to give the reader better idea of what the actual solution will behave. From the simulation, we can observe some behavior of $\Delta\theta$ with respect to η . Some conjectures about the behavior are posed here.

2.4.1 $\lambda \geq 0$ case

When $\lambda \geq 0$, the range for $\Delta\theta$ contains $(\pi, \pi\sqrt{2}\sqrt{\frac{\lambda}{\sqrt{\lambda^2+4}} + 1}]$ and $\Delta\theta > \pi$. There will not be embedded solutions. The following are some of the closed solutions for the case $\lambda = 0.19$ and $\lambda = 0.726$. The energy η increases from left to right. Note that for certain η , the solution passes through the origin. If we keep increasing η , unlike the case where $\lambda = 0$, the origin will not be on the same side of the solution anymore.

Conjecture 2.4.1. *When $\lambda \geq 0$, $\Delta\theta$ is monotonically decreasing with respect to η as in the case of Abresch and Langer[2].*

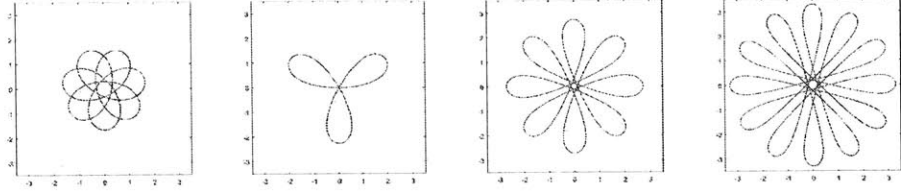


Figure 2-1: Solutions for $\lambda = 0.19$, $\Delta\theta = \frac{10\pi}{7}, \frac{4\pi}{3}, \frac{5\pi}{4}, \frac{7\pi}{6}$, respectively.

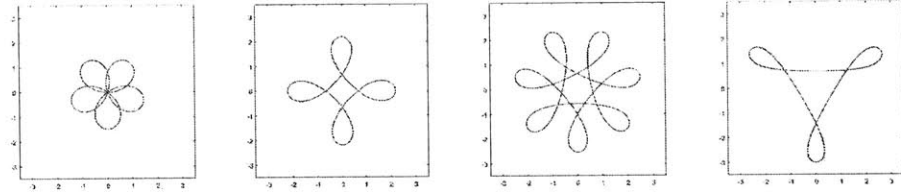


Figure 2-2: Solutions for $\lambda = 0.726$, $\Delta\theta = \frac{8\pi}{5}, \frac{3\pi}{2}, \frac{10\pi}{7}, \frac{4\pi}{3}$, respectively.

2.4.2 $\lambda < 0$ case

The $\lambda < 0$ cases are more interesting. For each $\frac{-2}{\sqrt{3}} < \lambda < 0$, there exist an η such that $\Delta\theta = \pi$. The corresponding solution is embedded and have 2-symmetry. The following are some of the examples. From left to right, $\lambda = -0.2, -0.3, -0.4, -0.5, -0.6, -0.7, -0.8, -0.9$, respectively.

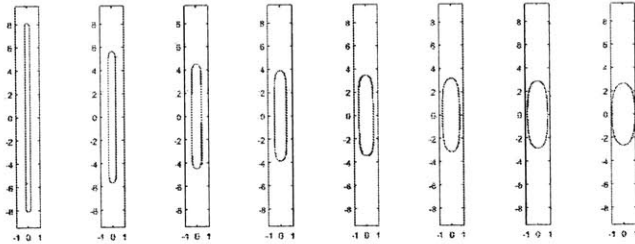


Figure 2-3: Embedded solutions for different λ 's

Conjecture 2.4.2. *There is a unique $\delta > 0$ such that for $\frac{-7}{2\sqrt{2}} + \delta < \lambda \leq \frac{-2}{\sqrt{3}}$, there are no embedded solutions and there is a 3-symmetry embedded solution when $\lambda \leq \frac{-7}{2\sqrt{2}} + \delta$.*

The following is the case $\lambda = -2$. From the numerical solutions, the lowest possible $\Delta\theta$ is around $\frac{3\pi}{4}$ and no embedded solution is found.

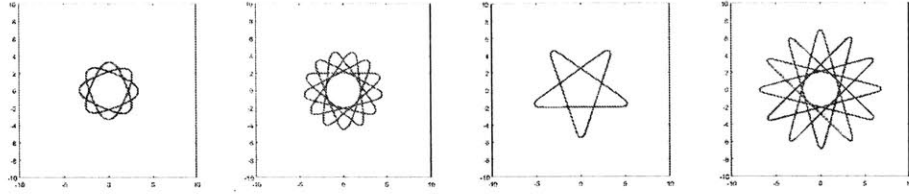


Figure 2-4: Solutions for $\lambda = -2$, $\Delta\theta = \frac{3\pi}{4}, \frac{10\pi}{13}, \frac{4\pi}{5}, \frac{5\pi}{6}$, respectively.

For $\lambda \leq \frac{-2}{\sqrt{3}}$, we have no embedded solutions with 2-symmetry. However, as $\lambda < \frac{-7}{2\sqrt{2}}$, there are embedded solutions with m -symmetry for some $m > 2$. The following are the cases where $\lambda = -3, -5$. The energy η increases from left to right. Unlike the case $\lambda > 0$, when $\lambda \leq \frac{-2}{\sqrt{3}}$, even though $\Delta\theta$ should be decreasing near $\min V(B)$, it appears that after some η , $\Delta\theta$ is increasing while η is increasing.

Conjecture 2.4.3. *When $\lambda < 0$, there is a function $\eta_{crit}(\lambda)$ such that $\Delta\theta$ is decreasing when $\eta < \eta_{crit}$ and $\Delta\theta$ is increasing when $\eta > \eta_{crit}(\lambda)$. Moreover, $\eta_{crit}(\lambda)$ goes to infinity when λ goes to zero, $\eta_{crit}(\lambda) - \min V_\lambda$ goes to zero when λ goes to negative infinity.*

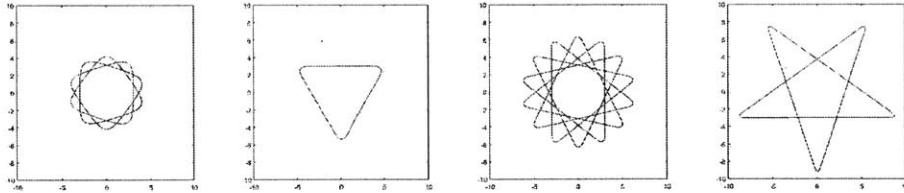


Figure 2-5: Solutions for $\lambda = -3$, $\Delta\theta = \frac{3\pi}{5}, \frac{2\pi}{3}, \frac{5\pi}{7}, \frac{4\pi}{5}$, respectively.

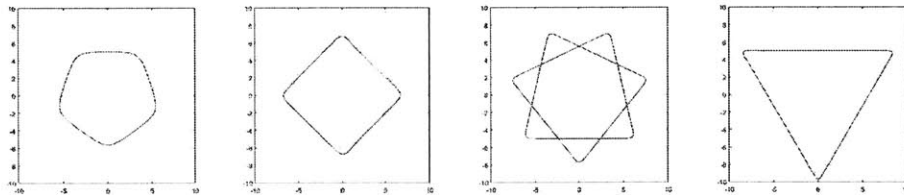


Figure 2-6: Solutions for $\lambda = -5$, $\Delta\theta = \frac{2\pi}{5}, \frac{\pi}{2}, \frac{4\pi}{7}, \frac{2\pi}{3}$, respectively.

Chapter 3

Background of spectral geometry

3.1 Eigenvalue and eigenfunction of Laplace operator

Let M be a compact n -dimensional Riemannian manifold without boundary. In terms of local coordinate (x_1, x_2, \dots, x_n) , the metric can be written as $ds^2 = g_{ij}dx_idx_j$. The Laplace-Beltrami operator is given by

$$\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_j} (\sqrt{g} g^{ij} \frac{\partial}{\partial x_i}), \quad (3.1.1)$$

where $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$.

The Laplace operator is a self-adjoint operator on the function space $H^1(M)$. From the theory developed to solve the Poisson equation, we know that the map

$$I - \Delta : H^1(M) \rightarrow H^1(M)^* \quad (3.1.2)$$

is bijective. If we denote the inverse by T and restrict it to $L^2(M)$, it will be a compact, self-adjoint operator from $L^2(M)$ to $L^2(M)$. Now, we can apply the spectral theory about compact, self-adjoint operators. The eigenspaces are finite dimensional and the only accumulation point of the eigenvalues is 0.

Therefore, the eigenvalues of Δ are discrete with finite multiplicity. Together with

the fact that $-\Delta$ is nonnegative, we have the sequence of eigenvalues $0 = \lambda_1^2 < \lambda_2^2 \leq \lambda_3^2 \cdots$ and the corresponding eigenfunctions $\{e_{\lambda_i}\}$ which satisfy $\Delta e_{\lambda_i} = -\lambda_i^2 e_{\lambda_i}$.

The eigenvalues can also be considered using a max-min principle. The Rayleigh quotient is

$$\lambda_k^2 = \sup_E \left(\inf_{u \in E} \frac{\int_M |\nabla u|^2 dV}{\int_M u^2 dV} \right), \quad (3.1.3)$$

where E ranges from all codimension $k - 1$ subspace in $H^1(M)$. This characterization helps us to obtain bounds of the eigenvalue. For example, most of the shape optimization problems are proved by using the Rayleigh quotient for appropriate test functions.

Remark 3.1.1. *In some of the literatures, the eigenvalue and eigenfunction are defined as $\Delta e_\lambda = -\lambda e_\lambda$. We choose λ^2 instead of λ to reflect that Laplacian is a second order operator. It is easier to compare this result to other self-adjoint operators with different differentiation order.*

The study of the eigenfunctions and eigenvalues of Laplacian is closely related to other partial differential equations such as the heat equation. The following theorem concerns the relation between the heat kernel and the eigenfunctions.

Theorem 3.1.2. *Let $\{e_{\lambda_i}\}$ be an orthonormal basis of $L^2(M)$ consisting of eigenfunctions, with λ_i^2 be the corresponding eigenvalues, then*

$$H(x, y, t) = \sum e^{-\lambda_i^2 t} e_{\lambda_i}(x) e_{\lambda_i}(y). \quad (3.1.4)$$

and

$$\text{Tr} e^{\Delta t} = \int_M H(x, x, t) dx = \sum e^{-\lambda_i^2 t}. \quad (3.1.5)$$

We have similar formulas for the wave equation and the Schrödinger's equation. The eigenfunctions give us special solutions so that we can separate the space and the time variables in each of the partial differential equations.

The study of the spectral behavior focuses on two different directions. One focuses on the geometric dependence of the lowest eigenvalues. The other focuses on the asymptotic behavior of the eigenfunction as the eigenvalue goes to infinity.

3.1.1 Asymptotic behavior of the eigenfunctions

We are interested in the behavior of the eigenfunction when the eigenvalue goes to infinity. Using the heat kernel and Tauberian theorem, Weyl prove the asymptotic formula concerning the distribution of eigenvalues on the real line:

$$\lambda_k \approx c_n \left(\frac{k}{V}\right)^{\frac{1}{n}} \quad (3.1.6)$$

as k goes to infinity, where $c_n = 2\pi \left(\frac{n}{\omega_{n-1}}\right)^{\frac{1}{n}}$, $\omega_{n-1} = \text{Vol}(\mathbb{S}^{n-1})$ and $V = \text{Vol}(M)$.

As the eigenvalue goes to infinity, the corresponding eigenfunction may concentrate on a small region. The following L^p estimate from [48], [47] describes such phenomenon.

Theorem 3.1.3. *Let e_λ be an eigenfunction of the Laplace operator. For $p \geq 2$, we have*

$$\|e_\lambda\|_{L^p} \leq C \lambda^{\sigma(n,p)} \|e_\lambda\|_{L^2}, \quad (3.1.7)$$

where C is a constant independent of λ and the power $\sigma(n, p)$ is given by

$$\sigma(n, p) = \begin{cases} n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}, & \frac{2(n+1)}{n-1} \leq p \leq \infty \\ \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right), & 2 \leq p \leq \frac{2(n+1)}{n-1}. \end{cases} \quad (3.1.8)$$

The bounds of L^p norms given above are sharp. They can be attained by the famous examples of the eigenfunctions on the standard sphere \mathbb{S}^n . The L^∞ upper bound is attained by the zonal harmonics on the sphere, with the property that the maximal value is attained at the north pole and the south pole. On the other hand, the L^p , $p < \frac{2(n+1)}{n-1}$, upper bound is attained by the sectoral harmonics, with the property that the mass of the eigenfunctions concentrate on a closed geodesic. The different behaviors when eigenvalue goes to infinity are the main difficulty in the study of the overall asymptotic behavior.

Jakobson and Nadirashvili[35] focus on the symmetry of the L^p norm for the positive part and the negative part of the eigenfunctions. If we defined the positive

part and the negative part of the eigenfunction by

$$\begin{aligned} e_\lambda^+ &= e_\lambda \cdot \chi(\{e_\lambda \geq 0\}) \\ e_\lambda^- &= e_\lambda \cdot \chi(\{e_\lambda \leq 0\}). \end{aligned} \tag{3.1.9}$$

We have the following theorem

Theorem 3.1.4 ([35]). *For any $p \geq 1$, there exists $C > 0$, depending only on p and the manifold M , such that for any nonconstant eigenfunction e_λ of the Laplacian,*

$$\frac{1}{C} \leq \frac{\|e_\lambda^+\|_{L^p}}{\|e_\lambda^-\|_{L^p}} \leq C. \tag{3.1.10}$$

3.1.2 Nodal set and nodal domains

The nodal set is the set on which the eigenfunction vanishes and the nodal domains are the connected components of the complement of the nodal set. The study of nodal patterns is first posed by the German physicist Chladni. He visualized the nodal lines of a vibrating metal plate covered with sand and recorded the resulting patterns. The nodal set can be thought of as the equilibrium points in a vibrating mode. This idea can be generalized to higher dimensional manifolds.

First, the following theorem concerns the local structure of nodal sets.

Theorem 3.1.5 (S.Y. Cheng). *Let M be an n -dimensional smooth Riemannian manifold. If a smooth function f satisfies $(\Delta + h(x))f = 0$ for a smooth function h . Then $f^{-1}(0)$ forms an $(n - 1)$ -dimensional manifold, except on a closed set of lower dimension.*

For the 2-dimensional case, we have further understanding of the nodal lines.

Theorem 3.1.6. *If M is a compact Riemannian surface, e_λ is an eigenfunction, then*

1. *The nodal set of f consists of a finite number of C^2 -immersed circles.*
2. *The critical points on the nodal set are isolated.*
3. *At each critical point, the nodal lines divide the whole angle 2π equally.*

The nodal set behaves like an $(n - 1)$ -dimensional manifold almost everywhere. We can use $(n - 1)$ -dimensional Hausdorff measure to describe how large the nodal set is.

Intuitively, when the eigenvalue goes to infinity, the function will oscillate more frequently. We expect there should be more zero crossing points and more nodal domains due to more frequent oscillation. One of the earliest theorem is given by Courant, which gives an upper bound of the number of nodal domains.

Theorem 3.1.7 (Courant). *The n -th eigenfunction can have at most n nodal domains.*

Remark 3.1.8. *One immediate corollary of this theorem is about the eigenfunctions corresponding to the lowest eigenvalues. The lowest eigenvalue of the Laplacian on a compact manifold is $\lambda_1 = 0$ which corresponds to the constant eigenfunction. This eigenspace have multiplicity 1. The eigenfunction which corresponds to the second lowest eigenvalue has exactly 2 nodal domains.*

Even though we might expect the number of nodal domains should grow with respect to the eigenvalue λ , this is not true. For arbitrary large k , we can create a manifold such that there exists $l > k$ such that the l -th eigenfunction only admits two nodal domains.

Let us consider the problem on the circle S^1 . The Laplacian eigenvalues are non-negative integers. The eigenspace which corresponds to positive integer k is spanned by $\cos(kx)$ and $\sin(kx)$. There are exactly $2k$ zeroes of the eigenfunction corresponds to $\lambda = k$. Therefore, the size of the nodal set, which is the vanishing set of the eigenfunction, should have growth rate of order λ . For the higher dimensional case, Yau[57], [1] proposes the following:

Conjecture 3.1.9. *Let e_λ be an eigenfunction of the Laplace-Beltrami operator on a closed smooth manifold M . For the nodal set $Z_\lambda = \{e_\lambda = 0\}$,*

$$c\lambda \leq |Z_\lambda| \leq C\lambda \tag{3.1.11}$$

for some constant c, C which only depend on M .

The lower bound in the conjecture 3.1.9 is established by Brüning[10] and Yau for the 2-dimensional case. Donnelly and Fefferman[22] prove both bounds for analytic manifolds. However, for n -dimensional smooth manifolds, the conjecture remains open. In the smooth case, some weaker bounds are established for the nodal sets. Colding and Minicozzi[20] give the currently best lower bound.

Theorem 3.1.10 (Colding, Minicozzi[20]). *Given a closed smooth n -dimensional Riemannian manifold M , there exists c so that*

$$|Z_\lambda| \geq c\lambda^{\frac{3-n}{2}}. \quad (3.1.12)$$

For alternative proofs, see Hezari and Sogge[32] and Sogge and Zelditch[50]. The lower bounds for the eigenfunction problem on smooth manifolds are summarized in [42]. We sketch the proofs in [20] and [32] here for comparison.

Sketch of the proof in [20]. Step 1. There exists $a > 0$ which depends on M such that e_λ has a zero in any ball of radius $\frac{a}{3\lambda}$.

Step 2. Cover the manifold with balls $\{B_i\}$ with radius $a\lambda^{-1}$ such that each point is contained in at most C_M of the balls $\{2B_i\}$.

Step 3. Define a ball B to be a d_M -good ball if

$$\int_{2B} e_\lambda^2 \leq 2^{d_M} \int_B e_\lambda^2. \quad (3.1.13)$$

There exist d_M which only depends on C_M such that there are at least $C\lambda^{\frac{n+1}{2}}$ d_M -good balls.

Step 4. Since the L^2 norm of the doubling of a d_M -good ball is controlled, the ratio of the measure of the positive domain and the negative domain is uniformly bounded away from zero. We can use the isoperimetric inequality to conclude the nodal set in a d_M -good ball, which separate the positive and negative domain, must admit measure bounded below by $c\lambda^{1-n}$. Combining all the estimation on the good balls yields the result. \square

Remark 3.1.11. *In the step 1, the fact that we can find a zero in any ball of radius comparable to λ^{-1} is consistent with our intuition that as λ goes to infinity, the corresponding eigenfunction will oscillate more and more rapidly. It is an important question to consider whether this property holds for other eigenfunction problems.*

Sketch of the proof in [32]. Step 1. Use integration by parts on each nodal domain to establish the equality

$$\int_M |e_\lambda|(\Delta + \lambda^2)f dV = 2 \int_{Z_\lambda} |\nabla e_\lambda| f d\sigma. \quad (3.1.14)$$

Step 2. Plug $f = 1$ in the equality to establish

$$\lambda^2 \int_M |e_\lambda| dV = 2 \int_{Z_\lambda} |\nabla e_\lambda| d\sigma. \quad (3.1.15)$$

Step 3. Plug $f = \sqrt{1 + |\nabla e_\lambda|^2}$ to establish

$$2 \int_{Z_\lambda} |\nabla e_\lambda|^2 d\sigma \leq C\lambda^3 \int e_\lambda^2 dV. \quad (3.1.16)$$

Step 4. From the estimations above, together with Holder's inequality and the L^1 lower bound of the eigenfunction, we can obtain the desired result. \square

Remark 3.1.12. *The two proofs above is quite different. For the first proof, the idea of covering the manifold with small balls is employed and we get an estimate in the balls with good behavior. For the second proof, a global integration formula on the nodal sets is used. Both of the proofs use the L^p estimates in theorem 3.1.3, which is known to be optimal. Therefore, in order to get a better lower bound, we need some methods other than the L^p estimates.*

There are some other interesting results concerning the asymptotic behavior of the nodal sets and nodal domains. Mangoubi[41] focuses on the local asymmetry of nodal domains and establishes the following,

Theorem 3.1.13 (Mangoubi[41]).

$$\frac{\text{Vol}(\{e_\lambda > 0\} \cap B)}{\text{Vol}(B)} \geq \frac{C_1}{\lambda^{n-1}}, \quad (3.1.17)$$

for all geodesic balls $B \subset M$ such that $\{e_\lambda = 0\} \cap \frac{1}{2}B \neq \emptyset$. Here $\frac{1}{2}B$ is a concentric ball of half radius of B , and C_1 is a constant which depends only on M .

There is a result focusing on only one nodal domain.

Theorem 3.1.14 ([14], [22]). *Let M be a closed Riemannian manifold of dimension n . Let D_λ be a λ -nodal domain. Then*

$$\frac{\text{Vol}(D_\lambda > 0 \cap B)}{\text{Vol}(B)} \geq \frac{C_2}{\lambda^{6n^2}}, \quad (3.1.18)$$

for all geodesic balls $B \subset M$ such that $D_\lambda \cap \frac{1}{2}B \neq \emptyset$.

Unlike the uniform bound in [35] about the ratio of the L^p norms between the positive part and the negative part of the eigenfunctions, this tells us that the eigenfunctions may be mostly positive or mostly negative on a small ball as the eigenvalues go to infinity.

3.2 General setting of spectral problems

In spectral theory, the relationships between the geometric structure and spectra of canonically defined differential operators are considered. Even though they may be defined in different context, we can usually use the PDE and the function space theory to relate the operators with compact operators and proceed as in the case of Laplacian.

The other operators we are interested in, such as the Laplace-Beltrami operator on a compact manifold with boundary or the Dirichlet-to-Neumann operator on the boundary of a compact manifold, may share similar spectral behaviors. Some of the most common properties are the following:

1. There is an orthonormal basis in $L^2(M)$ consisting of eigenfunctions.

2. The eigenvalues are discrete.
3. The only accumulation point of the eigenvalues is infinity.
4. Each eigenspace is finite dimensional.

The eigenvalues and the eigenfunctions are important in the study of the geometry analysis. They can be used in the study of other PDEs which involve evolution in time. If we arrange the eigenvalues to form a nondecreasing sequence, this sequence, the spectrum, carries information about the original manifold. For example, we can recover the volume, the integral of the scalar curvature and the dimension of a manifold from the spectrum of the Laplace-Beltrami operator. On the other hand, each of the eigenfunctions represents a steady state of the system. For example, in quantum mechanics, each eigenfunction corresponds to a particular state of the wave equation which can be interpreted as the probability density function of the particle in that state.

The lowest eigenvalue depends on the geometry. An important problem is to get an a priori estimate of the lowest eigenvalue, which is related to the best Sobolev constant in the case of the Laplace-Beltrami operator. The shape optimization problem asks: Which manifold attains the bound for the lowest eigenvalue? The answer usually poses some symmetry conditions or curvature conditions for the manifold.

On the other hand, the behavior of the spectra induced from different manifolds are similar near infinity. For example, the distribution of the eigenvalues on the real line is given by Weyl's formula below. The asymptotic distribution of eigenvalues only depends on the dimension n .

One of the most powerful tool is the pseudo-differential operator theory. On a compact manifold, the theory of elliptic, self-adjoint pseudo-differential operators establishes important results about the spectral behavior. In [48], Weyl's formula which concerns the number of eigenvalues below λ , is stated as follows.

Theorem 3.2.1 (Weyl's formula). *Let P be an elliptic, self-adjoint classical pseudo-differential operator of order 1 and $p(x, \xi)$ be its principle symbol. If $N(\lambda)$ denotes the*

number of eigenvalues of P which are less than λ (counted with respect to multiplicity), as λ goes to infinity,

$$N(\lambda) = c\lambda^n + \mathcal{O}(\lambda^{n-1}), \quad (3.2.1)$$

where

$$c = (2\pi)^{-n} \int \int_{\{(x,\xi) \in T^*M \mid p(x,\xi) \leq 1\}} d\xi dx. \quad (3.2.2)$$

If we apply Weyl's formula to the spectra of the Laplace-Beltrami eigenfunction problem or the harmonic Steklov eigenfunction problem, we can see that the dimension n and the volume of the manifold determine the asymptotic behavior of eigenvalues.

Another important result from pseudo-differential operator theory is the following L^p bound for the eigenfunctions for an order 1 self-adjoint elliptic operator in [48], [47]. By comparing the L^p norm, we can understand how much the mass of an eigenfunction concentrates.

Theorem 3.2.2. *Let P be as in the theorem above. Assume further that for each $x \in M$ the cospheres $\{\xi \mid p(x, \xi) = 1\}$ have non-vanishing curvature. Let e_λ be an eigenfunction corresponding to λ . We have the same L^p estimates as in theorem 3.1.3.*

In the following sections, other eigenfunction problems will be introduced. There are some problems we will focus on:

1. Courant's nodal domain theorem.
2. Shape optimization problem.
3. Bounds of the measure of nodal sets.

3.3 Spectral problems on a compact manifold with boundary

In this section, we explore some spectral problems on a compact manifold with boundary. The difference between the problem defined on manifolds with boundary and

the problem defined on manifolds without boundary is that we need to pose certain boundary condition to make the operator self-adjoint.

3.3.1 Laplacian spectral problems

On a manifold with boundary. There are two famous laplacian eigenvalue problems:

$$\begin{cases} \Delta e_\lambda = -\lambda^2 e_\lambda & \text{in } M, \\ e_\lambda = 0 & \text{on } \partial M. \end{cases} \quad (3.3.1)$$

$$\begin{cases} \Delta e_\lambda = -\lambda^2 e_\lambda & \text{in } M, \\ \partial_\nu e_\lambda = 0 & \text{on } \partial M. \end{cases} \quad (3.3.2)$$

They are called the Dirichlet and the Neumann eigenvalue problem, respectively. The corresponding Rayleigh quotients are given by

$$\lambda_k^2 = \sup_E \left(\inf_{u \in E} \frac{\int_M |\nabla u|^2 dV}{\int_M u^2 dV} \right), \quad (3.3.3)$$

where E ranges from all codimension $k - 1$ subspace in $H_0^1(M)$ for Dirichlet problem and $H^1(M)$ for the Neumann problem. The different function spaces are used to describe different boundary conditions: fixed boundary condition for the Dirichlet problem and free boundary condition for the Neumann problem.

These problems are similar to the eigenvalue problem on a compact manifold without boundary. Some results of the compact case have analogues in the noncompact case. For example, the Courant's theorem concerning the upper bound of the number of nodal domains is also established for the Dirichlet problem and the Neumann problem. From this, the first eigenfunction should be positive and therefore the eigenspace is 1-dimensional. This is interesting in the Dirichlet problem since the lowest eigenfunction is not a constant in this case.

For the study of the lowest eigenvalues and eigenfunctions, one important problem is the shape optimization problem: Finding an Euclidean domain which attains the extremal value of λ_1 . We expect a domain should admit some symmetry to attain

the extremal value. For Dirichlet problem, the result is given by the Faber-Krahn theorem.

Theorem 3.3.1 (Faber-Krahn). *Let $M \subset \mathbb{R}^n$ be a domain, $B(R)$ a ball in \mathbb{R}^n of radius R such that $\text{Vol}(M) = \text{Vol}(B(R))$, then for the Dirichlet eigenvalue problem, we have the inequality*

$$\lambda_1(M) \geq \lambda_1(B(R)), \quad (3.3.4)$$

with equality if and only if M is a ball.

Since the Laplacian operator is studied extensively with different boundary conditions, we also include the Robin eigenvalue problem, which can be think of as the intermediate problem between the Dirichlet problem and the Neumann problem. This problem is defined as

$$\begin{cases} \Delta e_\lambda = -\lambda^2 e_\lambda & \text{in } M, \\ \alpha e_\lambda + (1 - \alpha) \partial_\nu e_\lambda = 0 & \text{on } \partial M, \end{cases} \quad (3.3.5)$$

where $0 < \alpha < 1$ is a parameter. The cases $\alpha = 0, 1$ correspond to the Neumann and the Dirichlet boundary condition, respectively. The Rayleigh quotient is given by

$$\lambda_k^2 = \sup_E \left(\inf_{u \in E} \frac{\int_M |\nabla u|^2 dV + \frac{\alpha}{1-\alpha} \int_{\partial M} u^2 d\sigma}{\int_M u^2 dV} \right). \quad (3.3.6)$$

For this problem, Bossel and Daners establish the following:

Theorem 3.3.2 (Bossel-Daners). *The ball minimizes the first eigenvalue of the Robin problem among open sets with a given volume for every value of $\alpha \in (0, 1]$.*

For the case $\alpha = 0$, the boundary condition becomes the Neumann boundary condition. The first eigenvalue is always zero in this problem. There is a shape optimization result concerning the second eigenvalue λ_2 , which is proven by Szegő for the 2-dimensional case and is completed by Weinberger for Euclidean domain in all dimensions.

Theorem 3.3.3 (Szegő-Weinberger). *Let $M \subset \mathbb{R}^n$ be a domain, $B(R)$ a ball in \mathbb{R}^n of radius R such that $\text{Vol}(M) = \text{Vol}(B(R))$, then for the Neumann eigenvalue problem,*

we have the inequality

$$\lambda_2(M) \leq \lambda_2(B(R)), \quad (3.3.7)$$

with equality if and only if M is a ball.

The reader can refer to [5] for the proof.

Remark 3.3.4. *In some of the literatures, λ_1 is used to call the first nonzero eigenvalue. We always use λ_1 to refer to the lowest eigenvalue to make the comparison between different problems easier.*

Now, let's turn to the asymptotic behavior of nodal sets. For the Dirichlet and the Neumann problems, Donnelly and Fefferman[23] establish Yau's conjecture for analytic manifolds. Ariturk[4] establishes lower bounds for the Neumann problem and the Dirichlet problem on smooth manifolds.

Theorem 3.3.5 (Ariturk[4]). *If e_λ is a Neumann eigenfunction, then*

$$|Z_\lambda| \geq C\lambda^{\frac{5-2n}{3}}. \quad (3.3.8)$$

If e_λ is a Dirichlet eigenfunction and $n \leq 3$, then

$$|Z_\lambda| \geq C\lambda^{\frac{5-2n}{3}}. \quad (3.3.9)$$

If the boundary is strictly geodesically concave and e_λ is a Dirichlet eigenfunction, then for $n \leq 4$,

$$|Z_\lambda| \geq C\lambda^{\frac{3-n}{2}}. \quad (3.3.10)$$

The idea of the proof is making another copy of manifold M and then glueing it with the original manifold along the boundary. By doing so, we obtain a compact manifold. The metric is only Lipschitz on the glueing of the boundaries. By odd or even extension, any Dirichlet or Neumann eigenfunction can be extended as an eigenfunction on the new manifold. Using the L^p estimates for low regularity metrics, we can get the desired bound.

3.3.2 Bilaplacian spectral problems

Let M be an Euclidean domain. We have the following bilaplacian eigenvalue problems:

$$\begin{cases} \Delta^2 e_\lambda - T\Delta e_\lambda = \lambda^4 e_\lambda & \text{in } M, \\ \partial_\nu^2 e_\lambda = T\partial_\nu e_\lambda - \mathbf{div}_{\partial M}(P_{\partial M}[(D^2 e_\lambda)\nu]) - \partial_\nu \Delta e_\lambda = 0 & \text{on } \partial M, \end{cases} \quad (3.3.11)$$

$$\begin{cases} \Delta^2 e_\lambda - T\Delta e_\lambda = \lambda^4 e_\lambda & \text{in } M, \\ \partial_\nu^2 e_\lambda = e_\lambda = 0 & \text{on } \partial M, \end{cases} \quad (3.3.12)$$

$$\begin{cases} \Delta^2 e_\lambda - T\Delta e_\lambda = \lambda^4 e_\lambda & \text{in } M, \\ \partial_\nu e_\lambda = e_\lambda = 0 & \text{on } \partial M, \end{cases} \quad (3.3.13)$$

where T is the tension on the plate. They are called the free plate problem, the hinged plate problem, the clamped plate problem, respectively. These problems arise from the study of the vibration of an elastic plates subjected to different boundary conditions. The boundary conditions are obtained by considering the Rayleigh quotient,

$$\lambda_k^4 = \sup_E \left(\inf_{u \in E} \frac{\int_M |\nabla^2 u|^2 + T|\nabla u|^2 dV}{\int_M u^2 dV} \right). \quad (3.3.14)$$

This time, E ranges from all codimension $k-1$ subspace in $H^2(M)$, $H^2 \cap H_0^1(M)$ and $H_0^2(M)$, respectively. These are different degrees of freedom of boundary conditions. From the regularity theorems, the critical functions of the Rayleigh quotients should be smooth and admit the respective boundary conditions.

We also have the buckling plate problem given as below

$$\begin{cases} \Delta^2 e_\lambda = -\lambda^2 \Delta e_\lambda & \text{in } M, \\ \partial_\nu e_\lambda = e_\lambda = 0 & \text{on } \partial M, \end{cases} \quad (3.3.15)$$

and the corresponding Rayleigh quotient given by

$$\lambda_k^2 = \sup_E \left(\inf_{u \in E} \frac{\int_M |\nabla^2 u|^2 dV}{\int_M |\nabla u|^2 dV} \right), \quad (3.3.16)$$

where E ranges from all codimension $k - 1$ subspace in $H_0^2(M)$.

Remark 3.3.6. *We require the domain M to be Euclidean in order to make the operator self-adjoint. However, for the boundary condition of clamped plate problem and the buckling plate problem, we have*

$$\int_M |\nabla^2 u|^2 dV = \int_M (\Delta u)^2 dV \quad (3.3.17)$$

in Euclidean domain. We can generalize the problem to arbitrary Riemannian manifold. In that case, the $|\nabla^2 u|^2$ term in the Rayleigh quotient is replaced by $(\Delta u)^2$.

In the biharmonic problems, Courant's theorem concerning the number of nodal domains fails to hold. For example, in the clamped plate problem and the buckling problem, the eigenfunction corresponding to the lowest eigenvalue may not have fixed sign. There is an important problem of finding conditions which guarantee the first eigenfunction to admit fixed sign.

As in the Laplacian case, we can consider the shape optimization problem. For the free plate problem, the behavior is similar to the Neumann problem, Chasman[15] establishes the following,

Theorem 3.3.7. *For all smoothly bounded regions of a fixed volume, the fundamental tone, λ_2 of the free plate with a given positive tension is maximal for a ball. That is, if $M \subset \mathbb{R}^n$ be a domain, $B(R)$ a ball in \mathbb{R}^n of radius R such that $\text{Vol}(M) = \text{Vol}(B(R))$. The fundamental tone λ_2 satisfies*

$$\lambda_2(M) \leq \lambda_2(B(R)), \quad (3.3.18)$$

with equality if and only is M is a ball.

Remark 3.3.8. *In this theorem, we need the tension $T > 0$. For the case $T = 0$, the constant function and the coordinate functions are eigenfunctions corresponding to $\lambda_1 = 0$. The lowest eigenspace is $n + 1$ dimensional and the fundamental tone is λ_{n+2} .*

For the clamped plate problem with $T = 0$, it is conjectured that

$$\lambda_1(M) \geq \lambda_1(B(R)), \quad (3.3.19)$$

for any Euclidean domain M and the ball with the same volume. This is established for $n \leq 3$ by Ashbaugh and Benguria. Szegő also prove this under the assumption the eigenfunction corresponds to the lowest eigenvalue is of fixed sign. For $n \geq 4$, the best result is given by Laugesen that

$$\lambda_1(M) \geq d_n \lambda_1(B(R)) \quad (3.3.20)$$

for some constant d_n which only depends on the dimension n . Also, the conjecture holds asymptotically in the sense that d_n goes to 1 as n goes to infinity.

Buoso and Lamberti[12] consider the shape optimization problem for the hinged plate problem with $T = 0$. They show that balls are critical domains for the first eigenvalue of the hinged plate problem by the domain perturbation method.

The buckling problem is closely related to the clamped plate problem with $T = 0$. The conjecture is also proven under the assumption the eigenfunction corresponds to the lowest eigenvalue is of fixed sign. For the general case, the best result is

$$\lambda_1(M) \geq c_n \lambda_1(B(R)) \quad (3.3.21)$$

for some constant c_n and c_n also goes to 1 as n goes to infinity.

For the asymptotic distribution of eigenvalues, even though extra work is needed to take care the boundary condition, the same Weyl's formula that λ_k is asymptotic to $c_n(\frac{k}{V})^{\frac{1}{n}}$ as in the previous section is established in this case. The reader may refer to [46] as a good reference. There are some work comparing different spectrum for the different problems on the same compact manifold. Liu[39] compare the k -th eigenvalue for the Neumann problem, the Dirichlet problem, the clamped plate problem with $T = 0$ and the buckling plate problem.

Theorem 3.3.9. *Let λ_k^N , λ_k^D , λ_k^C and λ_k^B be the k -th eigenvalue of the problem,*

respectively. The following inequality holds

$$\lambda_k^N < \lambda_k^D < \lambda_k^C < \lambda_k^B. \quad (3.3.22)$$

This is established by constructing k -dimensional space spanned by the first k eigenfunctions of one problem and put it into the Rayleigh quotient as a test space of another problem.

3.4 Spectral problem of boundary Steklov problems

Now, we turn our attention to the Steklov eigenvalue problems. To be consistent with the language above, let Ω be a compact n -dimensional smooth manifold with smooth boundary $M = \partial\Omega$. Notice that M has dimension $n - 1$.

3.4.1 Harmonic Steklov problem

For the harmonic Steklov eigenfunction problem, e_λ is defined as the solution of

$$\begin{cases} \Delta e_\lambda = 0 & \text{in } \Omega, \\ \partial_\nu e_\lambda = \lambda e_\lambda & \text{on } M, \end{cases} \quad (3.4.1)$$

and the corresponding Rayleigh quotient is given by

$$\lambda_k = \sup_E \left(\inf_{u \in E} \frac{\int_\Omega |\nabla u|^2 dV}{\int_M u^2 dV} \right), \quad (3.4.2)$$

where E ranges from all codimension $k - 1$ subspace in $H^1(\Omega)$. From this characterization, this problem can also be realized as the weighted Neumann eigenfunction problem with mass uniformly distributed on the boundary. Therefore, these two problems admit some common properties.

If we restrict e_λ to the boundary, it satisfies the eigenvalue problem

$$\Lambda e_\lambda = \lambda e_\lambda, \quad (3.4.3)$$

where Λ is the Dirichlet-to-Neumann operator on the boundary.

The shape optimization problem in this case is more complicated. We have two different constraints: fixing the area of the boundary or fixing the volume of the domain. For the case that the boundary area is fixed, Weinstock[55] proof that for simply-connected domains in \mathbb{R}^2 ,

$$\lambda_2 |\partial\Omega| \leq 2\pi, \tag{3.4.4}$$

with the equality if and only if Ω is a disk. However, this does not hold for general domain. For example, there are planar annulus which make the Weinstock inequality fails. This counterexample can be generalized to higher dimensional Euclidean spaces. Therefore, further condition such as convexity or simply-connectedness may be needed for higher dimensional domains.

For the case that the volume of the domain is fixed, the problem behaves like the Neumann problem in the Euclidean domain. Brock[9] proves the following stronger result:

Theorem 3.4.1. *The ball minimizes the following sum of inverse Steklov eigenvalues:*

$$\sum_{i=2}^{n+1} \frac{1}{\lambda_i(\Omega)} \tag{3.4.5}$$

among open sets Ω of given volume in \mathbb{R}^n . Here $\lambda_i(\Omega)$ denotes the i -th Steklov eigenvalue.

Note that in the n -dimensional Euclidean ball B_R centered at the origin, we have

$$\lambda_2 = \lambda_3 = \dots = \lambda_{n+1} = \frac{1}{R} \tag{3.4.6}$$

and the corresponding eigenspace is spanned by the coordinate functions. Therefore, it is an easy corollary that the ball maximize λ_2 among all domain with the same volume.

In the setting of Steklov eigenvalue problems, there are two nodal sets we are

concerned: The interior nodal set and the boundary nodal set. These nodal sets may behave differently. For the interior nodal set, some of the behaviors are similar to the Dirichlet problem. For example, the Courant's theorem concerning the upper bounds of the number of interior nodal domains can be established. The lowest eigenfunction is constant which admits only one nodal domain, and the second lowest eigenfunction must have exactly two nodal domains. This is not true for the boundary nodal domains. It is hard to consider the number of boundary nodal domains since the boundary M may not even be connected.

Concerning the nodal set for the harmonic Steklov problem, Bellova and Lin[6] first establish an upper bound $C\lambda^6$ for the boundary nodal set on analytic domains in \mathbb{R}^n by using the frequency function. Later, Zelditch[58] gives the sharp upper bound $C\lambda$ on analytic manifolds with an analytic boundary. For the study of lower bounds on the boundary of a smooth manifold, Wang and Zhu[54] establish a lower bound $c\lambda^{\frac{4-n}{2}}$. Notice that this is the same order as in the Laplace-Beltrami eigenfunction problem since the dimension of M is $n - 1$. For the interior nodal set in Ω , Sogge, Wang and Zhu[49] establish a lower bound $c\lambda^{\frac{2-n}{2}}$. All the current best lower bounds on smooth manifolds employ the theory of pseudo-differential operators to obtain L^p estimates of the eigenfunctions.

3.4.2 Biharmonic Steklov problems

We have the following three biharmonic Steklov eigenfunction problems:

$$\begin{cases} \Delta^2 e_\lambda = 0 & \text{in } \Omega \\ \partial_\nu e_\lambda = \partial_\nu \Delta e_\lambda + \lambda^3 e_\lambda = 0 & \text{on } M; \end{cases} \quad (3.4.7)$$

$$\begin{cases} \Delta^2 e_\lambda = 0 & \text{in } \Omega \\ e_\lambda = \Delta e_\lambda - \lambda \partial_\nu e_\lambda = 0 & \text{on } M; \end{cases} \quad (3.4.8)$$

$$\begin{cases} \Delta^2 e_\lambda = 0 & \text{in } \Omega \\ e_\lambda = \partial_\nu^2 e_\lambda - \lambda \partial_\nu e_\lambda = 0 & \text{on } M. \end{cases} \quad (3.4.9)$$

and the corresponding Rayleigh quotient given by

$$\lambda_k^3 = \sup_E \left(\inf_{u \in E} \frac{\int_\Omega |\Delta u|^2 dV}{\int_M u^2 dV} \right), \quad (3.4.10)$$

$$\lambda_k = \sup_E \left(\inf_{u \in E} \frac{\int_\Omega |\Delta u|^2 dV}{\int_M \partial_\nu u^2 dV} \right), \quad (3.4.11)$$

where E ranges from all codimension $k - 1$ subspace in $H^2(\Omega)$ for the first two problems.

The problems arise in elastic mechanics. When the weight of the body M is the only body force, the stress function must be biharmonic in M . In addition, the problem (3.4.8) is referred to as the Dirichlet eigenvalue problem in [45] and it is related to the study of Poisson ratio in theory of elasticity, see [25]. Kutter and Sigillito[37], Payne[45], Wang and Xia[56] focus on giving bounds for the first eigenvalues, which are closely related to the geometry of the manifold.

All the Steklov problems above are also important in the inverse problem. The inverse problem was initially studied by Calderón[13]. The boundary operators for biharmonic Steklov problems are defined similarly as in the harmonic Steklov problem. We can obtain the well-known “Dirichlet to Neumann Laplacian” map and the “Neumann to Laplacian” map for biharmonic equation, respectively. These maps concern the relation between different boundary data.

Unlike the Laplace-Beltrami eigenfunction problems, the Steklov eigenfunction problems are not defined locally on the boundary. Therefore, most of the standard local results for elliptic PDEs such as the maximum principle fail in this problem. For example, in the Laplace-Beltrami eigenfunction problem, there is a constant R such that there is a zero of e_λ in any geodesic ball of radius $R\lambda^{-1}$. But whether the Steklov eigenfunction problem admits this property is still unknown, either on the boundary or inside M .

Just as the harmonic Steklov problem, the biharmonic Steklov problems don't admit a shape optimization property in general. For example, Bucur, Ferrero and Gazzola[11] find a sequence of annulus in \mathbb{R}^2 with the first eigenvalue λ_1 of problem

(3.4.8) goes to zero. Therefore, we can only hope the shape optimization property holds for simply-connected domains or even only for convex domains.

By using the theory of pseudo-differential operators on the boundary, the spectrum is discrete and the only accumulation points of eigenvalues is infinity in each problem. In view of the important applications, one is interested in finding the asymptotic behavior for eigenvalues and corresponding eigenfunctions. Weyl's theorem that λ_k is asymptotic to $c'_n(\frac{k}{V})^{\frac{1}{n}}$ is a direct application of the theory of pseudo-differential operators. This is given by Liu in [38], [40].

In my work, the biharmonic Steklov problems are studied. Polynomial lower bounds are established for size of the boundary nodal sets, the vanishing sets of Δe_λ and the interior nodal sets for (3.4.7), (3.4.8) and (3.4.9).

Theorem 3.4.2. *If 0 is a regular value of e_λ on M for (3.4.7) case, or 0 is a regular value of $\partial_\nu e_\lambda$ on M for (3.4.8), (3.4.9) case, we have*

$$|\tilde{Z}_\lambda| \geq c\lambda^{\frac{4-n}{2}}. \quad (3.4.12)$$

where

$$\begin{aligned} \tilde{Z}_\lambda &= \{x \in M | e_\lambda = 0\} \quad \text{for problem (3.4.7),} \\ \tilde{Z}_\lambda &= \{x \in M | \partial_\nu e_\lambda = 0\} \quad \text{for problems (3.4.8), (3.4.9).} \end{aligned} \quad (3.4.13)$$

Theorem 3.4.3. *For e_λ satisfying (3.4.7), (3.4.8) or (3.4.9), if 0 is a regular value of Δe_λ , we have*

$$|\hat{Z}_\lambda| \geq c\lambda^{\frac{2-n}{2}}, \quad (3.4.14)$$

where

$$\hat{Z}_\lambda = \{x \in \Omega | \Delta e_\lambda = 0\}. \quad (3.4.15)$$

Theorem 3.4.4. *For e_λ satisfying (3.4.8) or (3.4.9), if 0 is a regular value of e_λ , we have*

$$|Z_\lambda| \geq c\lambda^{\frac{2-n}{2}}. \quad (3.4.16)$$

For e_λ satisfying (3.4.7), if 0 is a regular value of e_λ , we have

$$|Z_\lambda| \geq c\lambda^{-\frac{n}{2}}, \quad (3.4.17)$$

where

$$Z_\lambda = \{x \in \Omega | e_\lambda = 0\}. \quad (3.4.18)$$

Remark 3.4.5. *The reader may compare theorem 3.4.4 to the lower bound of interior Steklov nodal sets given in [49], which is of the same order except the bound for $|Z_\lambda|$ in (3.4.7) case. Also, the reader may compare theorem 3.4.2 to the lower bound of boundary Steklov nodal sets given in [54]. Again, we get a lower bound with the same order.*

Chapter 4

Nodal sets for biharmonic Steklov problems

This chapter is organized as follows.

In section 1, we introduce related boundary operators and establish important equations for biharmonic functions. We also recall the Green's formula for biharmonic functions in terms of the boundary data.

In section 2, using the method of layer potentials as in [52], we show that the boundary operators are elliptic pseudo-differential operators on ∂M , which is different from the proof given in [40]. By the pseudo-differential operator theory, we establish the L^p estimates from the theorem in [48]. From this, L^∞ , L^2 , L^1 bounds for $|\nabla e_\lambda|$ is given on the sets which we want to find a lower bound as in [54] and [32].

In section 3, 4 we focus on the vanishing set of Δe_λ , \hat{Z}_λ , and the interior nodal set, Z_λ . The estimation is similar as in the harmonic Steklov problem. We need to do the biharmonic version of estimation.

In section 5, we use the theory of pseudo-differential operator to argue generally of the boundary nodal set, \tilde{Z}_λ , for all the eigenfunction problems.

4.1 Some basic properties for the biharmonic Steklov problem

The biharmonic Steklov problems are related to the boundary operators. The eigenfunctions e_λ in (3.4.7), (3.4.8) and (3.4.9) satisfy the eigenvalue problems

$$\begin{aligned}\Theta e_\lambda &= \lambda^3 e_\lambda, \\ \Xi \partial_\nu e_\lambda &= \lambda \partial_\nu e_\lambda, \\ \Pi \partial_\nu e_\lambda &= \lambda \partial_\nu e_\lambda,\end{aligned}\tag{4.1.1}$$

on M , respectively, for the Dirichlet-to-Neumann-Laplacian operator Θ , Neumann-to-Laplacian operator Ξ , Neumann-to-double-Neumann operator Π , which are defined below. For $f \in C^\infty(M)$, define

$$\begin{aligned}\Theta f &= -\partial_\nu \Delta(K_1 f)|_M, \\ \Xi f &= \Delta(K_2 f)|_M, \\ \Pi f &= \partial_\nu^2(K_2 f)|_M,\end{aligned}\tag{4.1.2}$$

where $K_1 f = u$ is the unique biharmonic function in Ω with $u|_M = f$, $\partial_\nu u|_M = 0$ and $K_2 f = v$ is the unique biharmonic function in Ω with $v|_M = 0$, $\partial_\nu v|_M = f$.

First, let us show the operator Π is well defined and establish the relation between Ξ and Π .

Theorem 4.1.1. *The ∂_ν^2 in the definition of Π is well defined. We have $\Xi f = \Pi f + Hf$, where H is the mean curvature of M .*

Proof. Let F be a smooth function on M with $F|_M = 0$. Let N be any unit vector field defined in a neighborhood of M in Ω with $N|_M = \partial_\nu$. We have

$$NNF = N(dF(N)) = \nabla^2 F(N, N) + dF(\nabla_N N).\tag{4.1.3}$$

Since $\nabla_N N \perp N$, $\nabla_N N|_M$ is tangent to M . Using $F|_M = 0$, we have $dF(\nabla_N N) = 0$. Therefore, $NNF = \nabla^2 F(N, N)$ is tensorial and only depends on $N|_M = \partial_\nu$.

Now, let $\{e_i\}_{i=1}^{n-1} \cup \{N\}$ be an orthonormal frame in a neighborhood of the boundary M . We have

$$\Delta F = \nabla^2 F(N, N) + \sum_{i=1}^{n-1} \nabla^2 F(e_i, e_i) = \nabla^2 F(N, N) + \sum_{i=1}^{n-1} \left(e_i e_i F - dF(\nabla_{e_i} e_i) \right). \quad (4.1.4)$$

Given that $F|_M = 0$, $e_i|_M$ is tangent to M , we have $e_i e_i F = 0$.

$$\Delta F|_M = \partial_\nu^2 F - \sum_{i=1}^{n-1} dF(\nabla_{e_i} e_i) = \partial_\nu^2 F - dF\left(\sum_{i=1}^{n-1} \nabla_{e_i} e_i\right) = \partial_\nu^2 F + H \partial_\nu F. \quad (4.1.5)$$

Plug in $F = K_2 f$ and we can get the desired result. \square

Next, let us recall the Green's formula for biharmonic function:

Theorem 4.1.2. *Let u, v be biharmonic functions defined on Ω , we have*

$$\begin{aligned} 0 &= \int_M \partial_\nu \Delta uv - \Delta u \partial_\nu v + \partial_\nu u \Delta v - u \partial_\nu \Delta v d\sigma, \\ \int_\Omega (\Delta u)^2 &= \int_M \partial_\nu u \Delta u - u \partial_\nu \Delta u d\sigma. \end{aligned} \quad (4.1.6)$$

Proof. Integrate the equation

$$\begin{aligned} \Delta^2 uv - u \Delta^2 v &= \nabla \cdot (\nabla \Delta uv - \Delta u \nabla v + \nabla u \Delta v - u \nabla \Delta v), \\ (\Delta u)^2 - u \Delta^2 u &= \nabla \cdot (\nabla u \Delta u - u \nabla \Delta u). \end{aligned} \quad (4.1.7)$$

in Ω and use divergence theorem. \square

From the equation above, we can deduce that Θ, Ξ, Π are self-adjoint, Θ and Ξ are positive.

Now, let $\hat{E}(x, y)$ be a symmetric fundamental solution to the biharmonic equation:

$$\Delta_x^2 \hat{E}(x, y) = \delta_y(x). \quad (4.1.8)$$

From the symmetry, we also have $\Delta_y^2 \hat{E}(x, y) = \delta_x(y)$. We have the following Green's formula for the biharmonic functions.

Theorem 4.1.3. *If u is a biharmonic function on Ω , we have*

$$u(x) = \int_M \left[u \cdot \partial_\nu^y \Delta_y \hat{E}(x, y) - \partial_\nu u \cdot \Delta_y \hat{E}(x, y) + \Delta u \cdot \partial_\nu^y \hat{E}(x, y) - \partial_\nu \Delta u \cdot \hat{E}(x, y) \right] d\sigma(y). \quad (4.1.9)$$

4.2 Layer potentials

Now, to establish the result, we need some technical results for biharmonic boundary Steklov operators. Let Ω as above, we can extend the manifold across the boundary such that $\bar{\Omega} \subset \hat{\Omega}$, where $\hat{\Omega}$ is a smooth n -dimensional manifold. Let $\mathcal{O} \subset \hat{\Omega}$ be a precompact open neighborhood of $\bar{\Omega}$. Start with a symmetric fundamental solution $E^\circ(x, y)$ of the Laplacian operator,

$$\Delta_x E^\circ(x, y) = \delta_y(x), \quad (4.2.1)$$

where $E^\circ(x, y)$ is the Schwartz kernel of the operator $E^\circ(x, D) \in OPS^{-2}(\hat{\Omega})$. Now, let $\eta \in C_0^\infty(\hat{\Omega})$ be a cutoff function which is identically 1 in \mathcal{O} and $E(x, y) = \eta(x)\eta(y)E^\circ(x, y)$ be the Schwartz kernel of the compactly supported operator $E(x, D) \in OPS^{-2}(\hat{\Omega})$. We can construct the following fundamental solution for the biharmonic equation:

$$\hat{E}(x, y) = \int_{\hat{\Omega}} E(x, z)E(z, y)dV(z), \quad (4.2.2)$$

which satisfies

$$\Delta_x^2 \hat{E}(x, y) = \delta_y(x), \Delta_y^2 \hat{E}(x, y) = \delta_x(y) \quad (4.2.3)$$

in \mathcal{O} . $\hat{E}(x, y)$ is the Schwartz kernel of a compactly supported operator $\hat{E}(x, D)$, where $\hat{E}(x, D) = E(x, D)E(x, D) \in OPS^{-4}(\hat{\Omega})$. The Schwartz kernel $\hat{E}(x, y)$ is

smooth off the diagonal. As $x \rightarrow y$, we have the following expansion:

$$\hat{E}(x, y) = c_n \begin{cases} R(x, y) + d(x, y)^2 \log(d(x, y)) + \dots & n = 2, \\ R(x, y) + d(x, y) + \dots & n = 3, \\ \log(d(x, y)) + \dots & n = 4, \\ d(x, y)^{4-n} + \dots & n \geq 5, \end{cases} \quad (4.2.4)$$

where $R(x, y)$ is smooth, in dimension $n = 2, 3$, they are more significant than the part contribute to $\Delta_x^2 \hat{E}(x, y) = \delta_y(x)$, but they only contribute to a smoothing operator. The function $d(x, y)$ is the distance on the manifold, and the constant

$$c_n = \begin{cases} \frac{1}{4\omega_1} = \frac{1}{8\pi} & n = 2, \\ \frac{-1}{4\omega_3} & n = 4, \\ \frac{1}{2(4-n)(2-n)\omega_{n-1}} & n = 3, n \geq 5, \end{cases} \quad (4.2.5)$$

with $\omega_n = Vol(S^n)$. For a function f on M , follow the same approach as in [52], we define the following potentials in Ω .

$$\begin{aligned} L_1 f(x) &= \int_M f(y) \hat{E}(x, y) d\sigma(y), \\ L_2 f(x) &= \int_M f(y) \partial_{\nu, y} \hat{E}(x, y) d\sigma(y), \\ L_3 f(x) &= \int_M f(y) \Delta_y \hat{E}(x, y) d\sigma(y), \\ L_4 f(x) &= \int_M f(y) \partial_{\nu, y} \Delta_y \hat{E}(x, y) d\sigma(y). \end{aligned} \quad (4.2.6)$$

Given a function u on $\hat{\Omega} \setminus M$. For $x \in M$, define $u_+(x)$ and $u_-(x)$ be the limit of $u(z)$ as $z \rightarrow x$, from $z \in \Omega$ and $z \in \hat{\Omega} \setminus \bar{\Omega}$, respectively. Now, we can find the limit of the above layer potentials on M .

Proposition 4.2.1. *For $x \in M$, we have*

$$L_1 f_+(x) = L_1 f_-(x) = S_3 f(x), \quad (4.2.7)$$

$$L_2 f_+(x) = L_2 f_-(x) = S_2 f(x), \quad (4.2.8)$$

$$L_3 f_+(x) = L_3 f_-(x) = S_1 f(x), \quad (4.2.9)$$

$$L_4 f_\pm(x) = \pm \frac{1}{2} f(x) + N f(x), \quad (4.2.10)$$

where, for $x \in M$,

$$S_3 f(x) = \int_M f(y) \hat{E}(x, y) d\sigma(y), \quad (4.2.11)$$

$$S_2 f(x) = \int_M f(y) \partial_{\nu, y} \hat{E}(x, y) d\sigma(y), \quad (4.2.12)$$

$$S_1 f(x) = \int_M f(y) \Delta_y \hat{E}(x, y) d\sigma(y), \quad (4.2.13)$$

and

$$N f(x) = \int_M f(y) \partial_{\nu, y} \Delta_y \hat{E}(x, y) d\sigma(y). \quad (4.2.14)$$

Furthermore, for the operators defined above, we have $S_2, S_3 \in OPS^{-3}(M)$, $S_1, N \in OPS^{-1}(M)$. S_1 and S_3 are elliptic. The principle symbols of S_1 and S_3 are the same as that of $\frac{1}{2}\sqrt{-\Delta_M}^{-1}$ and $\frac{1}{4}\sqrt{-\Delta_M}^{-3}$, respectively, where Δ_M is the Laplacian operator on the boundary $\partial\Omega = M$.

Proof. Following [52], if $\sigma \in \mathcal{E}'(\hat{\Omega})$ is the surface measure on M , $f \in \mathcal{D}'(M)$, we have $f\sigma \in \mathcal{E}'(\hat{\Omega})$. Now, let $p(x, D) \in OPS^m(\hat{\Omega})$, define

$$v = p(x, D)(f\sigma). \quad (4.2.15)$$

When $m < -1$, v is continuous even across M and

$$v|_M = Qf, \quad Q \in OPS^{m+1}(M). \quad (4.2.16)$$

We need to compute the principle symbols of them. At any point on M , choose the coordinates such that $\{x_i\}_{i=1}^{n-1}$ are normal coordinates on M and x_n is the normal direction pointing into Ω . The symbol of $Q(x, D)$ is given by

$$q(x_n, x', \xi') = \frac{1}{2\pi} \int p(x, \xi', \xi_n) e^{ix_n \xi_n} d\xi_n. \quad (4.2.17)$$

In this coordinate, put $p_3(x, D) = \hat{E}(x, D)$, $p_2(x, D) = \hat{E}(x, D)X^*$, $p_1(x, D) = \hat{E}(x, D)\Delta$ respectively, where X is any vector field on $\hat{\Omega}$ which equals the outer normal ν on M and X^* its formal adjoint. The corresponding principle symbols are $p_3(x, \xi) = |\xi|^{-4}$, $p_2(x, \xi) = i\xi_n|\xi|^{-4}$, $p_1(x, \xi) = -|\xi|^2$. Use this, we can get

$$\begin{aligned} q_3(x_n, x', \xi') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\xi|^{-4} e^{ix_n \xi_n} d\xi_n = \frac{1}{4} \left(\frac{1}{|\xi'|^3} + \frac{|x_n|}{|\xi'|^2} \right) e^{-|x_n \xi'|}, \\ q_2(x_n, x', \xi') &= \frac{i}{2\pi} \int_{-\infty}^{\infty} \xi_n |\xi|^{-4} e^{ix_n \xi_n} d\xi_n = \frac{-x_n}{4|\xi'|} e^{-|x_n \xi'|}, \\ q_1(x_n, x', \xi') &= \frac{-1}{2\pi} \int_{-\infty}^{\infty} |\xi|^{-2} e^{ix_n \xi_n} d\xi_n = -\frac{e^{-|x_n \xi'|}}{2|\xi'|}, \end{aligned} \quad (4.2.18)$$

Taking the limit as x_n goes to 0. For $|\xi'| > 1$ the right hand side uniformly converge. Therefore, after restricting on M , the principle symbol of S_3, S_1 are $\frac{1}{4}|\xi'|^{-3}, -\frac{1}{2}|\xi'|^{-1}$ respectively. For q_2 , since the right side converge to 0, and the term with $O(|\xi|^{-4})$ only contribute to $OPS^{-3}(M)$, we can conclude the resulting operator $S_2 \in OPS^{-3}(M)$. We can establish (4.2.7), (4.2.8), (4.2.9) and the properties of S_3, S_2, S_1 .

Now, let us turn out attention to (4.2.10). Put $p(x_n, \xi', \xi_n) = -i\xi_n|\xi|^{-2} + p'(x_n, \xi', \xi_n)$, where $p'(x, D) \in OPS^{-2}(\hat{\Omega})$. Since

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} -i\xi_n |\xi|^{-2} d\xi_n = \begin{cases} \frac{e^{-x_n |\xi'|}}{2} & x_n > 0, \\ \frac{-e^{x_n |\xi'|}}{2} & x_n < 0. \end{cases} \quad (4.2.19)$$

Let x_n goes to 0, the contribution of p' will converge to the same limit from both positive and negative direction. Therefore, $v_{\pm} = Q_{\pm}f$, where $Q_{\pm} \in OPS^0(M)$. $Q_{\pm} = \pm \frac{1}{2}I + Q'$, with $Q' \in OPS^{-1}(M)$. Now, for $\partial_{\nu, y} \Delta_y \hat{E}(x, y)$, the expansion when x is near y is given by

$$\nabla_y \Delta_y \hat{E}(x, y) = -\frac{1}{\omega_{n-1}} d(x, y)^{1-n} V_{x, y} + \dots, \quad (4.2.20)$$

where $V_{x, y}$ denotes the unit vector at y in the direction of the geodesic from x to y . Therefore,

$$\partial_{\nu, y} \Delta_y \hat{E}(x, y) = -\frac{1}{\omega_{n-1}} d(x, y)^{1-n} \langle V_{x, y}, \nu_y \rangle + \dots. \quad (4.2.21)$$

Since $\langle V_{x,y}, \nu_y \rangle$ is Lipschitz on $M \times M$ and vanishes on the diagonal, $\partial_{\nu,y} \Delta_y \hat{E}(x,y)$ is integrable on $M \times M$. Q_{\pm} has Schwartz kernels equal to $\partial_{\nu,y} \Delta_y \hat{E}(x,y)$ on the compliment of the diagonal in $M \times M$, together with the knowledge of the principle symbol of Q_{\pm} , we establish (4.2.10). \square

Now, we investigate the relation between the boundary biharmonic Steklov operators and the operators defined above.

Theorem 4.2.2. *For the biharmonic Steklov operators, Θ , Ξ and Π , we have $\Xi, \Pi \in OPS^1(M)$, $\Theta \in OPS^3(M)$. All of them are elliptic. The principle symbol of Θ is equal to the principle symbol of $2\sqrt{-\Delta_M^3}$. The principle symbols of Ξ, Π are equal to the principle symbol of $2\sqrt{-\Delta_M}$.*

Proof. For $f \in C^\infty(M)$, let $u = K_1 f \in C^\infty(\bar{\Omega})$. Define operator θ on M to be $\theta f = \Delta u|_M$. Since $\Delta(\Delta u) = 0$, we have $\Lambda \theta f = -\Theta f$, where Λ is the Dirichlet to Neumann operator for the harmonic Steklov problem. From the Green's formula,

$$\begin{aligned} u(x) &= \int_M u \partial_{\nu,y} \Delta_y \hat{E}(x,y) - \partial_\nu u \Delta_y \hat{E}(x,y) + \Delta u \partial_{\nu,y} \hat{E}(x,y) - \partial_\nu \Delta u \hat{E}(x,y) d\sigma(y) \\ &= \int_M f \partial_{\nu,y} \Delta_y \hat{E}(x,y) + \theta f \partial_{\nu,y} \hat{E}(x,y) - \Lambda \theta f \hat{E}(x,y) d\sigma(y) \\ &= L_4 f(x) + L_2 \theta f(x) - L_1 \Lambda \theta f(x). \end{aligned} \tag{4.2.22}$$

for $x \in \Omega$. Taking the limit, let x goes to a boundary point, we have

$$f = \frac{1}{2} f + \frac{1}{2} N f + S_2 \theta f - S_3 \Lambda \theta f \tag{4.2.23}$$

on M , which can be written as

$$\left(\frac{1}{2} - \frac{1}{2} N\right) f = (S_2 - S_3 \Lambda) \theta f. \tag{4.2.24}$$

Note that $S_2, S_3 \in OPS^{-3}(M)$, $\Lambda \in OPS^1(M)$, the principle symbols of S_3 and Λ are given by $\frac{1}{4} |\xi'|^{-3}$, $|\xi'|$ respectively, we can conclude that $\theta \in OPS^2(M)$ and the

corresponding principle symbol is $-2|\xi'|^2$. Therefore, $\Theta = -\Lambda\theta \in OPS^3(M)$ with principle symbol $2|\xi'|^3$.

Now, we deal the operators Ξ and Π in a similar way. For $f \in C^\infty(M)$, let $v = K_2f \in C^\infty(\bar{\Omega})$. Using Green's formula,

$$\begin{aligned}
v(x) &= \int_M v \partial_{\nu,y} \Delta_y \hat{E}(x,y) - \partial_\nu v \Delta_y \hat{E}(x,y) + \Delta v \partial_{\nu,y} \hat{E}(x,y) - \partial_\nu \Delta v \hat{E}(x,y) d\sigma(y) \\
&= \int_M -f \Delta_y \hat{E}(x,y) + \Xi f \partial_{\nu,y} \hat{E}(x,y) - \Lambda \Xi f \hat{E}(x,y) d\sigma(y) \\
&= -L_3 f(x) + L_2 \Xi f(x) - L_1 \Lambda \Xi f(x)
\end{aligned} \tag{4.2.25}$$

for $x \in \Omega$. Taking the limit as x goes to a boundary point, we have

$$0 = -S_1 f + S_2 \Xi f - S_3 \Lambda \Xi f. \tag{4.2.26}$$

on M , which is the same as

$$S_1 f = (S_2 - S_3 \Lambda) \Xi f. \tag{4.2.27}$$

Use the argument as above, we can conclude that $\Xi \in OPS^1(M)$ with the principle symbol $2|\xi'|$. Finally, recall that $\Pi = \Xi + H$ on M , we have $\Pi \in OPS^1(M)$ with the same principle symbol. \square

Remark 4.2.3. *The operator θ defined in the proof above may not be self-adjoint. In the proof, we only need the ellipticity of the operator.*

4.2.1 L^p estimates

One of the most important ingredients for the proof is the L^p estimates for eigenfunctions. For simplicity, in the following, we use $A \lesssim B$ to mean there exist constant C independent of λ such that $A \leq CB$ when λ large enough. $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

We have that $\sqrt[3]{\Theta}$, Ξ and Π are classical order 1 pseudo-differential operators

with principle symbol equal to some nonzero constants times the principle symbol of $\sqrt{-\Delta_M}$. From the theory of pseudo-differential operators in the previous chapter, we have the following:

Theorem 4.2.4. *For the Steklov eigenfunctions e_λ satisfying (3.4.7) and $p \geq 2$, we have*

$$\|e_\lambda\|_{L^p(M)} \lesssim \lambda^{\sigma(n,p)} \|e_\lambda\|_{L^2(M)}. \quad (4.2.28)$$

For the Steklov eigenfunctions e_λ satisfying (3.4.8) or (3.4.9), $p \geq 2$, we have

$$\|\partial_\nu e_\lambda\|_{L^p(M)} \lesssim \lambda^{\sigma(n,p)} \|\partial_\nu e_\lambda\|_{L^2(M)}. \quad (4.2.29)$$

where

$$\sigma(n,p) = \begin{cases} (n-1)\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}, & \frac{2n}{n-2} \leq p \leq \infty \\ \frac{n-2}{2}\left(\frac{1}{2} - \frac{1}{p}\right), & 2 \leq p \leq \frac{2n}{n-2}. \end{cases} \quad (4.2.30)$$

Use this theorem for $p = \frac{2n}{n-2}$ and the Holder inequality, we have

$$\|e_\lambda\|_{L^1(M)} \gtrsim \lambda^{-\frac{n-2}{4}} \|e_\lambda\|_{L^2(M)} \quad (4.2.31)$$

for e_λ satisfying (3.4.7) and

$$\|\partial_\nu e_\lambda\|_{L^1(M)} \gtrsim \lambda^{-\frac{n-2}{4}} \|\partial_\nu e_\lambda\|_{L^2(M)} \quad (4.2.32)$$

for e_λ satisfying (3.4.8), (3.4.9).

Now, we establish bounds of L^p estimates when applying pseudo-differential operators to the eigenfunctions.

Lemma 4.2.5. *Fix $p \in (1, \infty)$, for any $P \in OPS^k(M)$, we have*

$$\|Pe_\lambda\|_{L^p(M)} \lesssim \lambda^k \|e_\lambda\|_{L^p(M)} \quad (4.2.33)$$

for (3.4.7) and

$$\|P\partial_\nu e_\lambda\|_{L^p(M)} \lesssim \lambda^k \|\partial_\nu e_\lambda\|_{L^p(M)} \quad (4.2.34)$$

for (3.4.8), (3.4.9).

Proof. Let e_λ satisfies (3.4.7). Since the inverse of $I + \sqrt[3]{\Theta}$ exist, we have $P(I + \sqrt[3]{\Theta})^{-k} \in OPS^0(M)$. Therefore,

$$\begin{aligned} \|Pe_\lambda\|_{L^p(M)} &= \|P(I + \sqrt[3]{\Theta})^{-k}(I + \sqrt[3]{\Theta})^k e_\lambda\|_{L^p(M)} \\ &= (1 + \lambda)^k \|P(I + \sqrt[3]{\Theta})^{-k} e_\lambda\|_{L^p(M)} \\ &\lesssim \lambda^k \|e_\lambda\|_{L^p(M)}. \end{aligned} \quad (4.2.35)$$

We can get the similar result for e_λ satisfying (3.4.8), (3.4.9). \square

For the case that $p = 1$, we need to take extra care.

Lemma 4.2.6. *Let $P \in OPS^k(M)$. Fix $\epsilon > 0$, e_λ satisfying (3.4.7), we have*

$$\|Pe_\lambda\|_{L^1(M)} \lesssim \lambda^{k+\epsilon} \|e_\lambda\|_{L^1(M)}. \quad (4.2.36)$$

If e_λ satisfying (3.4.8), (3.4.9), similarly,

$$\|P\partial_\nu e_\lambda\|_{L^1(M)} \lesssim \lambda^{k+\epsilon} \|\partial_\nu e_\lambda\|_{L^1(M)}. \quad (4.2.37)$$

Proof. We proof the case for $k = 0$ first. If e_λ satisfies (3.4.7), let $\delta > 0$. By Holder's inequality,

$$\begin{aligned} \|Pe_\lambda\|_{L^1(M)} &\lesssim \|Pe_\lambda\|_{L^{1+\delta}(M)} \lesssim \|e_\lambda\|_{L^{1+\delta}(M)} \\ &\leq \|e_\lambda\|_{L^2(M)}^{\frac{2\delta}{1+\delta}} \|e_\lambda\|_{L^1(M)}^{\frac{1-\delta}{1+\delta}} \\ &\lesssim \lambda^{\frac{2(n-1)\delta}{4(1+\delta)}} \|e_\lambda\|_{L^1(M)}. \end{aligned} \quad (4.2.38)$$

Choose δ such that $\frac{2(n-1)\delta}{4(1+\delta)} < \epsilon$, we can get the desired result. For general k , $P(I + \sqrt[3]{\Theta})^{-k} \in OPS^0(M)$. We can use the same argument as in the lemma above. The case for (3.4.8), (3.4.9) can be done in a similar manner. \square

It's convenient to write the L^p norms in terms of that of Δe_λ . We have the following corollary

Corollary 4.2.7. Fix any $p \in [1, \infty)$. For (3.4.7), we have

$$\|\Delta e_\lambda\|_{L^p(M)} \approx \lambda^2 \|e_\lambda\|_{L^p(M)}. \quad (4.2.39)$$

For (3.4.9), we have

$$\|\Delta e_\lambda\|_{L^p(M)} \approx \lambda \|\partial_\nu e_\lambda\|_{L^p(M)}. \quad (4.2.40)$$

Proof. Choose $\epsilon = \frac{1}{2}$. For (3.4.7), we have $\Delta e_\lambda|_M = \theta e_\lambda$. Using $\theta + \sqrt[3]{2\Theta^2} = P'_1 \in OPS^1(M)$, we have

$$\theta e_\lambda = (-\sqrt[3]{2\Theta^2} + P'_1)e_\lambda = (-\sqrt[3]{2}\lambda^2 + P'_1)e_\lambda \quad (4.2.41)$$

on M . Therefore,

$$\begin{aligned} & \sqrt[3]{2}\lambda^2 \|e_\lambda\|_{L^p(M)} - \|P'_1 e_\lambda\|_{L^p(M)} \\ & \leq \|\Delta e_\lambda\|_{L^p(M)} \\ & \leq \sqrt[3]{2}\lambda^2 \|e_\lambda\|_{L^p(M)} + \|P'_1 e_\lambda\|_{L^p(M)}. \end{aligned} \quad (4.2.42)$$

Since $\|P'_1 e_\lambda\|_{L^p(M)} \lesssim \lambda^{\frac{3}{2}} \|e_\lambda\|_{L^p(M)}$, we can get the desired result. The case for (3.4.9) is similar. \square

4.3 Lower bound for the vanishing set of Δe_λ

For (3.4.8), we can think Δe_λ as the extension of the boundary data into Ω . Thus it would be interesting to get a lower bound of its vanishing set. Let $\hat{Z}_\lambda^\alpha = \{x \in \Omega | \Delta e_\lambda = \alpha\}$ be the α -level set of Δe_λ . Define

$$\sigma_\alpha(x) = \begin{cases} 1 & x > \alpha \\ 0 & x = \alpha \\ -1 & x < \alpha. \end{cases} \quad (4.3.1)$$

We have the following equation.

Theorem 4.3.1. For any $f \in C^\infty(\bar{\Omega})$, any regular value α of Δe_λ , we have

$$\int_M f \sigma_\alpha(\Delta e_\lambda) \partial_\nu \Delta e_\lambda d\sigma - \int_\Omega \sigma_\alpha(\Delta e_\lambda) \langle \nabla f, \nabla \Delta e_\lambda \rangle dV = 2 \int_{\hat{Z}_\lambda^\alpha} f |\nabla \Delta e_\lambda| d\sigma. \quad (4.3.2)$$

Proof. Let $\{\hat{D}_k^{+, \alpha}\}_k$ be the collection of connected components of the set $\{\Delta e_\lambda > \alpha\} \cap \Omega$, $\hat{Z}_k^{+, \alpha} = \partial \hat{D}_k^{+, \alpha} \cap \Omega$, $\hat{Y}_k^{+, \alpha} = \partial \hat{D}_k^{+, \alpha} \cap M$. We have

$$\begin{aligned} \int_{\hat{D}_k^{+, \alpha}} \langle \nabla f, \nabla \Delta e_\lambda \rangle dV &= - \int_{\hat{D}_k^{+, \alpha}} f \Delta^2 e_\lambda dV - \int_{\hat{Z}_k^{+, \alpha}} f |\nabla \Delta e_\lambda| d\sigma + \int_{\hat{Y}_k^{+, \alpha}} f \partial_\nu \Delta e_\lambda d\sigma \\ &= - \int_{\hat{Z}_k^{+, \alpha}} f |\nabla \Delta e_\lambda| d\sigma + \int_{\hat{Y}_k^{+, \alpha}} f \partial_\nu \Delta e_\lambda d\sigma. \end{aligned} \quad (4.3.3)$$

Similarly, from the set $\{\Delta e_\lambda < \alpha\} \cap \Omega$, we can define $\hat{D}_k^{-, \alpha}$, $\hat{Z}_k^{-, \alpha}$, $\hat{Y}_k^{-, \alpha}$ together with a similar equation:

$$- \int_{\hat{D}_k^{-, \alpha}} \langle \nabla f, \nabla \Delta e_\lambda \rangle dV = - \int_{\hat{Z}_k^{-, \alpha}} f |\nabla \Delta e_\lambda| d\sigma - \int_{\hat{Y}_k^{-, \alpha}} f \partial_\nu \Delta e_\lambda d\sigma. \quad (4.3.4)$$

Summing over all the equation above and notice that almost every point on \hat{Z}_λ^α will appear once for some $\hat{Z}_k^{+, \alpha}$ and once for some $\hat{Z}_k^{-, \alpha}$, we can get the desired equation. \square

Plug in $f = 1$ in the theorem, we get the following:

Corollary 4.3.2. There exists a constant c such that for any biharmonic Steklov eigenfunciton e_λ satisfying (3.4.7), (3.4.8), (3.4.9), any regular value α of Δe_λ satisfying $|\alpha| < c\lambda^{\frac{2-n}{4}} \|\Delta e_\lambda\|_{L^2(M)}$, we have

$$\int_{\hat{Z}_\lambda^\alpha} |\nabla \Delta e_\lambda| d\sigma \gtrsim \lambda \|\Delta e_\lambda\|_{L^1(M)}. \quad (4.3.5)$$

Proof. For the eigenfunction satisfying (3.4.8), we have

$$\begin{aligned} 2 \int_{\hat{Z}_\lambda^\alpha} |\nabla \Delta e_\lambda| d\sigma &= \int_M \sigma_\alpha(\Delta e_\lambda) \partial_\nu \Delta e_\lambda d\sigma \\ &= \int_M \sigma_\alpha(\Delta e_\lambda) \Lambda \Delta e_\lambda d\sigma. \end{aligned} \quad (4.3.6)$$

Since $2\Lambda - \Xi = P \in OPS^0(M)$, we have $\Lambda \Delta e_\lambda = \frac{1}{2}(\Xi + P)\Delta e_\lambda = \frac{1}{2}\lambda \Delta e_\lambda + \frac{1}{2}P\Delta e_\lambda$.

Thus

$$\begin{aligned} 2 \int_{\hat{Z}_\lambda^\alpha} |\nabla \Delta e_\lambda| d\sigma &= \frac{1}{2} \int_M \sigma_\alpha(\Delta e_\lambda) (\lambda + P) \Delta e_\lambda d\sigma \\ &= \frac{\lambda}{2} \int_M \sigma_\alpha(\Delta e_\lambda) \Delta e_\lambda d\sigma + \frac{1}{2} \int_M \sigma_\alpha(\Delta e_\lambda) P \Delta e_\lambda d\sigma \\ &\geq \frac{\lambda}{2} \left(\|\Delta e_\lambda\|_{L^1(M)} - 2\alpha|M| \right) - \frac{1}{2} \|P\Delta e_\lambda\|_{L^1(M)} \\ &\geq \frac{\lambda}{2} \|\Delta e_\lambda\|_{L^1(M)} - \lambda\alpha|M| - \frac{C}{2} \lambda^\epsilon \|\Delta e_\lambda\|_{L^1(M)} \end{aligned} \quad (4.3.7)$$

Choose c which only depend on Ω such that for any $|\alpha| < c\lambda^{\frac{2-n}{4}} \|\Delta e_\lambda\|_{L^2(M)}$, $\alpha|M| < \frac{1}{8} \|\Delta e_\lambda\|_{L^1(M)}$. We can get the desired result when λ is large.

For the eigenfunction satisfying (3.4.9), use $\Delta e_\lambda = \Pi \partial_\nu e_\lambda + H \partial_\nu e_\lambda = \lambda \partial_\nu e_\lambda + H \partial_\nu e_\lambda$ and $2\Lambda - \Pi = P' \in OPS^0(M)$. We have

$$\begin{aligned} \Lambda \Delta e_\lambda &= \frac{1}{2}(\Pi + P')(\lambda \partial_\nu e_\lambda + H \partial_\nu e_\lambda) = \frac{1}{2}\lambda^2 \partial_\nu e_\lambda + \frac{\lambda}{2} P' \partial_\nu e_\lambda + \frac{1}{2} \Lambda H \partial_\nu e_\lambda \\ &= \frac{1}{2}\lambda(\Delta e_\lambda - H \partial_\nu e_\lambda) + \frac{\lambda}{2} P' \partial_\nu e_\lambda + \frac{1}{2} \Lambda H \partial_\nu e_\lambda \\ &= \frac{\lambda}{2} \Delta e_\lambda + \frac{1}{2} (H\Pi + P'\Pi + \Lambda H) \partial_\nu e_\lambda. \end{aligned} \quad (4.3.8)$$

Notice that $\frac{1}{2}(H\Pi + P'\Pi + \Lambda H) \in OPS^1(M)$. Therefore

$$\begin{aligned} \left\| \frac{1}{2} (H\Pi + P'\Pi + \Lambda H) \partial_\nu e_\lambda \right\|_{L^1(M)} &\lesssim \lambda^{1+\epsilon} \|\partial_\nu e_\lambda\|_{L^1(M)} \\ &\lesssim \lambda^\epsilon \|\Delta e_\lambda\|_{L^1(M)}. \end{aligned} \quad (4.3.9)$$

We can use the same approach as in (4.3.7) to get the estimation for (3.4.9).

For the eigenfunction satisfying (3.4.7), we have $\Delta e_\lambda = \theta e_\lambda$ and $\partial_\nu \Delta e_\lambda = \Theta e_\lambda =$

$-\lambda^3 e_\lambda \cdot \sqrt[3]{2\Theta^2} + \theta = P'_1 \in OPS^1(M)$. We have

$$\begin{aligned}
\Lambda \Delta e_\lambda &= -\lambda^3 e_\lambda = -\lambda \sqrt[3]{2\Theta^2} e_\lambda \\
&= \frac{\lambda \theta}{\sqrt[3]{2}} e_\lambda - \frac{\lambda P'_1}{\sqrt[3]{2}} e_\lambda \\
&= \frac{\lambda}{\sqrt[3]{2}} \Delta e_\lambda - \frac{\lambda P'_1}{\sqrt[3]{2}} e_\lambda,
\end{aligned} \tag{4.3.10}$$

also,

$$\left\| \frac{\lambda P'_1}{\sqrt[3]{2}} e_\lambda \right\|_{L^1(M)} \lesssim \lambda \cdot \lambda^{1+\epsilon} \|e_\lambda\|_{L^1(M)} \lesssim \lambda^\epsilon \|\Delta e_\lambda\|_{L^1(M)}. \tag{4.3.11}$$

Again, we can use the same approach as in (4.3.7) to get the estimation for (3.4.7) when λ is sufficiently large. \square

Remark 4.3.3. For the operator Θ, Π , the eigenfunctions are $e_\lambda|_M, \partial_\nu e_\lambda$ respectively. It's more nature to write the norm in terms of the eigenfunctions. We choose $\Delta e_\lambda|_M$ to make the result for all the cases look similar.

Next, we can plug in $f = \sqrt{1 + |\nabla \Delta e_\lambda|^2}$ and get the following proposition.

Proposition 4.3.4. For the eigenfunctions satisfying (3.4.7), (3.4.8), (3.4.9), we have the following estimation when λ is large enough:

$$\int_{\hat{Z}_\lambda^\alpha} |\nabla \Delta e_\lambda|^2 d\sigma \lesssim \lambda^2 \|\Delta e_\lambda\|_{L^2(M)}^2. \tag{4.3.12}$$

Proof. Plug in $f = \sqrt{1 + |\nabla \Delta e_\lambda|^2}$, we have

$$\begin{aligned}
2 \int_{\hat{Z}_\lambda^\alpha} |\nabla \Delta e_\lambda|^2 d\sigma &\leq 2 \int_{\hat{Z}_\lambda^\alpha} |\nabla \Delta e_\lambda| \sqrt{1 + |\nabla \Delta e_\lambda|^2} d\sigma \\
&\leq \int_M \sqrt{1 + |\nabla \Delta e_\lambda|^2} |\partial_\nu \Delta e_\lambda| d\sigma \\
&\quad + \int_\Omega |\langle \nabla \sqrt{1 + |\nabla \Delta e_\lambda|^2}, \nabla \Delta e_\lambda \rangle| dV.
\end{aligned} \tag{4.3.13}$$

We need to estimate both terms on the right hand side. First, let e_λ satisfies (3.4.8). $|\nabla \Delta e_\lambda|^2 = (\partial_\nu \Delta e_\lambda)^2 + |\nabla^T \Delta e_\lambda|^2$ on M , where ∇^T denotes the gradient on M . Since

$\Delta_M \in OPS^2(M)$,

$$\begin{aligned} \|\Delta_M \Delta e_\lambda\|_{L^2(M)} &= \|\Delta_M \Xi \partial_\nu e_\lambda\|_{L^2(M)} \\ &\lesssim \lambda^3 \|\partial_\nu e_\lambda\|_{L^2(M)} \lesssim \lambda^2 \|\Delta e_\lambda\|_{L^2(M)}. \end{aligned} \quad (4.3.14)$$

We can get the following:

$$\begin{aligned} \int_M |\nabla^T \Delta e_\lambda|^2 d\sigma &= - \int_M \Delta e_\lambda \Delta_M \Delta e_\lambda d\sigma \\ &\leq \|\Delta e_\lambda\|_{L^2(M)} \|\Delta_M \Delta e_\lambda\|_{L^2(M)} \\ &\lesssim \lambda^2 \|\Delta e_\lambda\|_{L^2(M)}^2 \end{aligned} \quad (4.3.15)$$

when λ is large enough. Similarly, using $\Lambda \in OPS^1(M)$, we have

$$\|\partial_\nu \Delta e_\lambda\|_{L^2(M)}^2 \lesssim \lambda^2 \|\Delta e_\lambda\|_{L^2(M)}^2, \quad (4.3.16)$$

and therefore

$$\int_M 1 + |\nabla \Delta e_\lambda|^2 d\sigma = \int_M 1 + (\partial_\nu \Delta e_\lambda)^2 + |\nabla^T \Delta e_\lambda|^2 d\sigma \lesssim \lambda^2 \|\Delta e_\lambda\|_{L^2(M)}^2. \quad (4.3.17)$$

The estimation of the first term is given as

$$\begin{aligned} \int_M \sqrt{1 + |\nabla \Delta e_\lambda|^2} |\partial_\nu \Delta e_\lambda| d\sigma &\leq \|\sqrt{1 + |\nabla \Delta e_\lambda|^2}\|_{L^2(M)} \|\partial_\nu \Delta e_\lambda\|_{L^2(M)} \\ &\lesssim \lambda^2 \|\Delta e_\lambda\|_{L^2(M)}^2. \end{aligned} \quad (4.3.18)$$

Now, let us estimate the second term.

$$\begin{aligned} \int_\Omega |\langle \nabla \sqrt{1 + |\nabla \Delta e_\lambda|^2}, \nabla \Delta e_\lambda \rangle| dV &= \int_\Omega \frac{|\nabla^2 \Delta e_\lambda (\nabla \Delta e_\lambda, \nabla \Delta e_\lambda)|}{\sqrt{1 + |\nabla \Delta e_\lambda|^2}} dV \\ &\leq \|\nabla^2 \Delta e_\lambda\|_{L^2(\Omega)} \|\nabla \Delta e_\lambda\|_{L^2(\Omega)} \left\| \frac{\nabla \Delta e_\lambda}{\sqrt{1 + |\nabla \Delta e_\lambda|^2}} \right\|_{L^\infty(\Omega)} \\ &\leq \|\nabla^2 \Delta e_\lambda\|_{L^2(\Omega)} \|\nabla \Delta e_\lambda\|_{L^2(\Omega)}. \end{aligned} \quad (4.3.19)$$

The L^2 norm of $\nabla\Delta e_\lambda$ and $\nabla^2\Delta e_\lambda$ on Ω is needed. We have

$$\begin{aligned}\int_{\Omega} |\nabla\Delta e_\lambda|^2 dV &= -\int_{\Omega} \Delta e_\lambda \Delta^2 e_\lambda dV + \int_M \Delta e_\lambda \partial_\nu \Delta e_\lambda d\sigma \\ &\lesssim \lambda \|\Delta e_\lambda\|_{L^2(M)}^2.\end{aligned}\tag{4.3.20}$$

Therefore, $\|\nabla\Delta e_\lambda\|_{L^2(\Omega)} \lesssim \sqrt{\lambda} \|\Delta e_\lambda\|_{L^2(M)}$.

To estimate $\|\nabla^2\Delta e_\lambda\|_{L^2(\Omega)}$, let us recall the Reilly's formula: for any smooth function f on Ω , we have

$$\begin{aligned}\int_{\Omega} |\nabla^2 f|^2 + \mathbf{Ric}(\nabla f, \nabla f) - (\Delta f)^2 dV \\ = \int_M A(\nabla^T f, \nabla^T f) - 2\partial_\nu f \Delta_M f + H(\partial_\nu f)^2 d\sigma,\end{aligned}\tag{4.3.21}$$

where A is the second fundamental form and H is the mean curvature of M .

Use this formula for Δe_λ , we have

$$\begin{aligned}\int_{\Omega} |\nabla^2 \Delta e_\lambda|^2 &= -\int_{\Omega} \mathbf{Ric}(\nabla \Delta e_\lambda, \nabla \Delta e_\lambda) dV \\ &\quad + \int_M A(\nabla^T \Delta e_\lambda, \nabla^T \Delta e_\lambda) - 2\partial_\nu \Delta e_\lambda \Delta_M \Delta e_\lambda + H(\partial_\nu \Delta e_\lambda)^2 d\sigma \\ &\leq \|\mathbf{Ric}\|_{L^\infty(\Omega)} \|\nabla \Delta e_\lambda\|_{L^2(\Omega)}^2 + \|A\|_{L^\infty(M)} \|\nabla^T \Delta e_\lambda\|_{L^2(M)}^2 \\ &\quad + 2\|\partial_\nu \Delta e_\lambda\|_{L^2(M)} \|\Delta_M \Delta e_\lambda\|_{L^2(M)} + \|H\|_{L^\infty(M)} \|\partial_\nu \Delta e_\lambda\|_{L^2(M)}^2 \\ &\lesssim \left(\|\mathbf{Ric}\|_{L^\infty(\Omega)} \cdot \lambda + (\|A\|_{L^\infty(M)} + 2\lambda) \cdot \lambda^2 + \lambda^2 \|H\|_{L^\infty(M)} \right) \|\Delta e_\lambda\|_{L^2(M)}^2 \\ &\lesssim \lambda^3 \|\Delta e_\lambda\|_{L^2(M)}^2.\end{aligned}\tag{4.3.22}$$

The estimation of the second term is given by

$$\begin{aligned}\left| \int_M \langle \nabla \sqrt{1 + |\nabla \Delta e_\lambda|^2}, \nabla \Delta e_\lambda \rangle dV \right| &\leq \|\nabla^2 \Delta e_\lambda\|_{L^2(\Omega)} \|\nabla \Delta e_\lambda\|_{L^2(\Omega)} \\ &\lesssim \sqrt{\lambda} \sqrt{\lambda^3} \|\Delta e_\lambda\|_{L^2(M)}^2 = \lambda^2 \|\Delta e_\lambda\|_{L^2(M)}^2.\end{aligned}\tag{4.3.23}$$

Combine the estimations together, we have

$$\int_{\tilde{Z}_\lambda} |\nabla \Delta e_\lambda|^2 d\sigma \lesssim \lambda^2 \|\Delta e_\lambda\|_{L^2(M)}^2 + \lambda^2 \|\Delta e_\lambda\|_{L^2(M)}^2 \approx \lambda^2 \|\Delta e_\lambda\|_{L^2(M)}^2.\tag{4.3.24}$$

For the eigenfunction satisfying (3.4.9), just replace the operator Ξ to Π and we can get the desired result.

For the eigenfunction satisfying (3.4.7), use similar method, we can get the following estimation on the boundary. For the first order derivatives, we have

$$\begin{aligned}\|\partial_\nu e_\lambda\|_{L^2(M)} &= 0, \\ \|\nabla^T e_\lambda\|_{L^2(M)} &= \lambda \|e_\lambda\|_{L^2(M)}.\end{aligned}\tag{4.3.25}$$

The second order derivatives are

$$\begin{aligned}\|\Delta e_\lambda\|_{L^2(M)} &\lesssim \lambda^2 \|e_\lambda\|_{L^2(M)}, \\ \|\Delta_M e_\lambda\|_{L^2(M)} &\lesssim \lambda^2 \|e_\lambda\|_{L^2(M)}.\end{aligned}\tag{4.3.26}$$

The third order derivatives are

$$\begin{aligned}\|\nabla^T \Delta e_\lambda\|_{L^2(M)} &\lesssim \lambda^3 \|e_\lambda\|_{L^2(M)}, \\ \|\partial_\nu \Delta e_\lambda\|_{L^2(M)} &= \lambda^3 \|e_\lambda\|_{L^2(M)}.\end{aligned}\tag{4.3.27}$$

The fourth order derivatives needed is

$$\|\Delta_M \Delta e_\lambda\|_{L^2(M)} \lesssim \lambda^4 \|e_\lambda\|_{L^2(M)}.\tag{4.3.28}$$

The estimations needed on Ω are:

$$\begin{aligned}\|\nabla e_\lambda\|_{L^2(\Omega)} &\approx \lambda^{\frac{1}{2}} \|e_\lambda\|_{L^2(M)}, \\ \|\Delta e_\lambda\|_{L^2(\Omega)} &= \lambda^{\frac{3}{2}} \|e_\lambda\|_{L^2(M)}, \\ \|\nabla \Delta e_\lambda\|_{L^2(\Omega)} &\lesssim \lambda^{\frac{5}{2}} \|e_\lambda\|_{L^2(M)}, \\ \|\nabla^2 \Delta e_\lambda\|_{L^2(\Omega)} &\lesssim \lambda^{\frac{7}{2}} \|e_\lambda\|_{L^2(M)}.\end{aligned}\tag{4.3.29}$$

From these, using $\|\Delta e_\lambda\|_{L^2(M)} \approx \lambda^2 \|e_\lambda\|_{L^2(M)}$ when λ is large, we can get the desired estimation. \square

Finally, we can establish a lower bound of $|\hat{Z}_\lambda^\alpha|$.

Theorem 4.3.5. For e_λ satisfying (3.4.7), (3.4.8) or (3.4.9), for $|\alpha| < c\lambda^{\frac{2-n}{4}} \|\Delta e_\lambda\|_{L^2(\Omega)}$, we have

$$|\hat{Z}_\lambda^\alpha| \gtrsim \lambda^{\frac{2-n}{2}}. \quad (4.3.30)$$

Proof. We have

$$\begin{aligned} \lambda \|\Delta e_\lambda\|_{L^1(M)} &\lesssim \int_{\hat{Z}_\lambda^\alpha} |\nabla \Delta e_\lambda| d\sigma \\ &\leq \left(\int_{\hat{Z}_\lambda^\alpha} |\nabla \Delta e_\lambda|^2 d\sigma \right)^{\frac{1}{2}} |\hat{Z}_\lambda^\alpha|^{\frac{1}{2}} \\ &\lesssim \lambda |\hat{Z}_\lambda^\alpha|^{\frac{1}{2}} \|\Delta e_\lambda\|_{L^2(M)}. \end{aligned} \quad (4.3.31)$$

Recall that when λ is large, $p = 1, 2$, $\|\Delta e_\lambda\|_{L^p(M)} \approx \lambda \|\partial_\nu e_\lambda\|_{L^p(M)}$ for (3.4.9), $\|\Delta e_\lambda\|_{L^p(M)} \approx \lambda^2 \|e_\lambda\|_{L^p(M)}$ for (3.4.7). Using the L^p estimate (4.2.4) for the eigenfunctions, we have

$$\|\Delta e_\lambda\|_{L^1(M)} \gtrsim \lambda^{-\frac{n-2}{4}} \|\Delta e_\lambda\|_{L^2(M)}. \quad (4.3.32)$$

Therefore,

$$\lambda^{\frac{2-n}{4}} \lesssim |\hat{Z}_\lambda^\alpha|^{\frac{1}{2}}, \quad (4.3.33)$$

which is the desired result. \square

Plug in $\alpha = 0$, we have the lower bound for the vanishing set of Δe_λ as in theorem 3.4.3.

4.4 Lower bound for the interior nodal set

In this section, we get an lower bound for the interior nodal set, Z_λ . For problem (3.4.8), (3.4.9), the α -level set is unstable near the boundary, since e_λ vanishes on the boundary. For simplicity, we only consider the nodal set in this section. Let $Z_\lambda = \{x \in \Omega | e_\lambda = 0\}$ and $\sigma(x) = \sigma_0(x)$.

We have the following equations.

Theorem 4.4.1. For the problem (3.4.7), let $f \in C^\infty(\bar{\Omega})$, if 0 is a regular value of

e_λ , we have

$$\int_M f\sigma(e_\lambda)\partial_\nu\Delta e_\lambda d\sigma - \int_\Omega \sigma(e_\lambda)\langle\nabla f, \nabla\Delta e_\lambda\rangle dV = 2 \int_{Z_\lambda} f\langle\nabla\Delta e_\lambda, N\rangle d\sigma. \quad (4.4.1)$$

For the problem (3.4.8), (3.4.9), if 0 is a regular value of e_λ , we have

$$- \int_M f\sigma(\partial_\nu e_\lambda)\partial_\nu\Delta e_\lambda d\sigma - \int_\Omega \sigma(e_\lambda)\langle\nabla f, \nabla\Delta e_\lambda\rangle dV = 2 \int_{Z_\lambda} f\langle\nabla\Delta e_\lambda, N\rangle d\sigma, \quad (4.4.2)$$

where N on Z_λ is defined to be the unit normal $\frac{\nabla e_\lambda}{|\nabla e_\lambda|}$.

Proof. The result follows by replacing $\{D_k^{+,\alpha}\}_k$ to be the collection of connected components of the set $\{\Delta e_\lambda > 0\}$ in the Theorem 4.3.1. \square

Plug in $f = 1$ in the theorem, we get the following:

Corollary 4.4.2. *There exists a constant c such that for any biharmonic Steklov eigenfunction e_λ satisfying (3.4.7), with 0 as a regular value, we have*

$$\int_{Z_\lambda} |\langle\nabla\Delta e_\lambda, N\rangle| d\sigma \geq \frac{\lambda^3}{2} \|e_\lambda\|_{L^1(M)}. \quad (4.4.3)$$

For any eigenfunction satisfying (3.4.9) or (3.4.8), with 0 as a regular value, we have

$$\int_{Z_\lambda} |\langle\nabla\Delta e_\lambda, N\rangle| d\sigma \gtrsim \lambda^2 \|\partial_\nu e_\lambda\|_{L^1(M)}. \quad (4.4.4)$$

Proof. For the eigenfunction satisfying (3.4.7), we have

$$\begin{aligned} 2 \int_{Z_\lambda} |\langle\nabla\Delta e_\lambda, N\rangle| d\sigma &\geq -2 \int_{Z_\lambda} \langle\nabla\Delta e_\lambda, N\rangle d\sigma \\ &= - \int_M \sigma(e_\lambda)\partial_\nu\Delta e_\lambda d\sigma = \int_M \sigma(e_\lambda)\lambda^3 e_\lambda d\sigma \\ &= \lambda^3 \int_M |e_\lambda| d\sigma = \lambda^3 \|e_\lambda\|_{L^1(M)}. \end{aligned} \quad (4.4.5)$$

For the eigenfunction of satisfying (3.4.8) or (3.4.9), we have

$$\begin{aligned} 2 \int_{Z_\lambda} |\langle \nabla \Delta e_\lambda, N \rangle| d\sigma &\geq -2 \int_{Z_\lambda} \langle \nabla \Delta e_\lambda, N \rangle d\sigma \\ &= \int_M \sigma(\partial_\nu e_\lambda) \partial_\nu \Delta e_\lambda d\sigma \gtrsim \lambda^2 \|\partial_\nu e_\lambda\|_{L^1(M)}. \end{aligned} \quad (4.4.6)$$

□

4.4.1 Upper bound for $|\nabla \Delta e_\lambda|$

Now, we need to get an upper bound for $|\nabla \Delta e_\lambda|$. The approach is the same as that in Proposition 3.1 of [49]: Applying the gradient estimates of elliptic equation in the interior and near the boundary separately and combine the results.

Proposition 4.4.3. *If e_λ satisfies (3.4.8) or (3.4.9), $d = d(x)$ be the distance from $x \in \Omega$ to $\partial\Omega = M$, we have*

$$\|(\lambda^{-1} + d)\nabla \Delta e_\lambda\|_{L^\infty(\Omega)} \lesssim \lambda^{\frac{n}{2}} \|\partial_\nu e_\lambda\|_{L^1(M)}. \quad (4.4.7)$$

Proof. On the boundary, $\Delta e_\lambda = \lambda \partial_\nu e_\lambda$ for problem (3.4.8) and $\Delta e_\lambda = \lambda \partial_\nu e_\lambda + H \partial_\nu e_\lambda$ for problem (3.4.9). We can argue as in [49], see also [50] that

$$\lambda^{-k} \|(D^T)^k \Delta e_\lambda\|_{L^\infty(M)} \lesssim \lambda^{\frac{n}{2}} \|\partial_\nu e_\lambda\|_{L^1(M)}, \quad (4.4.8)$$

where $(D^T)^k$ denotes k boundary derivatives.

For the interior estimate, start with

$$\|\Delta e_\lambda\|_{L^\infty(M)} \lesssim \lambda^{\frac{n}{2}} \|\partial_\nu e_\lambda\|_{L^1(M)}, \quad (4.4.9)$$

since Δe_λ is harmonic, from the gradient estimate, see corollary 6.3 of [27], for a fixed $\delta > 0$,

$$\|d \nabla \Delta e_\lambda\|_{L^\infty(\{d \geq \delta \lambda^{-1}\})} \leq C_\delta \lambda^{\frac{n}{2}} \|\partial_\nu e_\lambda\|_{L^1(M)}. \quad (4.4.10)$$

The constant C_δ depends on δ and Ω , but not on λ .

Now, for the boundary estimate for any $x_0 \in M$, use a local coordinate in a neighborhood of x_0 which map x_0 to 0, M to $\{x_n = 0\}$, and the neighborhood of x_0 into the upper half space. For simplicity, we also use e_λ to denote the function induced on the coordinate. Consider the ball of radius $2\delta\lambda^{-1}$ around 0 and define

$$u_\lambda(x) = \lambda^{-\frac{n}{2}} \Delta_\Omega e_\lambda(x\lambda^{-1}), \quad (4.4.11)$$

which is defined in the upper half of the ball of radius 2δ , $B_{2\delta}^+(0)$. We have the estimate

$$\|(D^T)^k u_\lambda\|_{L^\infty(\mathbb{R}^{n-1} \cap B_{2\delta}^+(0))} \leq C_k \|\partial_\nu e_\lambda\|_{L^1(M)}. \quad (4.4.12)$$

The partial differential equation satisfied by u has uniformly bounded coefficients. We can also find ϕ_λ in this coordinate which agree with u_λ on the boundary and is bounded in $C^{2,\alpha}(B_{2\delta}^+(0))$ by some constant times $\|\partial_\nu e_\lambda\|_{L^1(M)}$. Use Corollary 8.36 in [27], the $C^{1,\alpha}(B_\delta^+(0))$ norm of u_λ is bounded by $C_\alpha \|\partial_\nu e_\lambda\|_{L^1(M)}$, with C_α independent of λ . Thus, we have

$$\|Du_\lambda\|_{L^\infty(B_\delta^+(0))} \leq C_\alpha \|\partial_\nu e_\lambda\|_{L^1(M)}, \quad (4.4.13)$$

which is the desired result. \square

Proposition 4.4.4. *If e_λ satisfies (3.4.7), we have*

$$\|(\lambda^{-1} + d)\nabla\Delta e_\lambda\|_{L^\infty(\Omega)} \lesssim \lambda^{\frac{n+4}{2}} \|e_\lambda\|_{L^1(M)}. \quad (4.4.14)$$

Proof. On the boundary, $\partial_\nu \Delta e_\lambda = -\lambda^3 e_\lambda$ for problem (3.4.7), we have that

$$\lambda^{-k} \|(D^T)^k \partial_\nu \Delta e_\lambda\|_{L^\infty(M)} \lesssim \lambda^{\frac{n+4}{2}} \|\partial_\nu e_\lambda\|_{L^1(M)}, \quad (4.4.15)$$

where $(D^T)^\alpha$ denotes α boundary derivatives.

For the interior estimate, use the L^∞ estimate of a harmonic function in terms of its Neumann data together with $\int_\Omega \Delta e_\lambda dV = 0$, we have

$$\|\Delta e_\lambda\|_{L^\infty(M)} \leq C \|\partial_\nu \Delta e_\lambda\|_{L^\infty(M)} \lesssim \lambda^{\frac{n+4}{2}} \|e_\lambda\|_{L^1(M)} \quad (4.4.16)$$

and therefore for any given $\delta > 0$,

$$\|d\nabla\Delta e_\lambda\|_{L^\infty(\{d\geq\delta\lambda^{-1}\})} \leq C_\delta\lambda^{\frac{n+4}{2}}\|e_\lambda\|_{L^1(M)}. \quad (4.4.17)$$

Now, for the boundary estimate, for any $x_0 \in M$, use the same approach as above, define

$$u_\lambda(x) = \lambda^{-\frac{n+4}{2}}\Delta e_\lambda(x\lambda^{-1}), \quad (4.4.18)$$

which is defined in $B_{2\delta}^+(0)$. We have the estimate

$$\|(D^T)^k\partial_\nu u_\lambda\|_{L^\infty(M)} \leq C_k\|e_\lambda\|_{L^1(M)}. \quad (4.4.19)$$

From lemma 6.29 in [27], we have the following bound:

$$\|u_\lambda\|_{C^{2,\alpha}} \leq C(\|u_\lambda\|_{L^\infty} + \|\partial_\nu u_\lambda\|_{C^{1,\alpha}}). \quad (4.4.20)$$

Thus, we have

$$\|Du_\lambda\|_{L^\infty(B_\delta^+(0))} \leq C_\alpha\|e_\lambda\|_{L^1(M)}, \quad (4.4.21)$$

and therefore

$$\|\lambda^{-1}\nabla\Delta e_\lambda\|_{L^\infty(B_\delta^+(0))} \leq C\lambda^{\frac{n+4}{2}}\|e_\lambda\|_{L^1(M)}. \quad (4.4.22)$$

□

Now, we can estimate the interior set in each case.

Proof of Theorem 3.4.4. For problem (3.4.8), (3.4.9), we have (4.4.4) and (4.4.7).

Therefore

$$\begin{aligned} \lambda^2\|\partial_\nu e_\lambda\|_{L^1(M)} &\lesssim \int_{Z_\lambda} |\langle \nabla\Delta e_\lambda, N \rangle| d\sigma \\ &\leq \|\nabla\Delta e_\lambda\|_{L^\infty(\Omega)}|Z_\lambda| \\ &\lesssim \lambda^{\frac{n+2}{2}}\|\partial_\nu e_\lambda\|_{L^1(M)}|Z_\lambda|. \end{aligned} \quad (4.4.23)$$

Cancelling $\|\partial_\nu e_\lambda\|_{L^1(M)}$ from both side yields the desired result.

For problem (3.4.7), we can use a similar argument. \square

Remark 4.4.5. For problem (3.4.7), we can not get the L^∞ bound of Δe_λ on the boundary. We use the L^∞ bound of $\partial_\nu \Delta e_\lambda$ instead, thus losing a factor of λ .

4.5 Lower bound for the boundary nodal set

Let us turn our attention to the boundary, M , and get the estimations of the nodal sets for the operators Θ , Ξ and Π . Since all we need is the property of the operator on M , we can argue in an abstract way. Let $\Psi \in OPS^1(M)$ be classical and with the principle symbol equal to some nonzero constant times the principle symbol of $\sqrt{-\Delta_M}$. Let ϕ_λ be an eigenfunction of Ψ corresponds to λ . Note that the case we want is given by $\Psi = \sqrt[3]{\Theta}$, Ξ , Π and $\phi_\lambda = e_\lambda|_M$, $\partial_\nu e_\lambda$, $\partial_\nu e_\lambda$ respectively.

The proof is given in [54] to establish the lower bound of boundary nodal sets of harmonic Steklov eigenfunctions. From now on, all the argument are on M and all the L^p norm are $L^p(M)$. Let $\tilde{Z}_\lambda^\alpha = \{x \in M | \phi_\lambda = \alpha\}$ be the α -level set of ϕ_λ . We have the following equation.

Theorem 4.5.1. For any $f \in C^\infty(\bar{\Omega})$, any regular value α of ϕ_λ , we have

$$-\int_M \sigma_\alpha(\phi_\lambda) \left[\langle \nabla^T f, \nabla^T \phi_\lambda \rangle + f \Delta_M \phi_\lambda \right] dV = 2 \int_{\tilde{Z}_\lambda^\alpha} f |\nabla^T \phi_\lambda| d\sigma. \quad (4.5.1)$$

Proof. Let $\{\tilde{D}_k^{+, \alpha}\}_k$ be the collection of connected components of the set $\{\phi_\lambda > \alpha\}$, $\tilde{Z}_k^{+, \alpha} = \partial \tilde{D}_k^{+, \alpha}$. we have

$$-\int_{\tilde{D}_k^{+, \alpha}} \langle \nabla^T f, \nabla^T \phi_\lambda \rangle + f \Delta_M \phi_\lambda dV = \int_{\tilde{Z}_k^{+, \alpha}} f |\nabla^T \phi_\lambda| d\sigma. \quad (4.5.2)$$

Similarly, from the set $\{\phi_\lambda < \alpha\}$, we can define $\tilde{D}_k^{-, \alpha}$, $\tilde{Z}_k^{-, \alpha}$ together with a similar equation:

$$\int_{\tilde{D}_k^{-, \alpha}} \langle \nabla^T f, \nabla^T \phi_\lambda \rangle + f \Delta_M \phi_\lambda dV = \int_{\tilde{Z}_k^{-, \alpha}} f |\nabla^T \phi_\lambda| d\sigma. \quad (4.5.3)$$

Summing over all the equations and we can get the desired equation. \square

Choosing $f = 1$ gives the following:

Corollary 4.5.2. *There exists a constant c such that for any regular value α of ϕ_λ satisfying $|\alpha| < c\lambda^{\frac{2-n}{4}} \|\phi_\lambda\|_{L^2(M)}$, we have*

$$\int_{\tilde{Z}_\lambda^\alpha} |\nabla^T \phi_\lambda| d\sigma \gtrsim \lambda^2 \|\phi_\lambda\|_{L^1(M)}. \quad (4.5.4)$$

Proof. Put $f = 1$ yields

$$2 \int_{\tilde{Z}_\lambda^\alpha} |\nabla^T \phi_\lambda| d\sigma = - \int_M \sigma_\alpha(\phi_\lambda) \Delta_M \phi_\lambda dV. \quad (4.5.5)$$

Since $\sqrt{-\Delta_M} = a\Psi + P_0$, for some $a \neq 0$, $P_0 \in OPS^0(M)$,

$$\begin{aligned} \Delta_M \phi_\lambda &= -a^2 \Psi^2 \phi_\lambda - (a\Psi P_0 + aP_0 \Psi + P_0^2) \phi_\lambda \\ &= -a^2 \lambda^2 \phi_\lambda - (a\Psi P_0 + aP_0 \Psi + P_0^2) \phi_\lambda. \end{aligned} \quad (4.5.6)$$

Using $a\Psi P_0 + aP_0 \Psi + P_0^2 \in OPS^1(M)$, we can bound the second term by

$$\|(a\Psi P_0 + aP_0 \Psi + P_0^2) \phi_\lambda\|_{L^1(M)} \lesssim \lambda^{1+\epsilon} \|\phi_\lambda\|_{L^1(M)}. \quad (4.5.7)$$

Proceed as in (4.3.7), and choose the constant c as before, we can get the desired result. \square

Next, choosing $f = \sqrt{1 + |\nabla^T \phi_\lambda|^2}$ gives the following proposition.

Proposition 4.5.3. *We have the following estimation when λ large enough:*

$$\int_{\tilde{Z}_\lambda^\alpha} |\nabla^T \phi_\lambda|^2 d\sigma \lesssim \lambda^3 \|\phi_\lambda\|_{L^2(M)}^2. \quad (4.5.8)$$

Proof. Plug in $f = \sqrt{1 + |\nabla^T \phi_\lambda|^2}$,

$$\begin{aligned}
2 \int_{\tilde{Z}_\lambda^\alpha} |\nabla^T \Delta e_\lambda|^2 d\sigma &\leq 2 \int_{\tilde{Z}_\lambda^\alpha} |\nabla^T \phi_\lambda| \sqrt{1 + |\nabla^T \phi_\lambda|^2} d\sigma \\
&\leq \int_M \sqrt{1 + |\nabla^T \phi_\lambda|^2} |\Delta_M \phi_\lambda| \\
&\quad + |\langle \nabla^T \sqrt{1 + |\nabla^T \phi_\lambda|^2}, \nabla^T \phi_\lambda \rangle| dV.
\end{aligned} \tag{4.5.9}$$

Since $\Delta_M \in OPS^2(M)$, we can use the lemma for L^p bounds to get

$$\|\Delta_M \phi_\lambda\|_{L^2(M)} \lesssim \lambda^2 \|\phi_\lambda\|_{L^2(M)} \tag{4.5.10}$$

and

$$\begin{aligned}
\int_M \|\nabla^T \phi_\lambda\|^2 dV &= - \int_M \phi_\lambda \Delta_M \phi_\lambda \\
&\leq \|\phi_\lambda\|_{L^2(M)} \|\Delta_M \phi_\lambda\|_{L^2(M)} \lesssim \lambda^2 \|\phi_\lambda\|_{L^2(M)}^2.
\end{aligned} \tag{4.5.11}$$

Therefore, the first term is bounded by

$$\begin{aligned}
\int_M \sqrt{1 + |\nabla^T \phi_\lambda|^2} |\Delta_M \phi_\lambda| dV &\leq \|\sqrt{1 + |\nabla^T \phi_\lambda|^2}\|_{L^2(M)} \|\Delta_M \phi_\lambda\|_{L^2(M)} \\
&\lesssim \lambda^3 \|\phi_\lambda\|_{L^2(M)}^2.
\end{aligned} \tag{4.5.12}$$

For the the second term,

$$\begin{aligned}
\int_M |\langle \nabla^T \sqrt{1 + |\nabla^T \phi_\lambda|^2}, \nabla^T \phi_\lambda \rangle| dV &= \int_M \frac{|(\nabla^T)^2 \phi_\lambda (\nabla^T \phi_\lambda, \nabla^T \phi_\lambda)|}{\sqrt{1 + |\nabla^T \phi_\lambda|^2}} dV \\
&\leq \|(\nabla^T)^2 \phi_\lambda\|_{L^2(M)} \|\nabla^T \phi_\lambda\|_{L^2(M)} \left\| \frac{\nabla^T \phi_\lambda}{\sqrt{1 + |\nabla^T \phi_\lambda|^2}} \right\|_{L^\infty(M)} \\
&\lesssim \lambda \|(\nabla^T)^2 \phi_\lambda\|_{L^2(M)} \|\phi_\lambda\|_{L^2(M)}.
\end{aligned} \tag{4.5.13}$$

Since M is compact without boundary, for any smooth function f on M , we can apply the Reilly's formula and obtain

$$\int_M |(\nabla^T)^2 f|^2 dV = \int_M -\mathbf{Ric}_M(\nabla^T f, \nabla^T f) + (\Delta_M f)^2 dV. \tag{4.5.14}$$

Use this formula on ϕ_λ ,

$$\begin{aligned} \int_M |(\nabla^T)^2 \phi_\lambda|^2 dV &\leq \|\mathbf{Ric}_M\|_{L^\infty(M)} \|\nabla^T \phi_\lambda\|_{L^2(M)}^2 + \|\Delta_M \phi_\lambda\|_{L^2(M)}^2 \\ &\lesssim \lambda^4 \|\phi_\lambda\|_{L^2(M)}^2. \end{aligned} \quad (4.5.15)$$

Thus the second term is bounded by

$$\begin{aligned} \int_M |\langle \nabla^T \sqrt{1 + |\nabla^T \phi_\lambda|^2}, \nabla^T \phi_\lambda \rangle| dV &\lesssim \lambda \|(\nabla^T)^2 \phi_\lambda\|_{L^2(M)} \|\phi_\lambda\|_{L^2(M)} \\ &\lesssim \lambda^3 \|\phi_\lambda\|_{L^2(M)}^2. \end{aligned} \quad (4.5.16)$$

Combining the estimation, we can get the desired bound for $\int_{\tilde{Z}_\lambda^\alpha} |\nabla^T \phi_\lambda|^2 d\sigma$. \square

Finally, we can estimate the size of boundary nodal sets.

Theorem 4.5.4. *For ϕ_λ , α as above, we have*

$$|\tilde{Z}_\lambda^\alpha| \gtrsim \lambda^{\frac{4-n}{2}}. \quad (4.5.17)$$

Proof. From the bounds above,

$$\begin{aligned} \lambda^2 \|\phi_\lambda\|_{L^1(M)} &\lesssim \int_{\tilde{Z}_\lambda^\alpha} |\nabla^T \phi_\lambda| d\sigma \\ &\leq \left(\int_{\tilde{Z}_\lambda^\alpha} |\nabla^T \phi_\lambda|^2 d\sigma \right)^{\frac{1}{2}} |\tilde{Z}_\lambda^\alpha|^{\frac{1}{2}} \\ &\lesssim \lambda^{\frac{3}{2}} |\tilde{Z}_\lambda^\alpha|^{\frac{1}{2}} \|\phi_\lambda\|_{L^2(M)}. \end{aligned} \quad (4.5.18)$$

Using the L^p estimate (4.2.4) for the ϕ_λ ,

$$\|\phi_\lambda\|_{L^1(M)} \gtrsim \lambda^{-\frac{n-2}{4}} \|\phi_\lambda\|_{L^2(M)}. \quad (4.5.19)$$

Therefore,

$$\lambda^{\frac{4-n}{4}} \lesssim |\tilde{Z}_\lambda^\alpha|^{\frac{1}{2}}. \quad (4.5.20)$$

\square

We can get the theorem 3.4.2 by plugging in $\alpha = 0$.

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