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ON THE INVARIANT IMPEDANCE FUNCTION AND ITS ASSOCIATED GROUP OF NETWORKS

by

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A C K N O W L E D G M E N T S

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I N T R O D U C T I O N

The problems of calculating the driving-point impedance function Z(p) of a given two-terminal network is a familiar one in electric circuit theory. The usual method is to combine the impedances and admittances of the various branches in such manner as to give the total impedance between the two terminals. This impedance is of course the alternating-current impedance of the network obtained by dividing the a.c. voltage across the terminals of the network by the resulting driving or indicial a.c. current. Another method, and in more complicated networks a better method, is the use of the determinant of the network. The determinant of the network is the determinant whose elements are the coefficients of the currents in the Kirchhoff equations of the network. The elements are of the form $\lambda p + \rho + \frac{\sigma}{p}$, where λ, ρ and σ are positive constants and are the inductance, resistance and elastance terms. The determinant is symmetrical about the main diagonal, and consists essentially of two kinds of elements. The elements of the main diagonal are the <u>total</u> parameters, that is, they are terms of the form $\lambda_{jj} p + \rho_{jj} + \frac{\sigma_{jj}}{p}$,where λ_{jj} , ρ_{jj} and σ_{jj} are respectively the <u>total</u> inductance, resistance and elastance of the j mesh. All the other elements are the <u>mutual</u> parameters, that is, they are terms of the form $\lambda_{jk} p + \rho_{jk} + \frac{\sigma_{jk}}{p}$ where λ_{jk} , ρ_{jk} and σ_{jk} are respectively, the inductance, resistance and elastance <u>mutual</u> or <u>common</u> to the two meshes j and k. The driving-point impedance of the network is obtained by dividing the determinant of the network by the minor of the element in the first row and first column. Thus the impedance of a given network is a fraction, the numerator of which is the determinant of the network and the denominator of which is the minor of the element in the first row and first column of this determinant. The expansion of the determinatn and its minor, results in the following expression for the impedance function

$$Z(p) = \frac{a_{2n}p^{n} + a_{2n-1}p^{n-1} + a_{2n-2}p^{n-2} + \cdots + a_2 p^{-n+2} + a_1 p^{-n+1} + a_0 p^{-n}}{b_{2n-1}p^{n-1} + b_{2n-2}p^{n-2} + \cdots + b_2 p^{-n+2} + b_1 p^{-n+1}} \qquad (1)$$

where the a and b terms are real cons--tants.

Multiplying numerator and denominator by p^n the impedance function becomes

$$Z(p) = \frac{a_{2n}p^{2n} + a_{2n-1}p^{2n-1} + a_{2n-2}p^{2n-2} + \cdots + a_2 p^2 + a_1 p + a_0}{b_{2n-1}p^{2n-1} + b_{2n-2}p^{2n-2} + \cdots + b_2 p^2 + b_1 p} \qquad (2)$$

This, then, is the most general expression of the impedance function of a network of n meshes. Thus, the determination of the impedance function of a given two-terminal network of any number of meshes is not difficult, and results in an expression of the form (2). This expression is <u>unique</u> for a given network, that is, for a network with given elements, the coefficients of p are definite constants. To every given network therefore, there corresponds <u>one</u> and <u>only</u> one impedance function, which can be reduced to the form (2).

The converse of this proposition fortunately is not true, that is, it is not true that to every impedance function there corresponds one and only one network. To an impedance function determined from a given network, there may correspond an infinite number of other networks. That is, the impedance of every one of these networks, having its elements of inductance, resistance and elastance different from those of the given network is exactly the same as the impedance of the given network.

This is a very important fact, and yet, communication design engineers have in general disregarded it; and it is only within recent years, since 1924, that serious thought has been given the matter.[1] The terminal equipment of communication systems consists essentially of networks, such as wave-filters, corrective networks, etc. These networks are designed to have desired characteristics which are obtained by giving a definite configuration to the network and by assigning proper magnitudes to the circuit elements. From the standpoint of good design, it is important to know that the network that has been designed to have the given characteristics is better and more economical than any other network having the same characteristics. This applies to any part of the designed network, for any part of a network must be so designed as to

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1. See R. M. Foster "Theorems Regarding the Driving-point Impedance of Two Mesh Circuits", Bell System Technical Journal, vol. 3, 1924, p. 651, and "A Reactance Theorem", ibid., p. 259. See also W. Cauer "Die Verwirklichung von Wechselstrom-widerständen vorgeschriebener Frequenzabhängikeit", Archiv für Elektrotechnik, Heft 4, Band XVII, 1926, p. 355 and "Vierpole", Elektrischen Nachrichtentechnik Heft 7, Band 6, 1929, p. 272.

contribute in the best and most economical manner to the desired result. Thus, consider the mid-series equivalent m-derived band pass type of filter shown in figure 1.

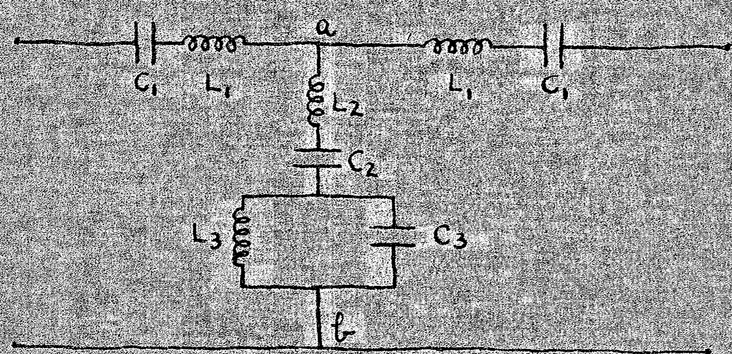

FIG. 1

The shunt arm a b is part of this network, and without being concerned about the rest of the network, this shunt arm may be considered by itself as a network with terminals a,b. The voltage across this shunt arm is the voltage e (t) across the terminals a, b when the filter is in operation. Figure 2 shows this shunt arm a,b, removed from the rest of the filter, with this voltage e(t) across its terminals a,b.

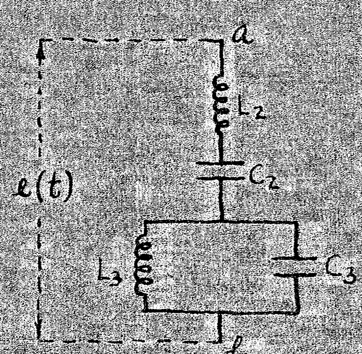

FIG.2

Now since there exists an infinite number of other networks which have exactly the same impedance function, these networks may be substituted for the given shunt arm without in any way affecting the operation of the filter.[2] The important question therefore arises, which is the best network to use? To answer this question, a knowledge of every one of the infinite number of networks having the same impedance as the given network is required. That is, it is necessary to be able to determine <u>all</u> the networks having the same impedance function as the given network.

When this has been accomplished the actual selection of the best network depends upon many factors, determined largely by experience and practical considerations. Small values of capacities are undesirable because they are difficult to measure accurately and because the capacity of the wiring may be large enough to be important, and large values of capacities are costly. Small values of inductances are difficult to measure accurately and large values of inductances are affected by shunt capacities between the windings. Finally, if there are present large quantities of certain standard condensers and coils, it seems that it would be more economical to choose those networks that can be made up of these standard elements.[3]

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2. Networks having the same driving-point impedances have identical indicial currents, both transient and steady state. This follows from the infinite integral theorem. See V. Bush, Operational Circuit Analysis, 1929, pp.34-75.

3. An excellent discussion of these factors is given in K. S. Johnson, Transmission Circuits for Telephone Communication, 1927, p. 197.

It is not enough therefore to design a network to perform a certain function in a communication system, and be satisfied when the network is finally built and performs its function satisfactorily. As long as there exists an infinite number of other networks which will perform identically the same function, the design is not a good one until the best and most economical network is selected.

The purpose of the thesis will be, therefore, to make a thorough investigation of the impedance function and its invariance to a change in network parameters, and to show how to obtain all networks equivalent to a given network . First, the general network theory will be developed from the Kirchhoff differential equations of the network and also from the energy relations and the use of Lagrange's equations. This latter method in arriving at the general network theory is the classic method used in dynamics in the theory of vibrations. The expressions for the total instantaneous energy in the magnetic fields of the coils, and the electrostatic fields in the condensers and the total instantaneous power lost in the resistances are very important quantities, and will be shown to play an important role in the theory of the impedance function and the networks which represent it. These quantities are the so-called quadratic forms, which are always positive and definite, that is, they are positive for all instantaneous values of current and charge, and are zero when and only when the corresponding currents and charges are zero. These fundamental quadratic forms of the electric circuit may be called respectively the inductance, resistance and elastance quadratic forms. It will be shown that the matrices containing the coefficients of these quadratic forms determine at once a definite network, the parameters of which are exactly the elements of the matrices. Second, general formulas will be obtained for expressing the impedance function of a network of n meshes directly from the elements of these matrices. A certain determinantal symbolic notation will be used, which seems to be a natural and convenient terminology for expressing the impedance function. This terminology simplifies considerably the usual method of calculating the impedance of a network. Third, the properties of the impedance function will be studied in detail. Questions of the roots and poles of the impedance function will be taken up, as well as the location of the roots and poles. Fourth, the "equivalence equations" will be obtained for networks of any number of meshes, and expressed in our symbolic notation. These equations will be the relations that must exist between the elements of two networks in order that they have the same impedance function. Fifth, conditions will be given for the invariance of the form of the impedance function in terms of the resultant of the numerator and denominator of the impedance function. The vanishing of the resultant will be shown to correspond to the short-circuiting of a network. By a removal of as many elements of the network as can be removed without violating the conditions for the invariance of the form of the impedance function, minimal networks, that is, networks with the least number of elements result. Sixth, the straight line equation in the mutual parameters of the two-mesh network, containing two kinds of elements will be plotted and a family of straight lines will be obtained, every point of which, within certain regions, will be a possible pair of mutual parameters of the network, the other parameters being obtained from the equivalence equations of the impedance function. The plane of the mutual parameters will be shown to be capable of division into regions which contain points representing six, five and four-element networks. Thus for example, the eight points corresponding to the eight minimal forms are indicated in the plane. Also, two regions of the plane may be images of each other, the interior of which regions may represent six-element networks and the boundary, five-element networks. This is carried through for networks containing all three kinds of elements, thereby considerably simplifying Foster's rather complicated procedure for this case. A vector notation is introduced, which appears to allow for generalization to networks of any number of meshes. Seventh, the condition that a function having the form of an impedance function, be in fact the impedance function of a physical network, will be given in terms of the resultant of the numerator and denominator of the impedance function. Finally, and most important, a study of quadratic forms will be made, and it will be shown that they are invariant to a linear affine transformation of the <u>instantaneous</u> currents or charges of the network. As pointed out above, the matrices containing the coefficients of the three fundamental forms of a given network have as elements the parameters of the network. Thus given these matrices, the network can at once be constructed from them. Now, by a linear affine transformation of the <u>instantaneous</u> currents or <u>charges</u>, the complete infinite set of quadratic forms will be obtained, the matrices of the coefficients of each of the forms representing an equivalent network. Thus the complete infinite group of networks, all having a given impedance function is readily obtained. These will be shown to be obtained more easily than through the above substitutions, by multiplying each matrix of the quadratic forms by the matrix of the transformation, and its conjugate. Thus if A represents the matrix, and C the transformation matrix, the matrix or tensor representing the complete infinite group of networks having a given impedance function will be given by

$$C^1 A C$$

where C^1 is the conjugate of C. By this means, no matter how complicated a network may be, it is a simple matter to obtain all of its equivalent networks. This equivalence is not limited only to equivalence of driving-point impedance, but also to transfer impedances, and holds for networks of any number of terminals as well. It is thus seen that the above transformation method is very powerful and gives, with surprising ease, the complete infinite group of networks having a given impedance function.

It will be useful at this point to briefly review the work of Foster and Cauer and point out what they have done.

In his "Reactance Theorem", Foster expresses essentially the impedance function of a two-terminal network composed of inductance and capacities in terms of its zeros and poles, that is, its resonant and anti-resonant frequences respectively. Following Routh in the identical dynamical problem of the small oscillations about a position of equilibrium, he shows that the zeros and poles of the impedance function (with the exception of the pole at the origin) are pure imaginaries, occuring in pairs with opposite signs,and that the zeros and poles separate each other.[4] Conversely,if he has an expression of the form of the impedance function in terms of zeros and poles, and these zeros and poles are pure imaginaries, occuring in pairs with opposite signs, and further if the zeros and poles separate each other, then he can construct certain networks having this impedance. These networks are constructed by combining, in parallel, resonant circuits, or by combining in series, anti-resonant circuits. He arrives at the network of resonant circuits in parallel simply by expanding the reciprocal of the given impedance, that is, the admittance in terms of partial fractions. Then each partial fraction represents a resonant circuit consisting of a condenser and an inductance in series. The sum of all of these gives the given admittance, and so this network represents the given impedance. The network of anti-resonant circuits in series is obtained by expanding the impedance function itself into partial fractions. Then each partial fraction represents an anti-resonant circuit consisting of a condenser and an inductance in parallel. This, in brief, is Foster's paper "A Reactance Theorem".

It is important to stress the fact at this point that Foster did essentially two things in this paper. First he showed that the impedance function of a given netwrk consisting of capacities and inductances could be expressed in a certain form in terms of its roots and poles,which was well known, but he gave from the dynamic analogy, certain properties of the roots

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4. See E. J. Routh "Advanced Rigid Dynamics", sixth edition, 1905, pages 44-45, or fourth edition, 1884, page 36.

and poles; their pure imaginary nature and their separation property. Secondly, and this is more important, he showed that an expression of the form of an impedance function in terms of zeros and poles, which zeros and poles had the above pure imaginary and separation properties, was, in fact, an impedance function. But he could construct only two networks having this impedance function; one by the partial fraction expansion of the admittance, and the other by the partial fraction expansion of the impedance function itself.[5] It is important to point this out because it is sometimes believed that this paper allowed the construction of all networks having the given impedance function. It turns out that these two networks that he can construct to have the given impedance contain the least number of elements realizing such an impedance. But there exist other networks containing the least number of elements which his formulas will not give. Thus, there exist four networks of the least number of elements realizing a given two mesh impedance, and an infinite number of five element and six element networks. Foster's paper allows us to construct but two of these networks, containing the least number of elements, but no more.

In his excellent paper "Theorems Regarding the Driving-Point Impedance of Two-Mesh Circuits", Foster considers the two-mesh network consisting of inductance, resistance and capacity elements. He shows first that the driving-point

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5. See Foster, "A Reactance Theorem", pp. 262-264.

impedance of such a network may be written in the form

$$Z(p) = \frac{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}{b_3 p^3 + b_2 p^2 + b_1 p} \qquad (3a)$$

which is the same as (2) for n = 2, or

$$Z(p) = k \frac{(p-\alpha_1)(p-\alpha_2)(p-\alpha_3)(p-\alpha_4)}{p(p-\beta_2)(p-\beta_3)} \qquad (3b)$$

which expresses Z(p) in terms of its zeros and poles. These zeros and poles are complex with negative reals; or negative reals. They occur, if complex, in pairs of conjugate complex roots. Further, the coefficients of (3) satisfy certain conditions, which are given in the form of equations and inequalities. Conversely, he shows that <u>any</u> expression of the form (3), whose coefficients and roots satisfy the above conditions, is, in fact, an impedance function of a two mesh network consisting of resistance, inductance and capacity elements. Foster arrives at the conditions that the coefficients must satisfy by obtaining the <u>equivalence equations</u> for the two-mesh network with resistance, inductance and capacity elements. It is important to emphasize the fact here, that the equivalence equations are essentially the important part of the paper. By means of these, he obtains not only the conditions that an expression like (3) should be an impedance function but also, he constructs networks of the least number of elements realizing such an impedance function. Again, Foster is not interested in obtaining expressions for <u>all</u> the networks having a given impedance function, but rather in those containing the least number of elements, and he obtains rather complicated expressions for the network elements of these equivalent circuits, obtained by solving the equivalence equations for these elements.

Cauer, like Foster, in his paper "Die Verwirklichung von Wechselstromwiderständen vorgeschriebener Frequenzabhängikeit", is primarily interested in finding the conditions that must exist on the coefficients of an expression like (2), it order that it represent an impedance function. Cauer begins first by <u>extending</u> the results of Foster's first paper, "A Reactance Theorem" to networks containing resistance and capacity elements and networks containing inductance and resistance elements. This extension is fairly obvious, and it is surprising that Foster did not make this extension himself. In the case of networks containing inductance and resistance elements or resistance and capacity elements, the impedance functions are, respectively, for the two mesh case, in the form

$$Z(p) = \frac{a_2 p^2 + a_1 p + a_0}{b_1 p + b_0} \qquad (4)$$

and

$$Z(p) = \frac{a_2 p^2 + a_1 p + a_0}{p\left[b_1 p + b_0\right]} \qquad (5)$$

For the two mesh case of inductance and capacity

$$Z(p) = \frac{a_4 p^4 + a_2 p^2 + a_0}{p(b_2 p^2 + b_0)} \qquad (6)$$

Thus, with the exception of the pole at zero, (4) and (5) are exactly in the same form as (6) except that p replaces p^2. Thus, (4) and (5) will have zeros and poles which are negative reals, instead of pure imaginaries as in (6). These zeros and poles in (4) and (5) will likewise separate each other.

Another extension which Cauer made to Foster's first paper, was to construct two networks in addition to the two that Foster could construct. It will be useful at this point to illustrate this for the two mesh network with inductance and capacity elements. The impedance of such a network will have the form

$$Z(p) = \frac{a_4 p^4 + a_2 p^2 + a_0}{p(b_2 p^2 + b_0)} \qquad (6)$$

Now Foster says, if you give me an expression like (6), where the a and b coefficients are real and the zeros and poles are pairs of pure imaginaries having the separation property, then I can construct the following two networks which will have

(6) for an impedance

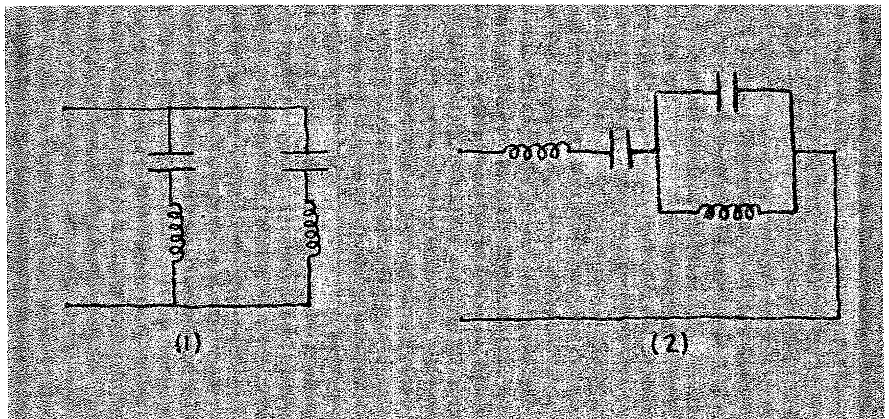

FIG. 3

Network (1) is obtained by expanding the given admittance function into partial fractions, and consists of two resonant circuits in parallel. Network (2) is obtained by expanding the given impedance function into partial fractions, after reducing the numerator to a lower power than the denominator by division. ✓ This therefore consists of the quotient, which is the circuit consisting of the inductance and capacity in series, and the partial fraction term, which is the anti-resonant circuit of capacity and inductance in parallel.

Cauer proceeds further and says that he can construct two more circuits having the impedance given by (6), and having only four elements. These additional circuits are shown in figure 4.

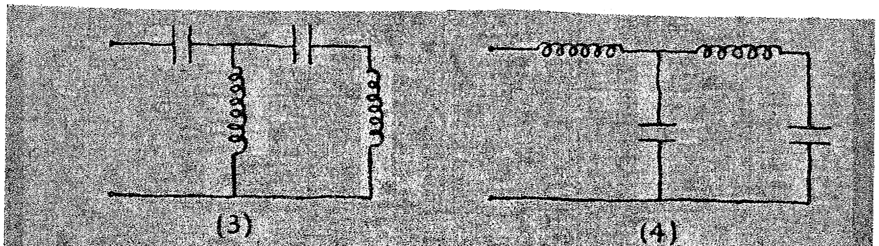

FIG. 4

Network (3) is obtained by expanding the admittance function into a continued fraction (finite Stieltjes continued fraction). Network (4) is obtained by expanding the impedance function into a continued fraction.

In brief then, if an expression like (6) satisfies certain conditions, then (a) an expansion of its reciprocal into partial fractions gives network (1), (b) an expansion of itself in partial fractions gives network (2), (c) an expansion of its reciprocal into a continued fraction gives network (3) and (d) an expansion of itself in a continued fraction gives network (4). The first two expansions were Foster's, the last two Cauer's. These four networks are called by Cauer the canonical forms.

This has been illustrated for the case of a two mesh network, but it holds as well for a network of any number of meshes, provided the network contains only two kinds of elements. Thus if the network of n meshes contains inductance and capacity elements only, its impedance function is of the form

$$Z(p) = \frac{a_{2n}p^{2n} + a_{2n-2}p^{2n-2} + a_{2n-4}p^{2n-4} + \cdots + a_4p^4 + a_2p^2 + a_0}{b_{2n-1}p^{2n-1} + b_{2n-3}p^{2n-3} + \cdots + b_5p^5 + b_3p^3 + b_1p} \qquad (7)$$

Foster's method in this case would be to factor both the numerator and denominator of (7), and if the zeros and poles were pairs of pure imaginaries with opposite signs and had the separation property, he would expand the reciprocal of (7) into partial fractions and obtain the network shown in figure 5.

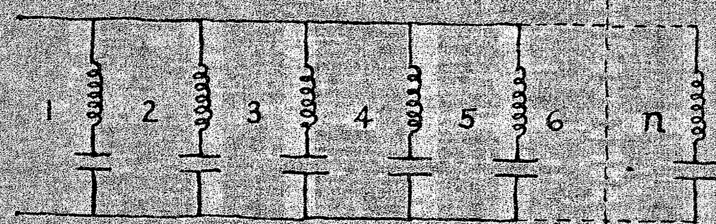

FIG.5

Then he would expand (7) itself into partial fractions and obtain the network shown in figure 6

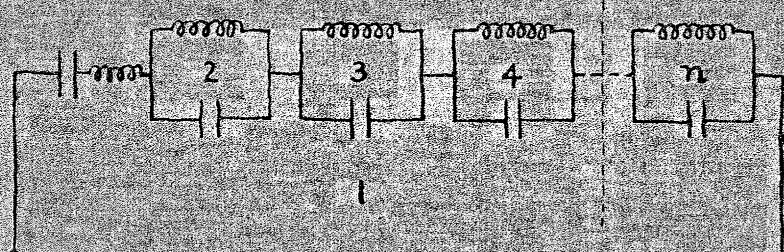

FIG.6

That is as far as Foster went. Then Cauer proceeds to expand the reciprocal of (7) into a continued fraction and obtains the network shown in figure 7.

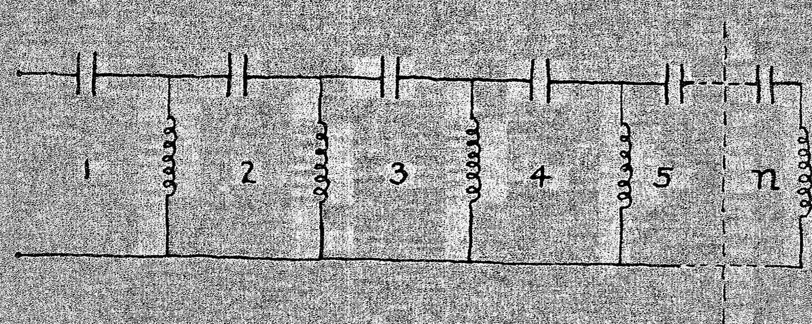

FIG. 7

Finally Cauer expands (7) itself into a continued fraction and obtains the network shown in figure 8.

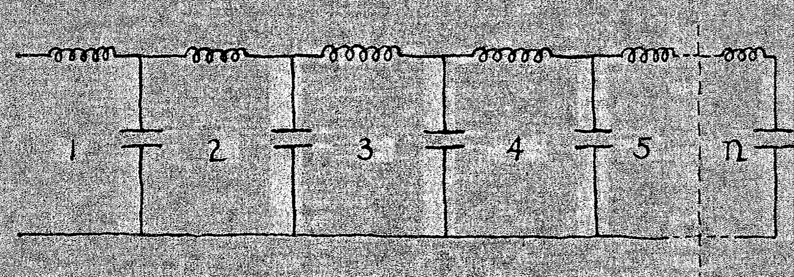

FIG. 8

In each case, it is important to remember that the network elements themselves are obtained from <u>each term</u> of the partial fraction expansion or the continued fraction expansion. In each of these cases, networks of the least number of elements are obtained that will have an impedance function given by (7).

As I have pointed out, Cauer extended the above to networks having inductance and resistance elements, and networks having resistance and capacity elements. The impedance function for a network of n meshes having inductance and resistance elements is of the form

$$Z(p) = \frac{a_n p^n + a_{n-1} p^{n-1} + a_{n-2} p^{n-2} + \cdots + a_2 p^2 + a_1 p + a_0}{b_{n-1} p^{n-1} + b_{n-2} p^{n-2} + \cdots + b_2 p^2 + b_1 p + b_0} \qquad (8)$$

The impedance function for a network of n meshes having resistance and capacity elements is of the form

$$Z(p) = \frac{a_n p^n + a_{n-1} p^{n-1} + a_{n-2} p^{n-2} + \cdots + a_2 p^2 + a_1 p + a_0}{b_n p^n + b_{n-1} p^{n-1} + b_{n-2} p^{n-2} + \cdots + b_2 p^2 + b_1 p} \qquad (9)$$

As was pointed out, Cauer extended Foster's conditions on the zeros and poles of an expression like (7), in order that it be in fact an impedance function, to expressions like (8) and (9). Cauer showed that an expression like (8) is in fact an impedance function if its zeros and poles are negative and in addition possess the separation property. If this is true, then he can proceed just as had been done in the case of networks having only inductance and capacity elements. Thus by expanding the reciprocal of (8) into partial fractions he obtains the network shown in figure 9.

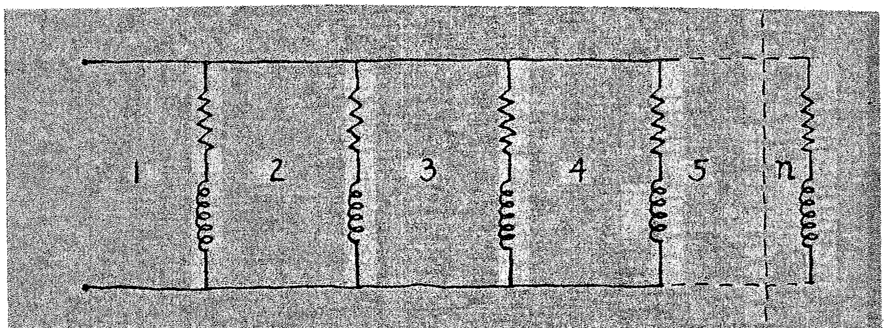

FIG. 9

By expanding (8) itself into partial fractions he obtains the network shown in figure 10.

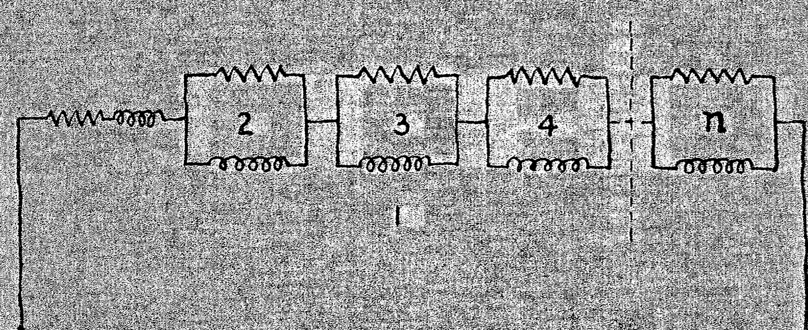

FIG 10

By expanding the reciprocal of (8) into a continued fraction he obtains the network shown in figure 11.

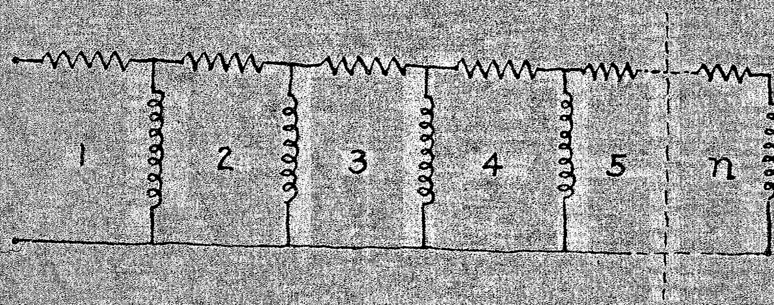

FIG.11

Finally by expanding (8) itself into a continued fraction, the network shown in figure 12 is obtained.

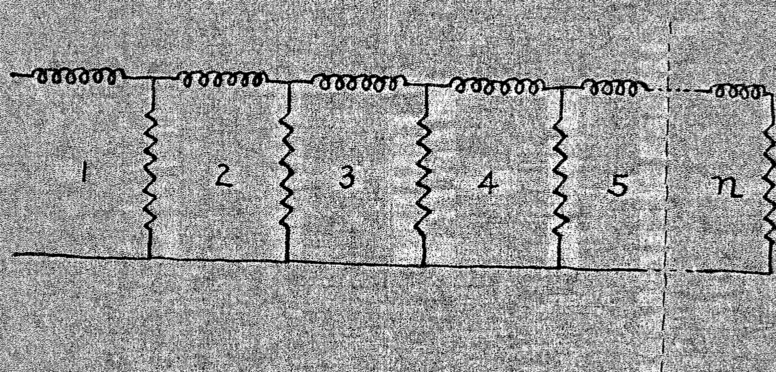

FIG.12

In the same way he showed that by each one of these four expansions (9) could be shown to be the impedance function of the following four networks.

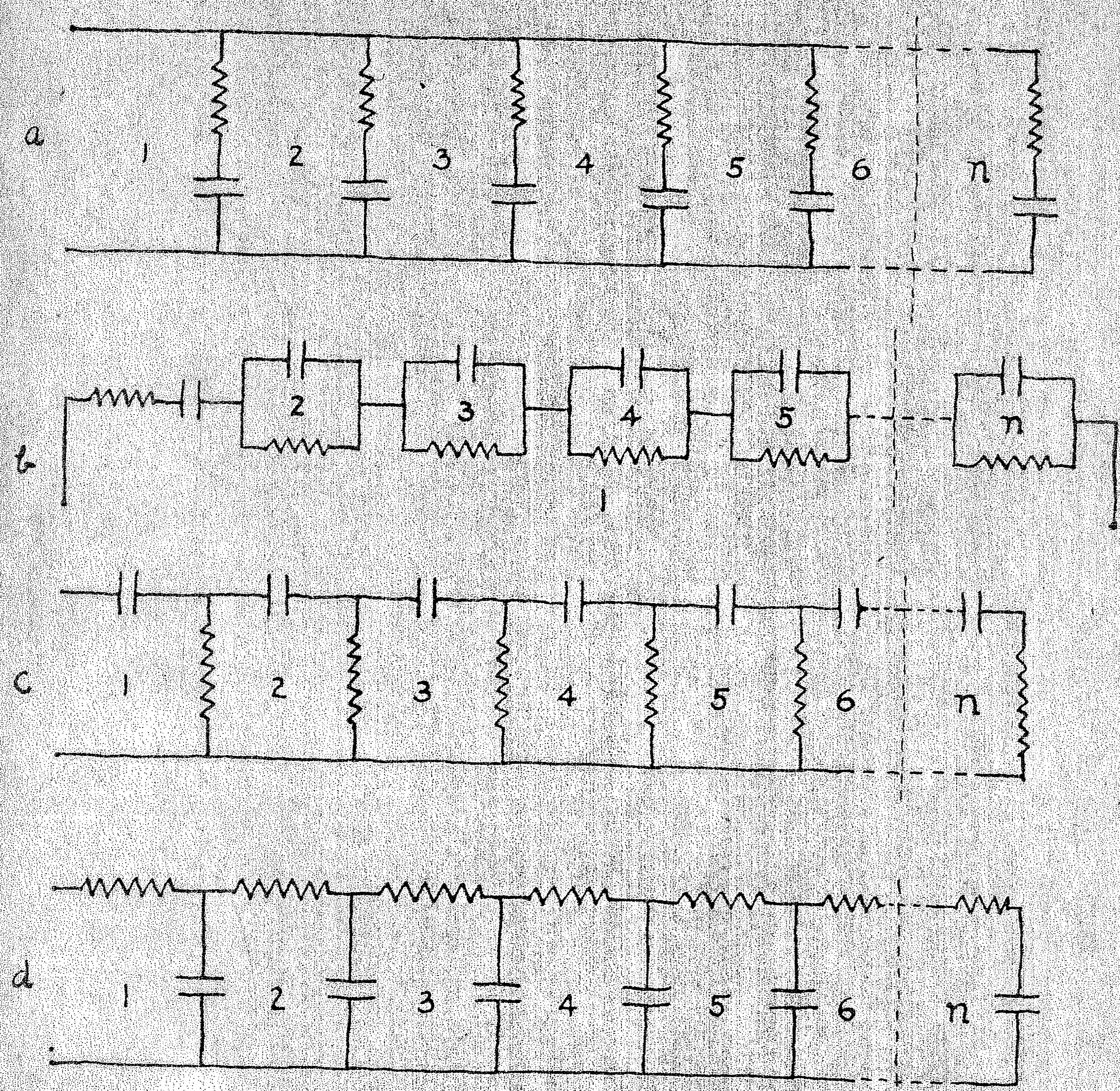

FIG. 13

Cauer proceeded still further and showed that expressions like (7), (8) and (9) could be shown to be in fact impedance functions if their coefficients a_i, b_i satisfied certain conditions. Foster had placed the conditions on the zeros and poles, whereas Cauer placed them upon the coefficients. In this, he makes use of an important theorem of Hurwitz which will be taken up later.

Finally Cauer, like Foster, considers the two mesh network containing all three elements, inductance, resistance and capacity and like Foster, obtains the equivalence equations. Cauer however proceeds further going more deeply into the theory, making use of a certain change of variable which permits him to use the Hurwitz condition in this case as well. He then expresses by means of it, the conditions which the coefficients of an expression like (3a) must satisfy in order that it be the impedance function of a two mesh network consisting of inductances, capacity and resistance elements. These conditions Cauer expresses in slightly simpler form than Foster.

It is thus seen that Foster and Cauer have been primarily interested in obtaining the conditions that the coefficients or roots of an expression that resembles an impedance function must satisfy in order that the expression be in fact the impedance function of some electrical network. Then, for networks with two kinds of elements, they would proceed by the expansion of the impedance function and the admittance function into partial fractions and continued fractions obtaining the caconical forms. They were <u>not</u> interested in obtaining the complete array of networks physically realizing the given impedance. In this case (networks with two kinds of network elements) the equivalence equations were not obtained, although all the equivalent networks could be obtained from them, and the canonical forms more readily than by the above expansions. Their interests were similar in the case of networks containing all three elements, but in this case, equivalence equations were obtained.

CHAPTER I

General Theory.

The general network theory will now be given first directly by means of Kirchhoff's differential equations of the network and second by means of Lagrange's equations.

The complete system of differential equations of a two-terminal network of n meshes containing as elements positive inductance, resistance and capacity elements, with the electromotive force e(t) applied to the terminals is given by

$$
\left.\begin{aligned}
&\lambda_{11}\frac{di_1}{dt}+\rho_{11}i_1+\sigma_{11}\int i_1\,dt+\lambda_{12}\frac{di_2}{dt}+\rho_{12}i_2+\sigma_{12}\int i_2\,dt+\cdots \\
&\qquad \cdots+\lambda_{1n}\frac{di_n}{dt}+\rho_{1n}i_n+\sigma_{1n}\int i_n\,dt &&=e(t)\\
&\lambda_{21}\frac{di_1}{dt}+\rho_{21}i_1+\sigma_{21}\int i_1\,dt+\lambda_{22}\frac{di_2}{dt}+\rho_{22}i_2+\sigma_{22}\int i_2\,dt+\cdots \\
&\qquad \cdots+\lambda_{2n}\frac{di_n}{dt}+\rho_{2n}i_n+\sigma_{2n}\int i_n\,dt &&=0\\
&------------------\\
&\lambda_{n1}\frac{di_1}{dt}+\rho_{n1}i_1+\sigma_{n1}\int i_1\,dt+\lambda_{n2}\frac{di_2}{dt}+\rho_{n2}i_2+\sigma_{n2}\int i_2\,dt+\cdots \\
&\qquad \cdots+\lambda_{nn}\frac{di_n}{dt}+\rho_{nn}i_n+\sigma_{nn}\int i_n\,dt &&=0
\end{aligned}\right\}\quad (10)
$$

In these equations λ_{kk}, ρ_{kk} and σ_{kk} are respectively the total inductance, resistance and elastance of mesh k and are called the total parameters of mesh k. The terms λ_{jk}, ρ_{jk} and σ_{jk} are respectively the inductance, resistance and elastance common or mutual to meshes j and k and are called the mutual parameters. From physical considerations of the network it is obvious that $\lambda_{jk} = \lambda_{kj}$, $\rho_{jk} = \rho_{kj}$ and $\sigma_{jk} = \sigma_{kj}$ and that these mutual parameters are contained in the total parameters, so that the following inequalities hold

$$\left.\begin{aligned} \lambda_{kk} &\geqq \lambda_{k1} + \lambda_{k2} + \text{-----} + \lambda_{k,k-1} + \lambda_{k,k+1} + \text{------} + \lambda_{kn} \\ \rho_{kk} &\geqq \rho_{k1} + \rho_{k2} + \text{------} + \rho_{k,k-1} + \rho_{k,k+1} + \text{------} + \rho_{kn} \\ \sigma_{kk} &\geqq \sigma_{k1} + \sigma_{k2} + \text{-----} + \sigma_{k,k-1} + \sigma_{k,k+1} + \text{-----} + \sigma_{kn} \end{aligned}\right\} \quad (11)$$

The quantities i_1, i_2 ----- i_n represent the mesh currents. The signs in the system of equations (10) are all taken positive but the signs of the mutual terms may be taken as negative depending upon the assumed directions of the mesh currents. The signs are readily checked by applying Kirchhoff's laws to the particular circuit in question. Mutual inductance has been omitted, but these additional elements are merely absorbed in the λ terms and its inclusion in these terms may be assumed.

To solve the systems of equations for the steady state currents, let

$$e(t) = E\varepsilon^{pt} \tag{12}$$

and assume that

$$\left.\begin{aligned} i_1 &= I_1 \varepsilon^{pt} \\ i_2 &= I_2 \varepsilon^{pt} \\ &- - - - - - \\ i_n &= I_n \varepsilon^{pt} \end{aligned}\right\} \tag{13}$$

Substituting these values of the currents in the system of equations (10), we have

$$\left.\begin{aligned} &\left(\lambda_{11} p + \rho_{11} + \frac{\sigma_{11}}{p}\right) I_1 + \left(\lambda_{12} p + \rho_{12} + \frac{\sigma_{12}}{p}\right) I_2 + \cdots + \left(\lambda_{1n} p + \rho_{1n} + \frac{\sigma_{1n}}{p}\right) I_n = E \\ &\left(\lambda_{12} p + \rho_{12} + \frac{\sigma_{12}}{p}\right) I_1 + \left(\lambda_{22} p + \rho_{22} + \frac{\sigma_{22}}{p}\right) I_2 + \cdots + \left(\lambda_{2n} p + \rho_{2n} + \frac{\sigma_{2n}}{p}\right) I_n = 0 \\ &- \\ &\left(\lambda_{1n} p + \rho_{1n} + \frac{\sigma_{1n}}{p}\right) I_1 + \left(\lambda_{2n} p + \rho_{2n} + \frac{\sigma_{2n}}{p}\right) I_2 + \cdots + \left(\lambda_{nn} p + \rho_{nn} + \frac{\sigma_{nn}}{p}\right) I_n = 0 \end{aligned}\right\} \tag{14}$$

Now let

$$a_{jk} = \lambda_{jk} p + \rho_{jk} + \frac{\sigma_{jk}}{p} \qquad (15)$$

Then equation (14) becomes

$$\left.\begin{aligned} a_{11} I_1 + a_{12} I_2 + \cdots + a_{1n} I_n &= E \\ a_{12} I_1 + a_{22} I_2 + \cdots + a_{2n} I_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{1n} I_1 + a_{2n} I_2 + \cdots + a_{nn} I_n &= 0 \end{aligned}\right\} \qquad (16)$$

The system of equations (16) being a set of linear equations, they are readily solved by the usual method of determinants. Thus

$$I_1 = \frac{\begin{vmatrix} E & a_{12} & \cdots & a_{1n} \\ 0 & & & \vdots \\ \vdots & & & \vdots \\ 0 & a_{2n} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & & & \vdots \\ \vdots & & & \vdots \\ a_{1n} & \cdots & \cdots & a_{nn} \end{vmatrix}} \qquad (17)$$

Let

$$D(p) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & & & \vdots \\ \vdots & & & \vdots \\ a_{1n} & \cdots & \cdots & a_{nn} \end{vmatrix} \qquad (18)$$

and let $M_{11}(p)$ be the cofactor or minor with proper sign of the element in the first row and first column of D(p). Then

$$I_1 = \frac{M_{11}(p)}{D(p)} E \qquad (19)$$

Likewise let $M_{12}(p)$ be the cofactor or minor with proper sign of the element in the first row and second column. Then

$$I_2 = \frac{M_{12}(p)}{D(p)} E \qquad (20)$$

Similarly

$$I_k = \frac{M_{1k}(p)}{D(p)} E \qquad (21)$$

Now the driving-point impedance function is defined as the ratio of the impressed voltage E to the entering current I. If Z(p) represents this driving-point impedance, then

$$Z(p) = \frac{D(p)}{M_{11}(p)} \qquad (22)$$

Similarly the transfer impedance of the k mesh, that is the ratio of the impressed voltage E to the mesh current in the k mesh is

$$Z_k(p) = \frac{D(p)}{M_{1k}(p)} \qquad (23)$$

The thesis will concern itself in detail with the driving-point impedance, although the theory will hold for the transfer impedance as well. The impedance function Z(p) is a very important quantity in electric circuit theory. Its importance in alternating-current theory is known and its use in symbolic complex notation is general. Its importance in the theory of transient currents is becoming more and more recognized in recent years through the exposition of Heaviside's work in operational circuit analysis.[6]

It is to be noted that the system of equations (10) were obtained directly by the use of Kirchhoff's laws applied to the network. It will be instructive to arrive at the system of equations (10) through the energy relations in the network and the use of Lagrange's equations. The analogy between the theory of vibrations in classical dynamics and electric circuit theory is well known. Yet, although there exists a mine of information in classical dynamic theory about electric circuit theory, little

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6. See V. Bush, loc. cit., p. 29.

has been done to convert this theoretical knowledge in dynamics into its proper language in circuit theory. Foster's first paper, for example, was essentially to translate Routh's classical treatment of the problems of small vibrations about a point of equilibrium into the proper terminology for the similar problem of the steady state oscillations of current in a network. Similarly, much of the inspiration and proof of his second paper came from Routh's derivation of the determinantal equation, not directly by Newton's laws of motion (which would correspond in electric circuit theory to Kirchhoff's laws), but from the energy relations of the dynamical system and the use of Lagrange's equations.

Let us proceed to give these energy relations, and to fix ideas, let us consider the case of the two mesh network with all three kinds of elements present as shown in figure 14.

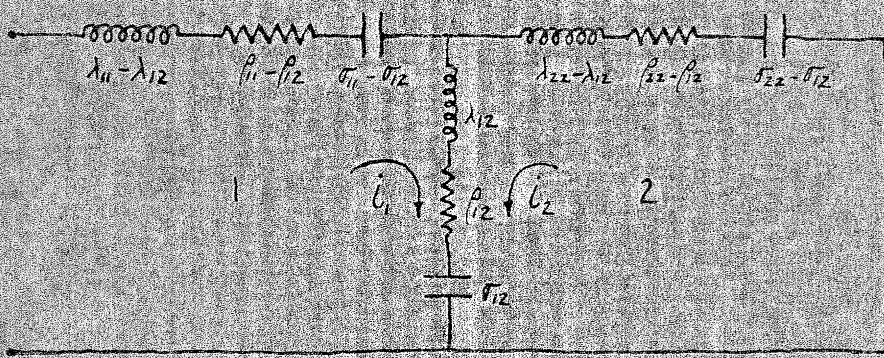

FIG. 14

The elements λ_{12}, ρ_{12} and σ_{12} are, as before, the elements common or mutual to mesh 1 and 2, λ_{11}, ρ_{11} and σ_{11} are the <u>total</u> parameters of mesh 1, that is they are respectively the total inductance, the total resistance and total elastance of mesh 1. Similarly λ_{22}, ρ_{22} and σ_{22} are the total parameters of mesh 2. Hence the elements in mesh 1 are $(\lambda_{11}-\lambda_{12})$, $(\rho_{11}-\rho_{12})$, $(\sigma_{11}-\sigma_{12})$ and similarly the elements in mesh 2 are $(\lambda_{22}-\lambda_{12})$, $(\rho_{22}-\rho_{12})$ and $(\sigma_{22}-\sigma_{12})$. The quantities i_1 and i_2 are the instantaneous mesh currents, the arrows indicating their directions. Let q_1 and q_2 be the corresponding mesh charges, so that

$$\left.\begin{aligned} i_1 &= \frac{dq_1}{dt} \\ i_2 &= \frac{dq_2}{dt} \end{aligned}\right\} \qquad (24)$$

The total instantaneous magnetic energy in the complete network is given by

$$T = \frac{1}{2}(\lambda_{11}-\lambda_{12})i_1^2 + \frac{1}{2}\lambda_{12}(i_1+i_2)^2 + \frac{1}{2}(\lambda_{22}-\lambda_{12})i_2^2 \qquad (25)$$

$$= \frac{1}{2}\lambda_{11}i_1^2 - \frac{1}{2}\lambda_{12}i_1^2 + \frac{1}{2}\lambda_{12}i_1^2 + \lambda_{12}i_1 i_2 + \frac{1}{2}\lambda_{12}i_2^2 + \frac{1}{2}\lambda_{22}i_2^2 - \frac{1}{2}\lambda_{12}i_2^2$$

$$= \frac{1}{2}\lambda_{11}i_1^2 + \lambda_{12}i_1 i_2 + \frac{1}{2}\lambda_{22}i_2^2$$

$$= \frac{1}{2}(\lambda_{11}i_1^2 + 2\lambda_{12}i_1 i_2 + \lambda_{22}i_2^2) \qquad (26)$$

Similarly, the total instantaneous electrostatic energy in the complete network is given by

$$V = \frac{1}{2}(\sigma_{11}-\sigma_{12})q_1^{\,2} + \frac{1}{2}\sigma_{12}(q_1+q_2)^2 + \frac{1}{2}(\sigma_{22}-\sigma_{12})q_2^{\,2}$$

$$= \frac{1}{2}(\sigma_{11}q_1^{\,2} + 2\sigma_{12}q_1q_2 + \sigma_{22}q_2^{\,2}) \qquad (27)$$

Finally, the total instantaneous power lost in the resistance of the complete network is given by

$$R = (\rho_{11}-\rho_{12})i_1^{\,2} + \rho_{12}(i_1+i_2)^2 + (\rho_{22}-\rho_{12})i_2^{\,2}$$

$$= \rho_{11}i_1^{\,2} + 2\rho_{12}i_1i_2 + \rho_{22}i_2^{\,2} \qquad (28)$$

In more compact notation T, V and R may respectively be written

$$T = \frac{1}{2}\sum_{jk=1}^{2}\lambda_{jk}i_ji_k \qquad (29)$$

$$V = \frac{1}{2}\sum_{jk=1}^{2}\sigma_{jk}q_jq_k \qquad (30)$$

$$R = \sum_{jk=1}^{2}\rho_{jk}i_ji_k \qquad (31)$$

Since $\lambda_{jk} = \lambda_{kj}$, $\sigma_{jk} = \sigma_{kj}$, $\rho_{jk} = \rho_{kj}$ it is readily seen that by giving j and k all possible values from 1 to 2, in any manner, equations (29), (30) and (31) reduce at once to equations (26), (27) and (28).

It might be well at this point to generalize equations (29), (30) and (31) for n meshes. This is done simply by changing the upper limit of the summation from 2 to n. For n meshes then, equations (29), (30) and (31) become

$$T = \frac{1}{2} \sum_{jk=1}^{n} \lambda_{jk} i_j i_k \qquad (32)$$

$$V = \frac{1}{2} \sum_{jk=1}^{n} \sigma_{jk} q_j q_k \qquad (33)$$

$$R = \sum_{jk=1}^{n} \rho_{jk} i_j i_k \qquad (34)$$

where j and k take on all possible values from 1 to n, in any manner.

The quantities T, V and R are the so-called quadratic forms[7] which are positive and definite, that is to say, they are positive for all values of the variable i or q, and they are zero when and only when all the variables are zero. The positiveness of these forms follows at once from physical considerations since the magnetic energy, the electrostatic energy and the power lost

7. See M. Bocher, Introduction to Higher Algebra, 1927, p. 150.

in the resistances of the network are positive quantities, and are zero when and only when all the currents or charges are respectively zero. These quadratic forms play an important role in dynamics, and very important results are obtained from their positive and definite character. A glance at Foster's and Cauer's papers will indicate their importance in electric circuit theory. It is interesting to point out the similar important role played by the quadratic form in differential geometry where the linear element of the surface ds is given by

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \tag{35}$$

The parametric representation of a curve on a surface is given by $f(u,v) = 0$. The differential arc on the surface is given by

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \tag{36}$$

where x_1, x_2 and x_3 are the rectangular coordinates. In vector notation

$$ds^2 = dx \cdot dx \tag{37}$$

$$= (x_u\,du + x_v\,dv) \cdot (x_u\,du + x_v\,dv)$$

$$= (x_u \cdot x_u)\,du^2 + 2(x_u \cdot x_v)\,du\,dv + (x_v \cdot x_v)\,dv^2 \tag{38}$$

Setting

$$E = (x_u \cdot x_u)$$

$$F = (x_u \cdot x_v)$$

$$G = (x_v \cdot x_v)$$

we have

$$ds^2 = E du^2 + 2F du dv + G dv^2 \qquad (35)$$

which, in more compact notation

$$ds^2 = \sum_{jk=1}^{2} \alpha_{jk} \, d\theta_j \, d\theta_k \qquad (39)$$

where now $\alpha_{11} = E$, $\alpha_{12} = \alpha_{21} = F$ and $\alpha_{22} = G$ and $d\theta_1 = du$ and $d\theta_2 = dv$.

As above, extending (39) now to n -dimension—al space instead of as in the electric circuit to n meshes, only the upper limit in (39) is changed from 2 to n and we have

$$ds^2 = \sum_{jk=1}^{n} \alpha_{jk} \, d\theta_j \, d\theta_k \qquad (40)$$

(39) and (40) are thus seen to be of exactly the same form as (29), (30) or (31), and (32), (33) and (34) respectively. The α_{jk}

terms are however not constant as are the λ_{jk}, σ_{jk} and ρ_{jk} term, and the differential $d\theta$'s replace the finite i's or q's. The quadratic forms (39) and (40) are likewise positive and definite, and play a most important role in differential geometry.[8] Thus the similarity of the quadratic forms in differential geometry, dynamics and electric circuit theory suggests the unification of all three branches in one theory.

Proceeding with our two mesh network, let us now make use of Lagrange's equation for a dissipative system of two degrees of freedom.

These are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial F}{\partial \dot{q}_r} + \frac{\partial V}{\partial q_r} = e_r \qquad (41)$$
$$r = 1, 2$$

where $F = \frac{1}{2}R$ and e_r represents the applied forces.[9]

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8. See W. Blaschke, Vorlesungen über Differentialgeometrie I, Springer, Berlin 1924.

9. See E. T. Whittaker, Treatise on the Analytical Dynamics of Particles and Rigid Bodies, 1917, page 232.

In our problem , these become

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial i_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial F}{\partial i_1} + \frac{\partial V}{\partial q_1} &= e_1 \\ \frac{d}{dt}\left(\frac{\partial T}{\partial i_2}\right) - \frac{\partial T}{\partial q_2} + \frac{\partial F}{\partial i_2} + \frac{\partial V}{\partial q_2} &= 0 \end{aligned}\right\} \quad (42)$$

From (26), (27) and (28) respectively,

$$\left.\begin{aligned} T &= \tfrac{1}{2}\left(\lambda_{11} i_1^{\,2} + 2\lambda_{12} i_1 i_2 + \lambda_{22} i_2^{\,2}\right) \\ V &= \tfrac{1}{2}\left(\sigma_{11} q_1^{\,2} + 2\sigma_{12} q_1 q_2 + \sigma_{22} q_2^{\,2}\right) \\ F &= \tfrac{1}{2}\left(\rho_{11} i_1^{\,2} + 2\rho_{12} i_1 i_2 + \rho_{22} i_2^{\,2}\right) \end{aligned}\right\} \quad (43)$$

To simplify the substitution of (43) in (42) note that

$$\frac{\partial T}{\partial i_1} = \lambda_{11} i_1 + \lambda_{12} i_2$$

$$\frac{\partial T}{\partial i_2} = \lambda_{12} i_1 + \lambda_{22} i_2$$

and

$$\frac{d}{dt}\left(\frac{\partial T}{\partial i_1}\right) = \lambda_{11}\frac{di_1}{dt} + \lambda_{12}\frac{di_2}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial i_2}\right) = \lambda_{12}\frac{di_1}{dt} + \lambda_{22}\frac{di_2}{dt}$$

Also

$$\frac{\partial T}{\partial q_1} = 0$$

$$\frac{\partial T}{\partial q_2} = 0$$

$$\frac{\partial F}{\partial i_1} = \rho_{11} i_1 + \rho_{12} i_2$$

$$\frac{\partial F}{\partial i_2} = \rho_{12} i_1 + \rho_{22} i_2$$

$$\frac{\partial V}{\partial q_1} = \sigma_{11} q_1 + \sigma_{12} q_2$$

$$\frac{\partial V}{\partial q_2} = \sigma_{12} q_1 + \sigma_{22} q_2$$

Thus equations (42) become

$$\left.\begin{aligned} \lambda_{11}\frac{di_1}{dt} + \lambda_{12}\frac{di_2}{dt} + \rho_{11} i_1 + \rho_{12} i_2 + \sigma_{11} q_1 + \sigma_{12} q_2 &= e_1 \\ \lambda_{12}\frac{di_1}{dt} + \lambda_{22}\frac{di_2}{dt} + \rho_{12} i_1 + \rho_{22} i_2 + \sigma_{12} q_1 + \sigma_{22} q_2 &= 0 \end{aligned}\right\} \quad (44)$$

But

$$q_1 = \int i_1\,dt$$

And

$$\left.\begin{aligned} q_1 &= \int i_1\,dt \\ q_2 &= \int i_2\,dt \end{aligned}\right\} \quad (45)$$

Hence equation (44) becomes

$$\left.\begin{aligned} \lambda_{11}\frac{di_1}{dt} + \rho_{11} i_1 + \sigma_{11}\int i_1\,dt + \lambda_{12}\frac{di_2}{dt} + \rho_{12} i_2 + \sigma_{12}\int i_2\,dt = e_1 \\ \lambda_{12}\frac{di_1}{dt} + \rho_{12} i_1 + \sigma_{12}\int i_1\,dt + \lambda_{22}\frac{di_2}{dt} + \rho_{22} i_2 + \sigma_{22}\int i_2\,dt = 0 \end{aligned}\right\} \quad (46)$$

But these equations are precisely the Kirchhoff's equations for the two-mesh network shown in figure 14, and corresponds to the system of equations (10) page 25 for n = 2

Generalizing then for the n mesh case, Lagrange's equations become in electric circuit theory

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial i_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial F}{\partial i_k} + \frac{\partial V}{\partial q_k} = e_k \\ k = 1, 2, ---- n \end{aligned}\right\} \quad (47)$$

where T is the <u>total</u> <u>instantaneous</u> magnetic energy in the complete network, V is the <u>total</u> <u>instantaneous</u> electrostatic energy in the complete network. F is the <u>dissipation</u> function which is equal to one half the total power lost in the resistances of the network and e_k represents the <u>instantaneous</u> mesh electromotive forces.

The substitution of the values of T, V and F for the n mesh network in the system of equations (47) will give precisely the system of equations (10). Substituting these values in (47) and omitting the term $\frac{\partial T}{\partial q_k}$, which is always zero for our case, we have

$$\left.\begin{array}{r}\frac{d}{dt}\left(\frac{\partial}{\partial i_r}\frac{1}{2}\sum_{jk=1}^{n}\lambda_{jk}\, i_j i_k\right)+\frac{\partial}{\partial i_r}\cdot\frac{1}{2}\sum_{jk=1}^{n}\rho_{jk}\, i_j i_k+\frac{\partial}{\partial q_r}\frac{1}{2}\sum_{jk=1}^{n}\sigma_{jk}\, q_j q_k=e_r \\ r=1,2\ -----\ n\end{array}\right\}\quad(48)$$

This is precisely the system of equations (10), page 25 obtained by the direct application of Kirchhoff's laws. By performing the operations indicated in (48), the system of equations (10) are thereby obtained.

The Kirchhoff law equations were derived from the energy relations by the use of Lagrange's equations in order to make clear the function that the quadratic forms (32), (33) and (34) perform in obtaining these equations. As will be seen later, much use will be made of these quadratic forms in arriving at certain very important properties of the impedance function,

It is instructive to point out here that the coefficients of the quadratic forms (29), (30) and (31), page 33 may be obtained directly from certain matrices. Thus, the coefficients of the quadratic form (29) are contained in the matrix

$$\left\Vert\begin{array}{cc}\lambda_{11} & \lambda_{12}\\ \lambda_{12} & \lambda_{22}\end{array}\right\Vert \qquad (49)$$

and the form is obtained at once by writing

$$\lambda_{11} i_1^2 + \lambda_{12} i_1 i_2 + \lambda_{12} i_1 i_2 + \lambda_{22} i_2^2$$

which is of course 2 T. Similarly the coefficients of the forms (30) and (31) respectively are contained in the matrices

$$\left\| \begin{matrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{matrix} \right\| \qquad \left\| \begin{matrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{matrix} \right\| \tag{50}$$

Similarly, in the n mesh case, the coefficients of the quadratic forms T, V and R are contained respectively in the matrices

$$\left\| \begin{matrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{12} & & & \vdots \\ \vdots & & & \vdots \\ \lambda_{1n} & \cdots & \cdots & \lambda_{nn} \end{matrix} \right\| \qquad \left\| \begin{matrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & & & \vdots \\ \vdots & & & \vdots \\ \sigma_{1n} & \cdots & \cdots & \sigma_{nn} \end{matrix} \right\| \qquad \left\| \begin{matrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & & & \vdots \\ \vdots & & & \vdots \\ \rho_{1n} & \cdots & \cdots & \rho_{nn} \end{matrix} \right\| \tag{51}$$

This is pointed out because these matrices and their corresponding determinants will be seen later to play an important role in our theory.

CHAPTER II.

The Two-Mesh Minimal Forms.

The driving-point impedance function was defined as the ratio of the impressed voltage to the entering current and was given by

$$Z(p) = \frac{D(p)}{M_{11}(p)} \qquad (22)$$

where D(p) was defined as the determinant of the network and $M_{11}(p)$ was the minor of the element of the first row and first column in D(p). Equation (22) expressed in determinant form is

$$Z(p) = \frac{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & & & \vdots \\ \vdots & & & \vdots \\ a_{1n} & \cdots & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{2n} & \cdots & a_{nn} \end{vmatrix}} \qquad (52)$$

where

$$a_{jk} = \lambda_{jk} p + \rho_{jk} + \frac{\sigma_{jk}}{p}$$

The expansion of the determinant and its minor in terms of powers of p will reduce Z(p) to the expression (2), page 2 . Let us therefore expand Z(p) thus and obtain an expression like (2), but noting at the same time how the a and b coefficients in (2) are expressed in terms of the elements λ, ρ and σ. To fix ideas, let us consider the general two-mesh network consisting of two kinds of elements, namely resistance and capacity elements. Such a network is shown in figure 15.

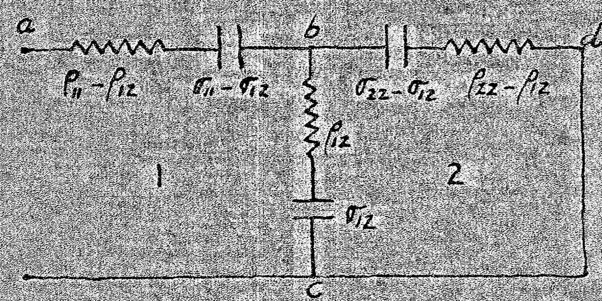

FIG.15

The determinant D(p) of this network is

$$D(p) = \begin{vmatrix} \rho_{11} + \dfrac{\sigma_{11}}{p} & \rho_{12} + \dfrac{\sigma_{12}}{p} \\ \\ \rho_{12} + \dfrac{\sigma_{12}}{p} & \rho_{22} + \dfrac{\sigma_{22}}{p} \end{vmatrix} \tag{53}$$

which may be written

$$D(p) = \begin{vmatrix} \dfrac{\rho_{11} p + \sigma_{11}}{p} & \dfrac{\rho_{12} p + \sigma_{12}}{p} \\ \\ \dfrac{\rho_{12} p + \sigma_{12}}{p} & \dfrac{\rho_{22} p + \sigma_{22}}{p} \end{vmatrix} \tag{54}$$

Removing the p in the denominator of the elements we have

$$D(p) = \frac{1}{p^2}\begin{vmatrix} \rho_{11}p+\sigma_{11} & \rho_{12}p+\sigma_{12} \\ \rho_{12}p+\sigma_{12} & \rho_{22}p+\sigma_{22} \end{vmatrix} \tag{54a}$$

This is readily reduced by expansion tothe <u>polynomial</u> of the network. Thus

$$\begin{vmatrix} \rho_{11}p+\sigma_{11} & \rho_{12}p+\sigma_{12} \\ \rho_{12}p+\sigma_{12} & \rho_{22}p+\sigma_{22} \end{vmatrix}$$

$$= \begin{vmatrix} \rho_{11}p & \rho_{12}p+\sigma_{12} \\ \rho_{12}p & \rho_{22}p+\sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \rho_{12}p+\sigma_{12} \\ \sigma_{12} & \rho_{22}p+\sigma_{22} \end{vmatrix}$$

$$= \begin{vmatrix} \rho_{11}p & \rho_{12}p \\ \rho_{12}p & \rho_{22}p \end{vmatrix} + \begin{vmatrix} \rho_{11}p & \sigma_{12} \\ \rho_{12}p & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \rho_{12}p \\ \sigma_{12} & \rho_{22}p \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix}$$

$$= \begin{vmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{vmatrix} p^2 + \left\{ \begin{vmatrix} \rho_{11} & \sigma_{12} \\ \rho_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \rho_{12} \\ \sigma_{12} & \rho_{22} \end{vmatrix} \right\} p + \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix}$$

Thus

$$D(p) = \frac{1}{p^2}\left[\begin{vmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{vmatrix} p^2 + \left\{ \begin{vmatrix} \rho_{11} & \sigma_{12} \\ \rho_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \rho_{12} \\ \sigma_{12} & \rho_{22} \end{vmatrix} \right\} p + \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} \right] \tag{55}$$

Now let us adopt the following symbolic notation

$$\Delta(\rho) = \begin{vmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{vmatrix} \qquad (56)$$

$$\Delta(\sigma) = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} \qquad (57)$$

$$\Delta_1(\rho,\sigma) = \begin{vmatrix} \rho_{11} & \sigma_{12} \\ \rho_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \rho_{12} \\ \sigma_{12} & \rho_{22} \end{vmatrix} \qquad (58)$$

$\Delta(\rho)$ is of course the determinant of the resistance parameters of the network and $\Delta(\sigma)$ is the determinant of the ~~capacity~~ elastance parameters of the network. These correspond to the matrices which contain the coefficients of the respective quadratic forms, as shown on page 42 , expressions (50). $\Delta_1(\rho,\sigma)$ is however the sum of two mixed determinants which contain both resistance and ~~capacity~~ elastance parameters. The rule of its formation is simple. Begin with the determinant $\Delta(\rho)$, and replace one column at a time the ρ terms of $\Delta(\rho)$ by σ terms. Add the two resulting mixed determinants and $\Delta_1(\rho,\sigma)$ is obtained. Thus

$$\Delta(\rho) = \begin{vmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{vmatrix}$$

Replacing the first column of the ρ terms by σ terms, we have

$$\begin{vmatrix} \sigma_{11} & \rho_{12} \\ \sigma_{12} & \rho_{22} \end{vmatrix} \qquad (59)$$

Now replace the second column of the ρ terms by σ terms

$$\begin{vmatrix} \rho_{11} & \sigma_{12} \\ \rho_{12} & \sigma_{22} \end{vmatrix} \qquad (60)$$

Adding (59) and (60) the right hand side of (58) is obtained.

The numeral 1 in $\Delta_1(\rho,\sigma)$ indicates that the replacement of the ρ terms by the σ terms in $\Delta(\rho)$ is made <u>one</u> column at a time. The ρ term is placed first in the parenthesis of $\Delta_1(\rho,\sigma)$ to indicate that we begin with the determinant $\Delta(\rho)$ and replace the ρ terms by the σ terms in $\Delta(\rho)$ one column at a time. Thus $\Delta_1(\sigma,\rho)$ would mean that we begin with the determinant $\Delta(\sigma)$ and replace the σ terms in $\Delta(\sigma)$ by the ρ terms one column at a time. From the above explanation it follows that

$$\Delta_1(\rho,\sigma) = \Delta_1(\sigma,\rho) \qquad (61)$$

With this symbolic notation (55) may be now written

$$D(p) = \frac{1}{p^2}\left[\Delta(\rho)p^2 + \Delta_1(\rho,\sigma)p + \Delta(\sigma)\right] \qquad (62)$$

The minor of the element in the first row and first column of $D(p)$ is obtained from (53) and is of course given by

$$M_{11}(p) = \rho_{22} + \frac{\sigma_{22}}{p}$$

$$= \frac{1}{p}\left(\rho_{22}p + \sigma_{22}\right) \qquad (63)$$

But ρ_{22} is the minor of the element in the first row and first column of $\Delta(\rho)$ and hence may be designated symbolically by $M_{11}(\rho)$ and σ_{22} is the minor of the element in the first row and first column of $\Delta(\sigma)$ and may be designated symbolically by $M_{11}(\sigma)$ In this terminology, (63) may be written

$$M_{11}(p) = \frac{1}{p}\left[M_{11}(\rho)p + M_{11}(\sigma)\right] \qquad (64)$$

Hence the impedance function, which is given by

$$Z(p) = \frac{D(p)}{M_{11}(p)} \qquad (22)$$

may be written in our symbolic notation

$$Z(p) = \frac{\frac{1}{p^2}\left[\Delta(\rho)p^2 + \Delta_1(\rho\sigma)p + \Delta(\sigma)\right]}{\frac{1}{p}\left[M_{11}(\rho)p + M_{11}(\sigma)\right]} \qquad (65)$$

Cancel~~l~~ing $\frac{1}{p}$ in the numerator and denominator we have

$$Z(p) = \frac{\Delta(\rho)p^2 + \Delta_1(\rho\sigma)p + \Delta(\sigma)}{p\left[M_{11}(\rho)p + M_{11}(\sigma)\right]} \qquad (66)$$

Thus the most general network of two meshes containing only resistance and capacity elements has an impedance function given by (66). Note that this formula (66) is exactly formula (5) page 13 where $a_2 = \Delta(\rho)$, $a_1 = \Delta_1(\rho\sigma)$ and $a_0 = \Delta(\sigma)$ and $b_1 = M_{11}(\rho)$ and $b_0 = M_{11}(\sigma)$.

Figure 15, page 44 gives this <u>most</u> general network, and it is noted that it contains six network elements, three resistances and three condensers. A study of the coefficients $\Delta(\rho)$ and $\Delta(\sigma)$ in (66) reveals at once what the <u>least</u> general network, that will still possess an impedance of the form (66) or (5) is. All that is necessary is to remove as many of the network elements as we please in the most general network shown in figure 15, but with the limitation that we preserve the form of the impedance function (66) or (5). Mathematically this means that we can make any changes we please in the network elements themselves, allowing them to take on all values in the real domain from zero to any positive quantity however great, but with the limitation that

$$\left.\begin{array}{l} \Delta(\rho) \neq 0 \\ \Delta(\sigma) \neq 0 \\ \Delta_1(\rho,\sigma) \neq 0 \\ M_{11}(\rho) \neq 0 \\ M_{11}(\sigma) \neq 0 \end{array}\right\} \qquad (67)$$

and that the numerator and denominator of (66) or (5) do not have a common factor, since this too would change the form of the impedance function. This latter condition means mathematically that the resultant (or eliminant) of the numerator and denominator of (66) or (5) be not zero.[10] The resultant of two equations of the second degree and first degree respectively

$$a_0 x^2 + a_1 x + a_2$$

$$b_1 x + b_2$$

10. For a good discussion of resultants (or eliminants) consult L. E. Dickson, Elementary Theory of Equations, 1914, p. 150. See also G. Salmon, Modern Higher Algebra, Fourth Edition, 1885, p. 76 and M. Bocher, Introduction to Higher Algebra, 1927, p. 196.

is the determinant[11]

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ b_1 & b_2 & 0 \\ 0 & b_1 & b_2 \end{vmatrix} \tag{68}$$

Hence the resultant of the numerator and denominator of the impedance function (66) is the determinant

$$\begin{vmatrix} \Delta(\rho) & \Delta_1(\rho,\sigma) & \Delta(\sigma) \\ M_{11}(\rho) & M_{11}(\sigma) & 0 \\ 0 & M_{11}(\rho) & M_{11}(\sigma) \end{vmatrix} \tag{69}$$

Evaluating (69) we have

$$\Delta(\rho)\, M_{11}^2(\sigma) - \Delta_1(\rho,\sigma)\, M_{11}(\rho)\, M_{11}(\sigma) + \Delta(\sigma)\, M_{11}^2(\rho) \tag{70}$$

The condition that Z(p) in (66) be not reducible, that is, that the numerator and denominator of (66) have no common factor is that the eliminant (70) be not zero. Since [illegible] by (67), this condition is that

$$\Delta(\rho)\, M_{11}^2(\sigma) - \Delta_1(\rho\sigma)\, M_{11}(\rho)\, M_{11}(\sigma) + \Delta(\sigma)\, M_{11}^2(\rho) \neq 0 \tag{71}$$

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11. See Dickson, loc.cit., p. 155.

Thus we may state that the necessary and sufficient conditions that an expression of the form (66) be in fact an impedance function of that form is that

$$\left.\begin{array}{l} \Delta(\rho)\neq 0,\ \ \Delta(\sigma)\neq 0,\ \ \Delta_1(\rho,\sigma)\neq 0\ ,\ M_{11}(\rho)\neq 0\ ,\ M_{11}(\sigma)\neq 0 \\ \\ \text{and the resultant} \\ \\ \Delta(\rho)\,M_{11}{}^2(\sigma)-\Delta_1(\rho,\sigma)\,M_{11}(\rho)\,M_{11}(\sigma)+\Delta(\sigma)\,M_{11}{}^2(\rho)\neq 0 \end{array}\right\} \qquad (72)$$

It is of course understood, as may be easily ascertained that $\Delta(\rho)$, $\Delta(\sigma)$, $\Delta_1(\rho,\sigma)$, $M_{11}(\rho)$, $M_{11}(\sigma)$ are all positive. This follows from the fact that the <u>total</u> parameters are greater than the <u>mutual</u> parameters.

Let us then proceed to remove as many network elements as we can in the network shown in figure 15, without violating the conditions (72). It will be seen that the maximum number of network elements that may thus be removed is two. To remove a network element we simply make its value zero. Thus, for example, we may remove the elements in branch a b, figure 15, by making

$$\left.\begin{array}{l} \rho_{11}-\rho_{12}=0 \\ \sigma_{11}-\sigma_{12}=0 \end{array}\right\} \qquad (73)$$

By this change, the network shown in figure 15 becomes that shown in figure 16.

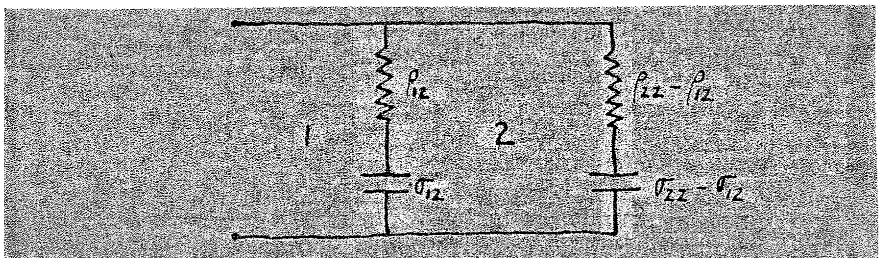


FIG. 16

It is a simple matter to verify the fact that conditions (72) are not violated by this transformation of the network.

Now let us remove the mutual elements ρ_{12} and σ_{12} in branch bc (fig.15). Physically, this really means that we are short-circuiting the network between the points b and c, and it is to be expected, merely from physical considerations, that the form of the impedance function (66) will not be preserved. This follows because it is readily seen that short-circuiting the network between the points b and c makes the resulting impedance

$$Z(p) = \rho_{11} + \frac{\sigma_{11}}{p} \tag{74}$$

Let us however see what Z(p) in (66) does become when in the network figure 15 we remove the mutual elements, that is we make

$$\left.\begin{aligned} \rho_{12} &= 0 \\ \sigma_{12} &= 0 \end{aligned}\right\} \tag{75}$$

Let us calculate the coefficients of (66)

$$\Delta(\rho) = \begin{vmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{vmatrix} = \begin{vmatrix} \rho_{11} & 0 \\ 0 & \rho_{22} \end{vmatrix} = \rho_{11}\rho_{22}$$

$$\Delta(\sigma) = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} = \begin{vmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{vmatrix} = \sigma_{11}\sigma_{22}$$

$$\Delta_1(\rho,\sigma) = \begin{vmatrix} \rho_{11} & \sigma_{12} \\ \rho_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \rho_{12} \\ \sigma_{12} & \rho_{22} \end{vmatrix}$$

$$= \begin{vmatrix} \rho_{11} & 0 \\ 0 & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{vmatrix}$$

$$= \rho_{11}\sigma_{22} + \sigma_{11}\rho_{22}$$

$$M_{11}(\rho) = \rho_{22}$$

$$M_{11}(\sigma) = \sigma_{22}$$

Thus the impedance function (66) becomes under the conditions (75), that is, by removing the mutual elements

$$Z(p) = \frac{\rho_{11}\rho_{22}p^2 + (\rho_{11}\sigma_{22} + \sigma_{11}\rho_{22})p + \sigma_{11}\sigma_{22}}{p(\rho_{22}p + \sigma_{22})} \qquad (76)$$

At a first glance it appears that the form of the impedance function (66) is preserved, but note that the numerator of (76) can be factored. Thus

$$Z(p) = \frac{\rho_{11} p(\rho_{22} p + \sigma_{22}) + \sigma_{11}(\rho_{22} p + \sigma_{22})}{p(\rho_{22} p + \sigma_{22})}$$

$$= \frac{(\rho_{11} p + \sigma_{11})(\rho_{22} p + \sigma_{22})}{p(\rho_{22} p + \sigma_{22})}.$$

Cancelling the common factor $\rho_{22} p + \sigma_{22}$ in the numerator and denominator, we have

$$Z(p) = \frac{\rho_{11} p + \sigma_{11}}{p}$$

$$= \rho_{11} + \frac{\sigma_{11}}{p} \qquad (77)$$

But this is precisely (74) which we predicted merely from physical considerations.

Since by making $\rho_{12} = \sigma_{12} = 0$ the numerator and denominator of the general impedance Z(p) in (66) have a common factor, namely $\rho_{22} p + \sigma_{22}$, it is to be expected that the resultant (71) must vanish, in other words one of the conditions for the preservation of the form of Z(p) in (66) has been violated. Let us see if this is so. The resultant is

$$\Delta(\rho) M_{11}{}^2(\sigma) - \Delta_1(\rho,\sigma) M_{11}(\rho) M_{11}(\sigma) + \Delta(\sigma) M_{11}{}^2(\rho) \qquad (78)$$

Substituting the values of $\Delta(\rho), \Delta(\sigma), M_{11}(\rho), M_{11}(\sigma), \Delta_1(\rho,\sigma)$ given on page 54 in (78), we have

$$\rho_{11}\rho_{22}\sigma_{22}^2 - \rho_{22}\sigma_{22}(\rho_{11}\sigma_{22}+\sigma_{11}\rho_{22}) + \sigma_{11}\sigma_{22}\rho_{22}^2$$

$$= \rho_{11}\rho_{22}\sigma_{22}^2 - \rho_{11}\rho_{22}\sigma_{22}^2 - \sigma_{11}\sigma_{22}\rho_{22}^2 + \sigma_{11}\sigma_{22}\rho_{22}^2$$

$$= 0$$

Thus, removing the mutual elements ρ_{12} and σ_{12} violates one of our conditions for the preservation of the form of the impedance function, namely that the eliminant be not zero.

Now let us remove the network elements in the branch bd, figure 15, by making $(\rho_{22}-\rho_{12})$ and $(\sigma_{22}-\sigma_{12})$ both zero. Physically, this is again short-circuiting the network between the points b and c of figure 15, and it is to be expected that, as in the previous case, the form of the impedance function (66) will not be preserved. In fact, the impedance will be

$$Z(p) = (\rho_{11}-\rho_{12}) + \frac{\sigma_{11}-\sigma_{12}}{p} \tag{79}$$

Let us then for this case, namely

$$\left.\begin{aligned} \rho_{22}-\rho_{12} &= 0 \\ \sigma_{22}-\sigma_{12} &= 0 \end{aligned}\right\} \tag{80}$$

calculate the impedance function (66). As before first calculate the coefficients of (66) for this case. Thus

$$\Delta(\rho) = \begin{vmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{vmatrix} = \begin{vmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{12} \end{vmatrix}$$

$$= \rho_{11}\rho_{12} - \rho_{12}^2$$

$$\Delta(\sigma) = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{12} \end{vmatrix}$$

$$= \sigma_{11}\sigma_{12} - \sigma_{12}^2$$

$$\Delta_1(\rho,\sigma) = \begin{vmatrix} \rho_{11} & \sigma_{12} \\ \rho_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \rho_{12} \\ \sigma_{12} & \rho_{22} \end{vmatrix}$$

$$= \begin{vmatrix} \rho_{11} & \sigma_{12} \\ \rho_{12} & \sigma_{12} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \rho_{12} \\ \sigma_{12} & \rho_{12} \end{vmatrix}$$

$$= \rho_{11}\sigma_{12} - \rho_{12}\sigma_{12} + \sigma_{11}\rho_{12} - \sigma_{12}\rho_{12}$$

$$M_{11}(\rho) = \rho_{12}$$

$$M_{11}(\sigma) = \sigma_{12}$$

Substituting these values of the coefficients $\Delta(\rho)$, $\Delta(\sigma)$, $\Delta_1(\rho,\sigma)$ $M_{11}(\rho)$, $M_{11}(\sigma)$ in the impedance function (66), we have

$$Z(p) = \frac{(\rho_{11}\rho_{12}-\rho_{12}^2)p^2+(\rho_{11}\sigma_{12}-\rho_{12}\sigma_{12}+\sigma_{11}\rho_{12}-\sigma_{12}\rho_{12})p+\sigma_{11}\sigma_{12}-\sigma_{12}^2}{p(\rho_{12}p+\sigma_{12})}$$

$$= \frac{\rho_{12}p(\rho_{11}-\rho_{12})p+\sigma_{12}(\rho_{11}-\rho_{12})p+\rho_{12}(\sigma_{11}-\sigma_{12})p+\sigma_{12}(\sigma_{11}-\sigma_{12})}{p(\rho_{12}p+\sigma_{12})}$$

$$= \frac{(\rho_{12}p+\sigma_{12})(\rho_{11}-\rho_{12})p+(\rho_{12}p+\sigma_{12})(\sigma_{11}-\sigma_{12})}{p(\rho_{12}p+\sigma_{12})}$$

$$= \frac{(\rho_{12}p+\sigma_{12})\left[(\rho_{11}-\rho_{12})p+(\sigma_{11}-\sigma_{12})\right]}{p(\rho_{12}p+\sigma_{12})}$$

Cancelling the factor $\rho_{12}p+\sigma_{12}$, common to both numerator and denominator, we have

$$Z(p) = \frac{(\rho_{11}-\rho_{12})p+(\sigma_{11}-\sigma_{12})}{p}$$

$$= (\rho_{11}-\rho_{12}) + \frac{\sigma_{11}-\sigma_{12}}{p}$$

But this is exactly (79), which we saw must be the value of Z(p) if the terminals b and c are short-circuited.

Again, as before, since by making

$$\rho_{22} - \rho_{12} = 0$$

$$\sigma_{22} - \sigma_{12} = 0$$

the impedance function Z(p), (66), has a common factor, namely $\rho_{12} p + \sigma_{12}$, in its numerator and denominator, it is to be expected that the resultant will vanish. Substituting the values obtained above for $\Delta(\rho)$, $\Delta(\sigma)$, $\Delta_1(\rho\sigma)$, $M_{11}(\rho)$, $M_{11}(\sigma)$ in the eliminant (78), we have

$$(\rho_{11}\rho_{12} - \rho_{12}^2)\sigma_{12}^2 - \rho_{12}\sigma_{12}(\rho_{11}\sigma_{12} - \rho_{12}\sigma_{12} + \sigma_{11}\rho_{12} - \sigma_{12}\rho_{12}) + (\sigma_{11}\sigma_{12} - \sigma_{12}^2)\rho_{12}^2$$

$$= \rho_{11}\rho_{12}\sigma_{12}^2 - \rho_{12}^2\sigma_{12}^2 - \rho_{11}\rho_{12}\sigma_{12}^2 + \rho_{12}^2\sigma_{12}^2 - \sigma_{11}\sigma_{12}\rho_{12}^2 + \sigma_{12}^2\rho_{12}^2 + \sigma_{11}\sigma_{12}\rho_{12}^2 - \sigma_{12}^2\rho_{12}^2$$

$$= 0$$

Thus, as was to be expected, the resultant is zero.

Now let us remove the resistance in branch ab and the condenser in branch bc of the general network shown in figure 15. This means that

$$\left.\begin{array}{l} \rho_{11} - \rho_{12} = 0 \\ \sigma_{12} = 0 \end{array}\right\} \qquad (81)$$

and the network thus obtained is shown in figure 17.

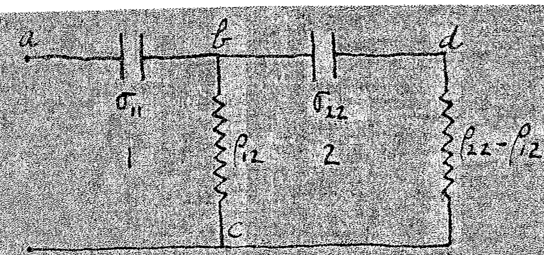


FIG. 17

It is a simple matter to verify the fact that conditions (72) are not violated by this transformation of the network.

Now let us remove the condenser in branch ab and the resistance in branch bc of the general network, figure 15. This means that

$$\left.\begin{aligned} \sigma_{11} - \sigma_{12} &= 0 \\ \rho_{12} &= 0 \end{aligned}\right\} \qquad (82)$$

and the network thus obtained is shown in figure 18

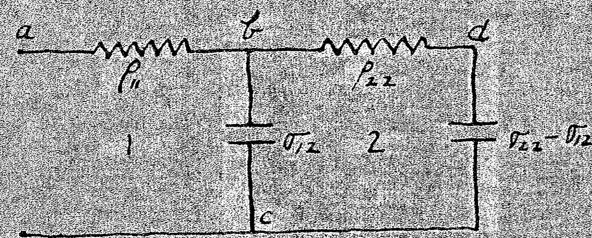


FIG. 18

Again, conditions (72) are not violated by this change.

Now let us remove the condenser in branch bc and the resistance in branch bd in the general network figure 15. This means that

$$\left.\begin{aligned} \rho_{12} &= 0 \\ \sigma_{22} - \sigma_{12} &= 0 \end{aligned}\right\} \qquad (83)$$

and the network thus obtained is shown in figure 19.

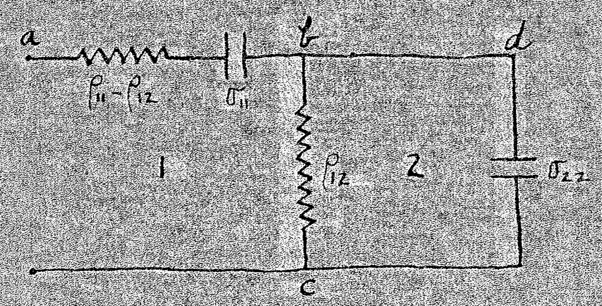


FIG. 19

Again it can be verified that conditions (72) are not violated by this change. It is readily seen also that making

$$\left.\begin{aligned} \sigma_{12} &= 0 \\ \rho_{22} - \rho_{12} &= 0 \end{aligned}\right\} \qquad (84)$$

will give exactly the same network as that shown in figure 19.

Thus we see that of all the possible ways of reducing the number of elements in the general network shown in figure 15, and yet preserving the form of the impedance function, that is, satisfying conditions (72), the least number of elements remaining in the network is four. Furthermore, there are only four different networks, namely those shown in figures 16, 17, 18, 19, that are made up of the least number of elements. These networks we shall call the minimal forms. Note that these networks are exactly those obtained by Foster and Cauer by means of partial fraction and continued fraction expansion, and called by Cauer the two-mesh canonical forms. It is hardly necessary to point out that neither Foster nor Cauer showed why there should exist but four of these networks containing the least number of elements, and preserving the form of the impedance function; and why the least number of elements should be but four.

Considerable work has been done in mathematics on the subject of resultants and the resultant of two polynomials of any degree has been expressed in determinant notation. Thus, for example, the eliminant of any two polynomials

$$a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \cdots + a_m$$

$$b_0 x^n + b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_n$$

is given by the determinant[12]

12. See L. E. Dickson, loc. cit., p. 154, where a simple proof is given by Sylvester's Dialytic Method of Elimination; and M. Bocher, loc. cit., p. 196.

$$\left|\begin{array}{cccccccc}
a_0 & a_1 & a_2 & \cdots & a_m & 0 & \cdots & 0 \\
0 & a_0 & a_1 & a_2 & \cdots & a_m & \cdots & 0 \\
0 & 0 & a_0 & a_1 & a_2 & \cdots & a_m & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & a_0 & a_1 & a_2 & \cdots & a_m \\
b_0 & b_1 & \cdots & \cdots & b_n & 0 & \cdots & 0 \\
0 & b_0 & b_1 & \cdots & \cdots & b_n & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & b_0 & b_1 & \cdots & \cdots & b_n
\end{array}\right|
\begin{array}{l}
\left.\vphantom{\begin{array}{c}a\\a\\a\\a\\a\end{array}}\right\} n \text{ rows} \\
\left.\vphantom{\begin{array}{c}b\\b\\b\\b\end{array}}\right\} m \text{ rows}
\end{array}
\qquad (86)$$

Thus, by the use of the resultant (86) we can extend our two-mesh theory with two kinds of network elements to the two-mesh theory with all three network elements present, and to the n-mesh theory with all three network elements present. By this means it is a simple matter to state the condition that an impedance function preserve its form with a change of network elements, and to find the least number of elements necessary in a network to preserve the form of its impedance function. It will be shown later that by extending our symbolic notation to the n-mesh theory, it will be possible to obtain by means of it the coefficients of the impedance function of any number of meshes. Hence it will be possible to write the resultant (86) for any impedance function in terms of our symbolic notation, thus obtaining the conditions for the preservation of the form of the impedance function in terms of the network elements themselves.

Thus far we have dealt entirely with the two-mesh network containing only two kinds of network elements, namely resistance and capacity elements. It will now be shown that everything given above concerning two-mesh networks with resistance and capacity elements holds just as well for two-mesh networks containing inductance and resistance elements, and for two-mesh networks containing inductance and capacity elements.

Let us proceed therefore to obtain the impedance function of a two-mesh network containing inductance and resistance elements. Such a network is shown in figure 20.

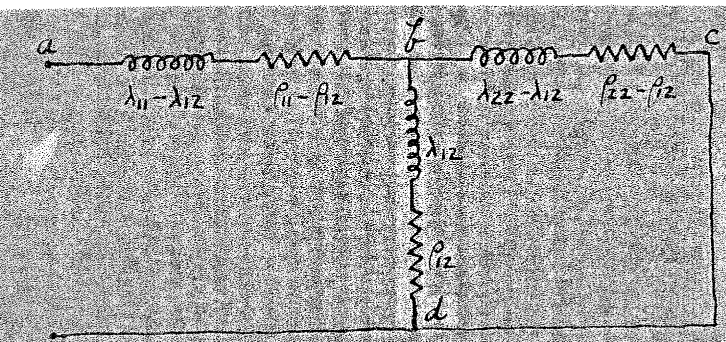


FIG. 20

The determinant D(p) of this network is

$$D(p) = \begin{vmatrix} \lambda_{11}p+\rho_{11} & \lambda_{12}p+\rho_{12} \\ \lambda_{12}p+\rho_{12} & \lambda_{22}p+\rho_{22} \end{vmatrix} \qquad (87)$$

Note however that this determinant is exactly the same as the determinant in (54a) for the two-mesh network containing resistance and capacity elements with the following modifications. There is no factor $\frac{1}{p^2}$ in (87), which appears in (54a), and the λ terms in (87) replace the ρ terms in (54a) and the ρ terms in (87) replace the σ terms in (54a). Hence, without going through the expansion of D(p) in (87) as we did for D(p) in (54a) we may obtain the polynomial of the network for this case by writing (55) with the following modifications. Omit the factor $\frac{1}{p^2}$ appearing in (55) and replace the ρ terms in (55) by λ terms and the σ terms in (55) by ρ terms. Thus we obtain for the polynomial of the network shown in figure 20 the following expression

$$D(p) = \begin{vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{vmatrix} p^2 + \left\{ \begin{vmatrix} \lambda_{11} & \rho_{12} \\ \lambda_{12} & \rho_{22} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} \\ \rho_{12} & \lambda_{22} \end{vmatrix} \right\} p + \begin{vmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{vmatrix} \qquad (88)$$

This becomes, using our symbolic notation as explained for the two-mesh network containing resistance and capacity elements pages 46 and 47

$$D(p) = \Delta(\lambda) p^2 + \Delta_1(\lambda, \rho) p + \Delta(\sigma) \qquad (89)$$

The minor of the element in the first row and first column of D(p), (87) is

$$M_{11}(p) = \lambda_{22} p + \rho_{22} \qquad (90)$$

which in our notation may be written

$$M_{11}(p) = M_{11}(\lambda)p + M_{11}(\rho) \tag{91}$$

Hence the impedance function for this case in our symbolic notation is

$$\begin{aligned} Z(p) &= \frac{D(p)}{M_{11}(p)} \\ &= \frac{\Delta(\lambda)p^2 + \Delta_1(\lambda,\rho)p + \Delta(\rho)}{M_{11}(\lambda)p + M_{11}(\rho)} \end{aligned} \tag{92}$$

Thus the most general network of two-meshes containing only inductance and resistance elements has an impedance given by (92). Note that this formula (92) is exactly formula (4) page 13, where $a_2 = \Delta(\lambda)$, $a_1 = \Delta_1(\lambda,\rho)$ and $a_0 = \Delta(\rho)$; and $b_1 = M_{11}(\lambda)$ and $b_0 = M_{11}(\rho)$

Proceeding as we did in the two-mesh network containing resistance and capacity elements we obtain of course the same conditions for the preservation of the form of our impedance function (92). These conditions are of course those in (72) with the following modifications; the ρ terms in (72) are replaced by λ terms and the σ terms in (72) are replaced by ρ terms. These conditions become then for this case

$$\left.\begin{aligned} &\Delta(\lambda) \neq 0, \quad \Delta(\rho) \neq 0, \quad \Delta_1(\lambda,\rho) \neq 0, \quad M_{11}(\lambda) \neq 0, \quad M_{11}(\rho) \neq 0 \\ &\text{and the eliminant} \\ &\Delta(\lambda)\, M_{11}^2(\rho) - \Delta_1(\lambda,\rho)\, M_{11}(\lambda)\, M_{11}(\rho) + \Delta(\rho)\, M_{11}^2(\lambda) \neq 0 \end{aligned}\right\} \tag{93}$$

In the same way as in the two-mesh network containing resistance and capacity elements, we find that by proceeding to remove as many elements as we can from the general network shown in figure 20 without violating conditions (93) for the preservation of the form of the impedance function, we finally arrive, as before, at just four different networks containing the least number of elements, namely four. These are of course the minimal forms for the two-mesh network containing inductance and resistance elements. These networks are the same as those shown in figures 16, 17, 18 and 19, with the following modifications. The resistance in figures 16-19 are replaced by inductances and the condensers in figures 16-20 are replaced by resistances. Figure 21 shows these four networks.

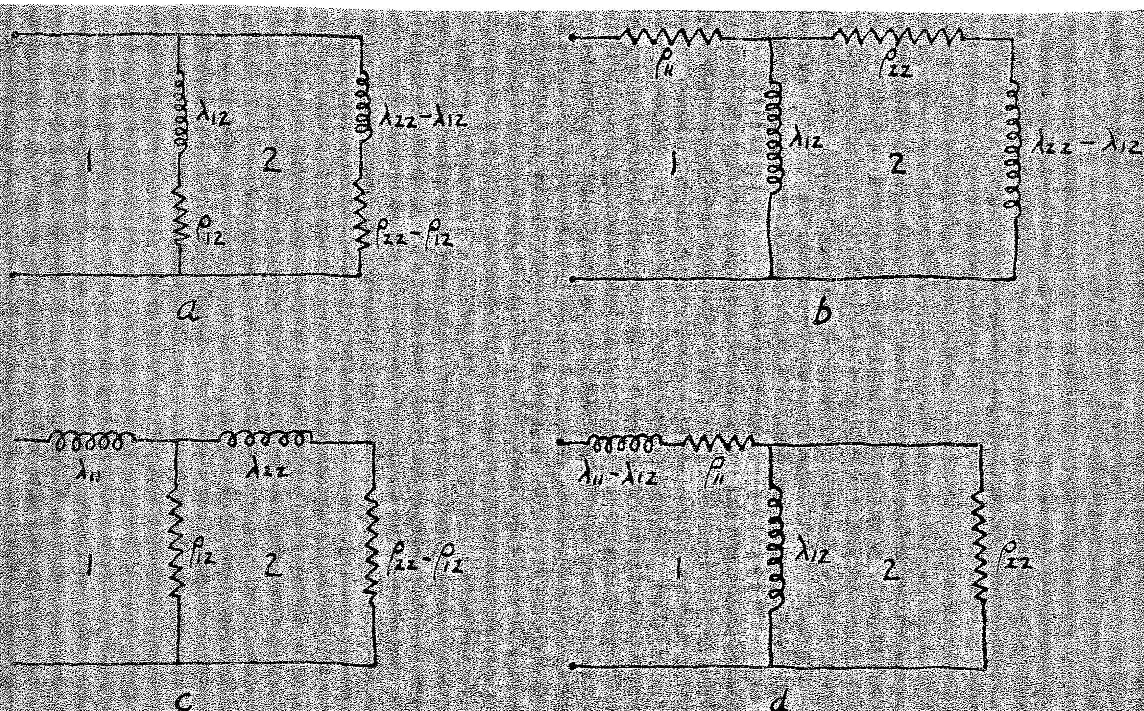


FIG. 21

Finally, let us proceed to obtain the impedance function of a two-mesh network containing inductance and capacity elements. Such a network is shown in figure 22.

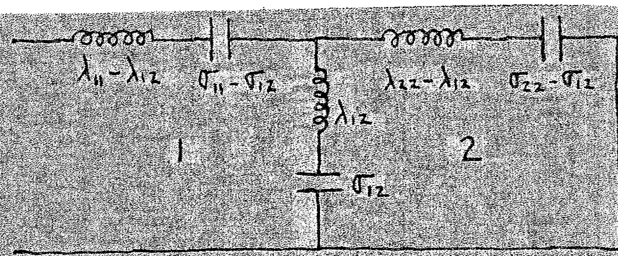


FIG. 22

The determinant D(p) of this network is

$$D(p) = \begin{vmatrix} \lambda_{11}p + \dfrac{\sigma_{11}}{p} & \lambda_{12}p + \dfrac{\sigma_{12}}{p} \\ \lambda_{12}p + \dfrac{\sigma_{12}}{p} & \lambda_{22}p + \dfrac{\sigma_{22}}{p} \end{vmatrix} \qquad (94)$$

which may be written

$$D(p) = \begin{vmatrix} \dfrac{\lambda_{11}p^2 + \sigma_{11}}{p} & \dfrac{\lambda_{12}p^2 + \sigma_{12}}{p} \\ \dfrac{\lambda_{12}p^2 + \sigma_{12}}{p} & \dfrac{\lambda_{22}p^2 + \sigma_{22}}{p} \end{vmatrix}$$

$$= \frac{1}{p^2} \begin{vmatrix} \lambda_{11}p^2 + \sigma_{11} & \lambda_{12}p^2 + \sigma_{12} \\ \lambda_{12}p^2 + \sigma_{12} & \lambda_{22}p^2 + \sigma_{22} \end{vmatrix} \qquad (95)$$

Note however that (95) is exactly the same as D(p) in (54a) for the two-mesh network containing resistance and capacity elements, with the following modifications. Inside the determinant, p in the determinant (54a) is replaced by p^2 in (95) and the ρ terms in (54a) are replaced by λ terms in (95). Hence we can write at once the polynomial of the network by writing (55) with the following modifications. Replace p by p^2 inside the brackets of (55) and the ρ terms by λ terms. Thus we obtain for the polynomial of the network shown in figure 22 the following expression

$$D(p) = \frac{1}{p^2}\left[\begin{vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{vmatrix} p^4 + \left\{\begin{vmatrix} \lambda_{11} & \sigma_{12} \\ \lambda_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \lambda_{12} \\ \sigma_{12} & \lambda_{22} \end{vmatrix}\right\} p^2 + \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix}\right] \qquad (96)$$

This becomes, using our symbolic notation

$$D(p) = \frac{1}{p^2}\left[\Delta(\lambda)p^4 + \Delta_1(\lambda,\sigma)p^2 + \Delta(\sigma)\right] \qquad (97)$$

The minor of the element in the first row and first column of D(p), (94) is

$$M_{11}(p) = \lambda_{22}p + \frac{\sigma_{22}}{p}$$

$$= \frac{1}{p}\left(\lambda_{22}p^2 + \sigma_{22}\right) \qquad (98)$$

This, in our notation may be written

$$M_{11}(p) = \frac{1}{p}\left[M_{11}(\lambda)p^2 + M_{11}(\sigma)\right] \tag{99}$$

Hence the impedance function is

$$Z(p) = \frac{D(p)}{M_{11}(p)}$$

$$= \frac{\Delta(\lambda)p^4 + \Delta_1(\lambda,\sigma)p^2 + \Delta(\sigma)}{p\left[M_{11}(\lambda)p^2 + M_{11}(\sigma)\right]} \tag{100}$$

Thus the most general two-mesh network containing only inductance and capacity elements, has an impedance function given by (100). Note that formula (100) is exactly formula (6) page 14, where $a_4 = \Delta(\lambda)$, $a_2 = \Delta(\lambda,\sigma)$ and $a_0 = \Delta(\sigma)$; $b_2 = M_{11}(\lambda)$ and $b_0 = M_{11}(\lambda)$.

In the same way as in the previous two cases we arrive at the conditions for the preservation of the form of our impedance function, which for this case are

$$\left.\begin{array}{l} \Delta(\lambda) \neq 0, \quad \Delta_1(\lambda,\sigma) \neq 0, \quad \Delta(\sigma) \neq 0, \quad M_{11}(\lambda) \neq 0, \quad M_{11}(\sigma) \neq 0 \\ \\ \text{and the eliminant} \\ \\ \Delta(\lambda)\, M_{11}^2(\sigma) - \Delta_1(\lambda,\sigma)\, M_{11}(\lambda)\, M_{11}(\sigma) + \Delta(\sigma)\, M_{11}^2(\lambda) \neq 0 \end{array}\right\} \tag{101}$$

Proceeding as in the previous two cases, to remove as many elements from the general network shown in figure 22 as we can, without violating the conditions for the preservation of the form of our impedance function (100), we arrive at the four different networks containing the least number of elements, shown in figure 23

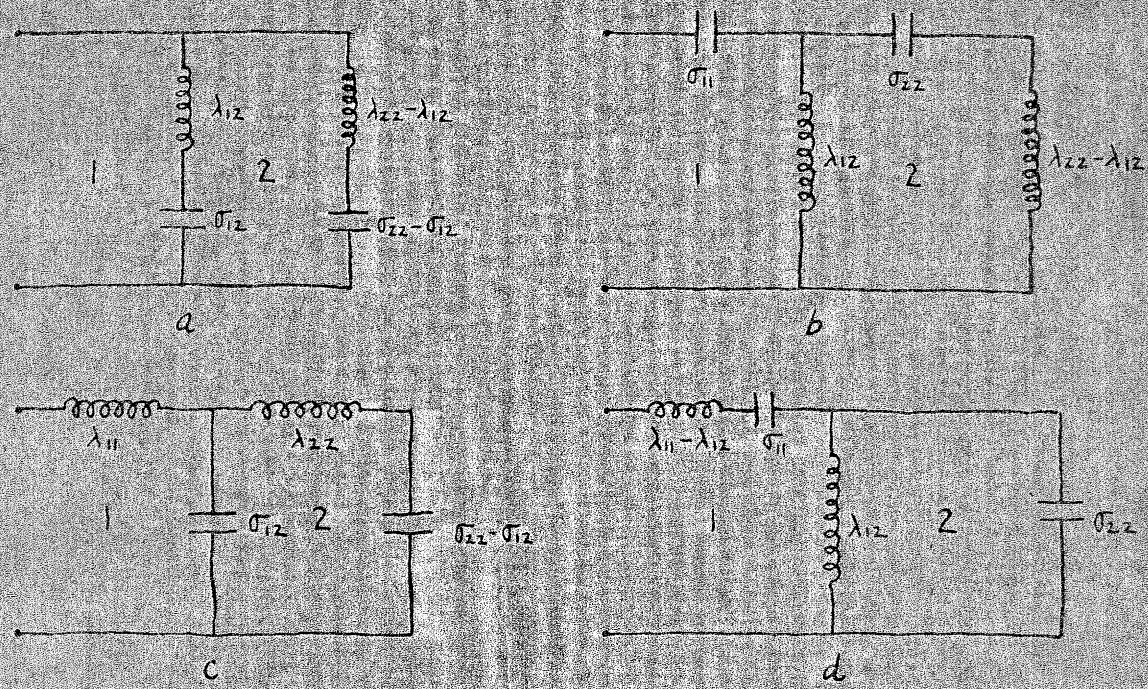


FIG. 23

It is useful at this point to bring together the three different formulas for the impedance functions of the two-mesh networks containing respectively resistance and capacity elements, inductance and resistance elements, and inductance and capacity elements. These are the formulas (66) page 49, (92) page 66 and (100) page 70. Let us tabulate these formulas for reference, since we will have occasion to use them often.

Impedance Function for Two-Mesh Networks with Two Kinds of Elements.

Network Elements	Impedance Function
Resistance and capacity	$Z(p) = \dfrac{\Delta(\rho)p^2 + \Delta_1(\rho,\sigma)p + \Delta(\sigma)}{p[M_{11}(\rho)p + M_{11}(\sigma)]}$ (102a)
Inductance and resistance	$Z(p) = \dfrac{\Delta(\lambda)p^2 + \Delta_1(\lambda,\rho)p + \Delta(\rho)}{M_{11}(\lambda)p + M_{11}(\rho)}$ (102b)
Inductance and capacity	$Z(p) = \dfrac{\Delta(\lambda)p^4 + \Delta_1(\lambda,\sigma)p^2 + \Delta(\sigma)}{p[M_{11}(\lambda)p^2 + M_{11}(\sigma)]}$ (102c)

These formulas (102) are in themselves quite important, since they give us the coefficients of the impedance function in terms of the actual network elements themselves. The nature of these coefficients will tell us for example whether an expression of the form (102) is in fact an impedance function. Many expressions of the form (102) are not impedance functions. Thus, take at random the expression

$$\frac{p^2 + 3p + 2}{p(4p+5)} \tag{103}$$

which is an expression of the form (102a). It is not difficult to show that (103) is not the impedance function of a physical network containing positive resistance and capacity elements. Here it should be pointed out that both Foster and Cauer use the expression "Conditions for the physical realizability of an impedance function", which expression I believe causes confusion. If a function is an impedance function, it <u>is</u> physically realizable, since an impedance function <u>is</u> the impedance of an actual physical network. Thus for example they would talk of the physical realizability of an impedance function (103), and finally say it is not physically realizable, which is of course confusing, since (103) is not and never was an impedance function, and it is meaningless to talk of its "physical realizability". Hence I have purposely avoided the use of this expression. It should also be pointed out that neither Foster nor Cauer realized that the coefficients of the impedance function could be expressed directly in terms of the network elements in the manner that I have shown, and so remained in the dark as to the actual nature of these coefficients.

Furthermore, expressions (102) are very useful in providing a short cut in the actual calculation of an impedance, as anyone who has computed impedances can verify. These formulas are easily extended to networks of any number of meshes, as we shall show later, the formulas saving considerable labor in the actual computation of any network as the complexity or the number of meshes increases.

CHAPTER III.

The Impedance Function and its Infinite Set of Networks.

It will be useful at this point to go thoroughly into the question of the nature of the coefficients, the zeros and the poles of the impedance function. We will limit the investigation for the present to impedance functions containing two kinds of elements only, resistance and capacity elements, inductance and resistance elements and inductance and capacity elements. In these cases, as we shall see and as was mentioned in the introduction, the zeros and poles are negative reals in the first two cases, and pairs of negative and positive pure imaginaries in the last case. It was also mentioned that these zeros and poles had the separation property. Questions such as these as well as what the nature of the coefficients of expressions of the form of impedance functions must be in order that these expressions be in fact impedance functions, will now be studied.

To fix ideas, consider the impedance function of the two mesh network containing inductance and resistance elements, the general network for which is shown in figure 20, page 64 The impedance of this network is of course given by

$$Z(p) = \frac{\Delta(\lambda) p^2 + \Delta_1(\lambda, \rho) p + \Delta(\rho)}{M_{11}(\lambda) p + M_{11}(\rho)} \qquad (102b)$$

where

$$\Delta(\lambda) = \begin{vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{vmatrix} \qquad \Delta(\rho) = \begin{vmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{vmatrix}$$

$$\Delta_1(\lambda,\rho) = \begin{vmatrix} \lambda_{11} & \rho_{12} \\ \lambda_{12} & \rho_{22} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} \\ \rho_{12} & \lambda_{22} \end{vmatrix}$$

$$M_{11}(\lambda) = \lambda_{22}$$

$$M_{11}(\rho) = \rho_{22}$$

Note that the six quantities $\lambda_{11}, \lambda_{12}, \lambda_{22}, \rho_{11}, \rho_{12}, \rho_{22}$ determine the nature of the coefficients, since the coefficients are functions of these quantities. Note also that all of these quantities are contained in the two matrices

$$\begin{Vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{Vmatrix} \qquad \begin{Vmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{Vmatrix}$$

the determinants of which are respectively $\Delta(\lambda)$ and $\Delta(\rho)$. These matrices are exactly the matrices which contain the coefficients of the respective inductance and resistance quadratic forms as shown on page 41. The determinant of the inductance matrix gives the first coefficient of the numerator of the impedance function (102a) and the determinant of the resistance matrix gives the last coefficient of the numerator of (102a), and by certain rules, the remaining coefficients of

(102a) are obtained from these matrices.

From physical considerations we know first that the six quantities $\lambda_{11}, \lambda_{12}, \lambda_{22}; \rho_{11}, \rho_{12}, \rho_{22}$ are all positive real numbers, since resistance and inductance are positive quantities. Hence it follows that the coefficients of the denominator of (102b) must be both positive. Also, as we pointed out before, the total parameters $\lambda_{11}, \lambda_{22}; \rho_{11}, \rho_{22}$ are greater or equal to the corresponding mutual parameters λ_{12} and ρ_{12} so that

$$\left.\begin{array}{ll} \lambda_{12} \leqq \lambda_{11} & \lambda_{12} \leqq \lambda_{22} \\ \rho_{12} \leqq \rho_{11} & \rho_{12} \leqq \rho_{22} \end{array}\right\} \qquad (104)$$

From (104), and the fact that $\Delta(\lambda) \neq 0$, $\Delta(\rho) \neq 0$

$$\left.\begin{array}{l} \Delta(\lambda) = \lambda_{11}\lambda_{22} - \lambda_{12}^2 > 0 \\ \Delta(\rho) = \rho_{11}\rho_{22} - \rho_{12}^2 > 0 \end{array}\right\} \qquad (105)$$

Thus the first and last coefficient of the numerator of (102a) must be positive. Finally

$$\Delta_1(\lambda,\rho) = (\lambda_{11}\rho_{22} - \lambda_{12}\rho_{12}) + (\rho_{11}\lambda_{22} - \rho_{12}\lambda_{12}) > 0 \qquad (106)$$

This follows from (104) so that $\lambda_{11}\rho_{22} \geqq \lambda_{12}\rho_{12}$ and $\rho_{11}\lambda_{22} \geqq \rho_{12}\lambda_{12}$ and also from the fact that $\Delta_1(\lambda,\rho) \neq 0$. Hence all the coefficients of the impedance function (102a) are real and positive.

Now let us ask the following question: From the complete domain of positive real numbers, what values may be assigned to the parameters $\lambda_{11}, \lambda_{12}, \lambda_{22}; \rho_{11}, \rho_{12}, \rho_{22}$ in order that these become the parameters of a physical network having (102b) as its impedance function? Suppose (102b) be written as

$$\frac{a_0 p^2 + a_1 p + a_2}{b_1 p + b_2} \qquad (107)$$

All we know now about the coefficients is that they are all positive. Is (107) the impedance function of some finite network containing inductance and resistance elements? If not, what other conditions must the coefficients satisfy besides their being positive reals, in order that (107) represent a network containing inductance and resistance elements.

Let us proceed to determine a set of values $\lambda_{11}, \lambda_{12}, \lambda_{22}; \rho_{11}, \rho_{12}, \rho_{22}$ satisfying (107). Comparing (107) with (102b), which represents the impedance of the most general network, we must have

$$\left.\begin{aligned} a_0 &= \Delta(\lambda) \\ a_1 &= \Delta_1(\lambda,\rho) \\ a_2 &= \Delta(\rho) \\ b_1 &= M_{11}(\lambda) \\ b_2 &= M_{11}(\rho) \end{aligned}\right\} \qquad (108)$$

Hence the parameters $\lambda_{11}, \lambda_{12}, \lambda_{22}; \rho_{11}, \rho_{12}, \rho_{22}$ must satisfy the following equations

$$\lambda_{11}\lambda_{22} - \lambda_{12}^2 = a_0 \qquad (109a)$$

$$\lambda_{11}\rho_{22} + \lambda_{22}\rho_{11} - 2\lambda_{12}\rho_{12} = a_1 \qquad (109b)$$

$$\rho_{11}\rho_{22} - \rho_{12}^2 = a_2 \qquad (109c)$$

$$\lambda_{22} = b_1 \qquad (109d)$$

$$\rho_{22} = b_2 \qquad (109e)$$

Let us proceed to obtain values of $\lambda_{11}, \lambda_{12}, \lambda_{22}; \rho_{11}, \rho_{12}, \rho_{22}$ satisfying these equations. Substitute (109d) and (109e) in (109a) and (109c).

$$b_1\lambda_{11} - \lambda_{12}^2 = a_0$$

$$b_2\rho_{11} - \rho_{12}^2 = a_2$$

Hence:

$$\lambda_{11} = \frac{a_0 + \lambda_{12}^2}{b_1}$$

$$\rho_{11} = \frac{a_2 + \rho_{12}^2}{b_2}$$

Substituting these values as well as (109d) and (109e) in (109b) we have

$$b_2\left(\frac{a_0 + \lambda_{12}^2}{b_1}\right) + b_1\left(\frac{a_2 + \rho_{12}^2}{b_2}\right) - 2\lambda_{12}\rho_{12} = a_1$$

Clearing fractions, we have

$$a_0 b_2^2 + b_2^2 \lambda_{12}^2 + a_2 b_1^2 + b_1^2 \rho_{12}^2 - 2\lambda_{12}\rho_{12} b_1 b_2 = a_1 b_1 b_2$$

$$\therefore \quad b_2^2 \lambda_{12}^2 - 2\lambda_{12}\rho_{12} b_1 b_2 + b_1^2 \rho_{12}^2 = a_1 b_1 b_2 - a_0 b_2^2 - a_2 b_1^2$$

And

$$(b_2 \lambda_{12} - b_1 \rho_{12})^2 = a_1 b_1 b_2 - a_0 b_2^2 - a_2 b_1^2 \qquad (110)$$

Note that $b_2 \lambda_{12} - b_1 \rho_{12}$ is a real number, so that $(b_2 \lambda_{12} - b_1 \rho_{12})^2$ is positive. Hence $a_1 b_1 b_2 - a_0 b_2^2 - a_2 b_1^2$ must be positive. Thus we have answered our first question. Not all expressions like (107), with coefficients which are positive reals are impedance functions. Expression (107) may represent an impedance function of some network if its coefficients a_0, a_1, a_2, b_1, b_2 satisfy the condition, besides their positiveness, that

$$a_1 b_1 b_2 - a_0 b_2^2 - a_2 b_1^2 > 0 \qquad (111)$$

If these conditions are satisfied, then any positive values may be assigned to the mutual parameters λ_{12} and ρ_{12} with the limitation that these be less than their total parameters and be so related that (110) is satisfied. Having the mutual parameters λ_{12} and ρ_{12} of the network, all the other parameters $\lambda_{11}, \lambda_{22}, \rho_{11}, \rho_{22}$ are determined from the system of equations (109). Thus if (107) satisfies the condition (111), it represents an infinite set of two-mesh networks containing inductance and resistance elements whose mutual elements are determined from (110), the other elements being determined from (109). This infinite set of networks will contain networks having all six elements; networks having five elements and finally networks having but four elements, but not less than four as we saw in Chapter II.

But note this interesting point. The right hand side of (110) is, except for a multiplying factor, the resultant of the numerator and denominator of (107). The resultant of the numerator and denominator of (107) is[13]

$$R = \begin{vmatrix} a_0 & a_1 & a_2 \\ b_1 & b_2 & 0 \\ 0 & b_1 & b_2 \end{vmatrix}$$

$$= a_0 b_2^2 - a_1 b_1 b_2 + a_2 b_1^2$$

$$= -(a_1 b_1 b_2 - a_0 b_2^2 - a_2 b_1^2)$$

---

13. See L. E. Dickson, Elementary Theory of Equations, 1914, p. 155, or use (86) p. 63)

But the expression in the parenthesis is exactly the left-hand side of (111). Hence, if R be the resultant

$$a_1 b_1 b_2 - a_0 b_2^2 - a_2 b_1^2 = -R$$

Hence from (111)

$$(b_2 \lambda_{12} - b_1 \rho_{12})^2 = -R \qquad (112)$$

Since the left-hand side of (112) is always positive, it follows that the resultant must be negative.

Hence we may state that the x) necessary and sufficient conditions that an expression like (107) with positive coefficients represent an impedance function is that the resultant of its numerator and denominator be negative. If that is so, then (107) represents the impedance function not of just one network, but an infinite set of them, containing inductance and resistance elements. This infinite set is obtained by assigning any positive values to the mutual parameters λ_{12} and ρ_{12} provided these values are such that (104) and (112) are satisfied. Having the mutual parameters λ_{12} and ρ_{12} all the other elements are determined from the system of equations (109).

It will be clarifying to illustrate the ideas above with a numerical problem. Consider the following function

$$\frac{p^2 + 4p + 3}{p + 2} \qquad (113)$$

x) Does not hold for networks having all three kinds of elements. Further are conditions [illegible] must be proved. Correct.

All the coefficients of (113) are positive, hence it might represent an impedance function. It will if the resultant of its numerator and denominator be negative. Let us see if this is so.

The resultant of the numerator and denominator of (113) is

$$R = \begin{vmatrix} 1 & 4 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= 4 - 5$$

$$= -1$$

Thus the resultant of (113) is negative, and so (113) represents the impedance function of an infinite set of different networks, the mutual parameters of which is given by (112), and the total parameters by the system of equation (109). Let us obtain the values of the network parameters. Substituting the values for b_2, b_1 and R in (112), that is $b_2 = 2$, $b_1 = 1$ and $R = -1$, we have

$$(2\lambda_{12} - \rho_{12})^2 = -(-1)$$

$$\therefore (2\lambda_{12} - \rho_{12})^2 = 1$$

and

$$2\lambda_{12} - \rho_{12} = \pm 1 \qquad (114)$$

Since the mutual parameters must be less than the total parameters,

$$\lambda_{12} \leqq 1$$

$$\rho_{12} \leqq 2$$

This limits our choice of mutual parameters. Let us plot equation (114). Writing this equation in slope-intercept form, we have

$$\lambda_{12} = \frac{1}{2}\rho_{12} \pm \frac{1}{2} \qquad (115)$$

This equation represents two straight lines with slope 1/2 and intercept $+\frac{1}{2}$ and $-\frac{1}{2}$. Figure 24 shows these two lines. Since $\lambda_{12} \leqq 1$ and $\rho_{12} \leqq 2$ the only parts of these lines which may be used as possible values of the mutual parameters are the segments AB and CD.

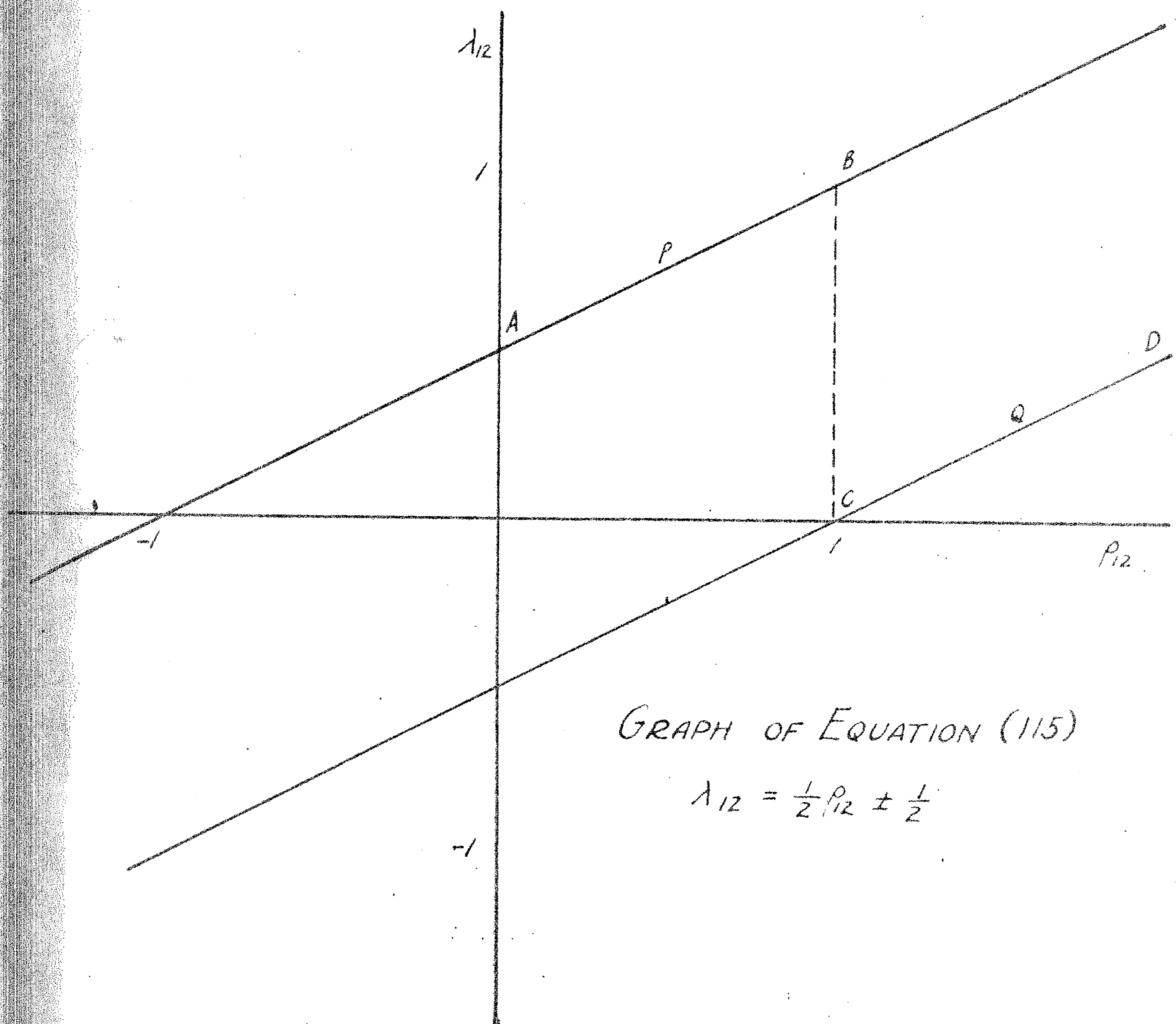
Let us select, at random, a point in each segment AB and CD, as the values of the mutual parameters and construct the two networks having (113) as an impedance function. Choose on AB the point P, that is $\rho_{12} = \frac{1}{2}$ and $\lambda_{12} = \frac{3}{4}$, and on CD the point $(\frac{3}{2}, \frac{1}{4})$, that is $\rho_{12} = \frac{3}{2}$, $\lambda_{12} = \frac{1}{4}$

Constructing the first network, we have for the mutual parameters

$$\rho_{12} = \frac{1}{2}, \qquad \lambda_{12} = \frac{3}{4}$$

The total parameters are given by (109). Thus from (109d) and (109e) we have

$$\lambda_{22} = 1, \qquad \rho_{22} = 2$$



GRAPH OF EQUATION (115)

$$\lambda_{12} = \frac{1}{2}\rho_{12} \pm \frac{1}{2}$$

FIG. 24

From (109a) we have

$$\lambda_{11}\lambda_{22} - \lambda_{12}^2 = a_0$$

$$\lambda_{11} = 1 + \left(\frac{3}{4}\right)^2$$

$$= 1 + \frac{9}{16}$$

And

$$\lambda_{11} = \frac{25}{16}$$

Finally from (109c)

$$\rho_{11}\rho_{22} - \rho_{12}^2 = \cancel{a_0}\ a_2$$

we have

$$2\rho_{11} = 3 + \left(\frac{1}{2}\right)^2$$

$$= \frac{13}{4}$$

and

$$\rho_{11} = \frac{13}{8}$$

The network having these parameters is shown in figure 25.

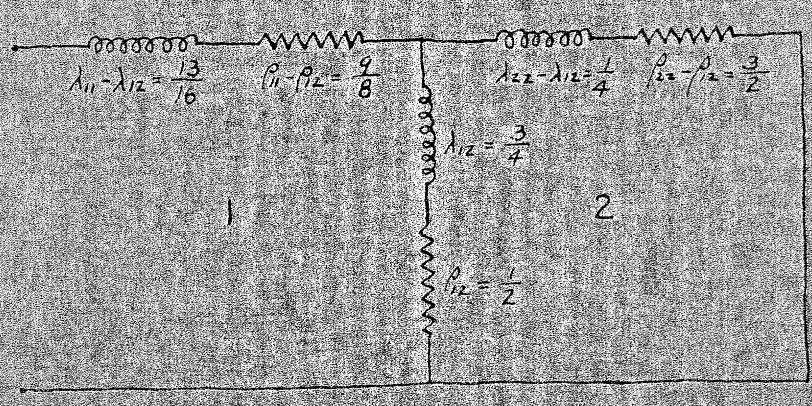


FIG 25

Let us calculate the impedance of this network by the classical method of combination of impedances

$$Z = \frac{13}{16}p + \frac{9}{8} + \frac{\left(\frac{3}{4}p+\frac{1}{2}\right)\left(\frac{1}{4}p+\frac{3}{2}\right)}{\left(\frac{3}{4}p+\frac{1}{2}\right)+\left(\frac{1}{4}p+\frac{3}{2}\right)}$$

$$= \frac{13}{16}p + \frac{9}{8} + \frac{\frac{3}{16}p^2 + \frac{10}{8}p + \frac{3}{4}}{p+2}$$

$$= \frac{\frac{13}{16}p^2 + \frac{9}{8}p + \frac{13}{8}p + \frac{18}{8} + \frac{3}{16}p^2 + \frac{10}{8}p + \frac{3}{4}}{p+2}$$

$$= \frac{p^2+4p+3}{p+2} \qquad (113\,a)$$

which is exactly (113)

Calculating the impedance by our formula (102b) we have

$$Z(p) = \frac{\begin{vmatrix} \frac{25}{16} & \frac{3}{4} \\ \frac{3}{4} & 1 \end{vmatrix} p^2 + \left\{ \begin{vmatrix} \frac{25}{16} & \frac{1}{2} \\ \frac{3}{4} & 2 \end{vmatrix} + \begin{vmatrix} \frac{13}{8} & \frac{3}{4} \\ \frac{1}{2} & 1 \end{vmatrix} \right\} p + \begin{vmatrix} \frac{13}{8} & \frac{1}{2} \\ \frac{1}{2} & 2 \end{vmatrix}}{p+2}$$

$$= \frac{p^2+4p+3}{p+2} \qquad (113\,b)$$

which of course checks the result obtained by the first method.

Let us construct networks corresponding to the extremeties of the segment AB. Thus consider the network corresponding to point A. At this point $\rho_{12}=0$, $\lambda_{12}=\frac{1}{2}$. As before $\lambda_{22}=1$ and $\rho_{22}=2$. Hence using the equations (109a) and (109c) we have for λ_{11} and ρ_{11} respectively

$$\lambda_{11} = 1 + \left(\frac{1}{2}\right)^2$$

$$= \frac{5}{4}$$

and

$$\rho_{11} = \frac{3+0}{2}$$

$$= \frac{3}{2}$$

Thus the parameters of the network are

$$\lambda_{11} = \frac{5}{4},\ \lambda_{22}=1,\ \lambda_{12}=\frac{1}{2}\ ;\ \rho_{11}=\frac{3}{2},\ \rho_{22}=2,\ \rho_{12}=0$$

and the corresponding network is shown in figure 26.

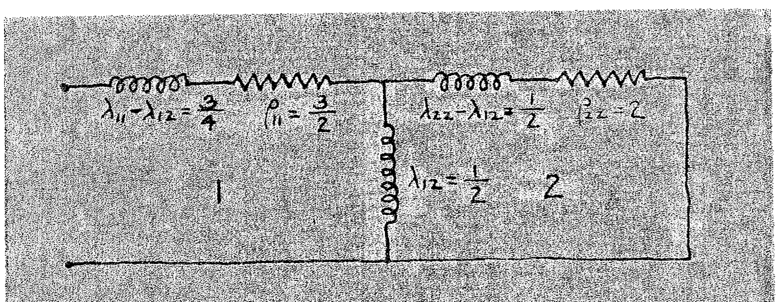


FIG. 26

Calculating the impedance of this network by the mutual method and by our formula (102a) we have by the first method

$$Z(p) = \frac{3}{4}p + \frac{3}{2} + \frac{\frac{1}{2}p\left(\frac{1}{2}p+2\right)}{\frac{1}{2}p+\left(\frac{1}{2}p+2\right)}$$

$$= \frac{3}{4}p + \frac{3}{2} + \frac{\frac{1}{4}p^2+p}{p+2}$$

$$= \frac{\frac{3}{4}p^2+\frac{3}{2}p+\frac{3}{2}p+3+\frac{1}{4}p^2+p}{p+2}$$

$$= \frac{p^2+4p+3}{p+2} \qquad (113c)$$

which again is exactly (113). Using formula (102b)

$$Z(p)=\frac{\begin{vmatrix}\frac{5}{4} & \frac{1}{2}\\ \frac{1}{2} & 1\end{vmatrix}p^2+\left\{\begin{vmatrix}\frac{5}{4} & 0\\ \frac{1}{2} & 2\end{vmatrix}+\begin{vmatrix}\frac{3}{2} & \frac{1}{2}\\ 0 & 1\end{vmatrix}\right\}p+\begin{vmatrix}\frac{3}{2} & 0\\ 0 & 2\end{vmatrix}}{p+2}$$

$$=\frac{p^2+4p+3}{p+2} \qquad (113d)$$

which checks the previous result.

Note that this end point A of the segment AB corresponds to a five element network. We shall see now that the other end point B of the segment likewise corresponds to a five element network. Let us construct this network. At this end point B, we have $\rho_{12}=1$ and $\lambda_{12}=1$. As before $\lambda_{22}=1$ and $\rho_{22}=2$ Calculating by means of (109a) and (109c) for the parameters λ_{22} and ρ_{22} we obtain $\lambda_{11}=2$ and $\rho_{11}=2$. Thus the parameters of the network are

$$\lambda_{11}=2,\ \lambda_{22}=1,\ \lambda_{12}=1\ ;\quad \rho_{11}=2,\ \rho_{22}=2,\ \rho_{12}=1$$

and the corresponding network is shown in figure 27.

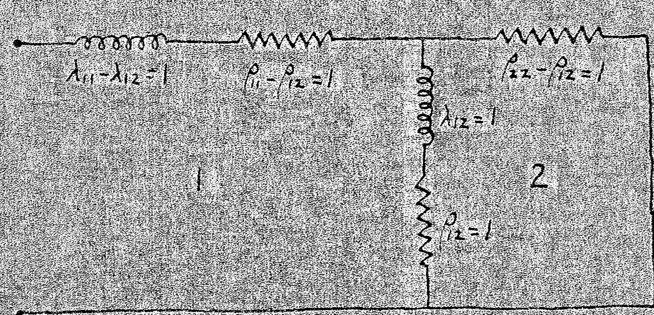


FIG 27

Calculating the impedance again by our two methods we have, by the impedance combination method

$$Z(p) = p+1 + \frac{(p+1)1}{(p+1)+1}$$

$$= p+1 + \frac{p+1}{p+2}$$

$$= \frac{p^2+4p+3}{p+2} \qquad (113e)$$

which is exactly (113).

By formula (102*b*), we have for the impedance

$$Z(p) = \frac{\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} p^2 + \left\{ \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \right\} p + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}{p+2}$$

$$= \frac{p^2+4p+3}{p+2} \qquad (113f)$$

which checks the previous result.

Thus it appears that the interior points of the segment AB correspond to six-element networks and the two end points A and B determine five-element networks.

Let us duplicate what we have done for the segment AB, for the segment CD, by constructing networks corresponding to an interior point and to both end points of the segment CD.

An interesting result follows. We find that the segment CD gives all the networks that AB does, but with the two branches in mesh 2 interchanged. Thus proceeding in the same way as we have for the segment AB, we obtain for the parameters of the network corresponding to the interior point Q, which corresponds to point P on the segment AB

$$\lambda_{11} = \frac{17}{16}, \quad \lambda_{22} = 1, \quad \lambda_{12} = \frac{1}{4}; \quad \rho_{11} = \frac{21}{8}, \quad \rho_{22} = 2, \quad \rho_{12} = \frac{3}{2}$$

and the corresponding network is shown in figure 28.

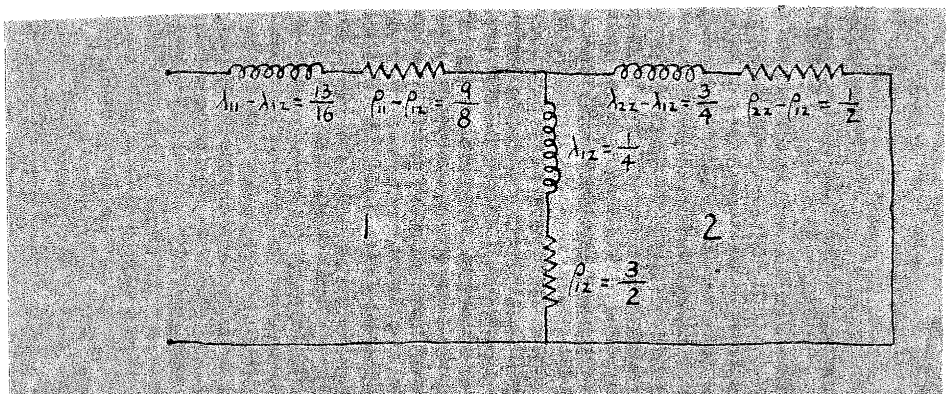


FIG. 28

But note that this network is exactly the network shown in figure 25, corresponding to point P on the segment AB, with the two branches in mesh 2 interchanged.

Let us now obtain the parameters of the network corresponding to the end point C. We find that these parameters are

$$\lambda_{11} = 1, \quad \lambda_{22} = 1, \quad \lambda_{12} = 0; \quad \rho_{11} = 2, \quad \rho_{22} = 2, \quad \rho_{12} = 1$$

and the corresponding network for this end point C on the segment CD, which corresponds to the point B on segment AB is shown in figure 29

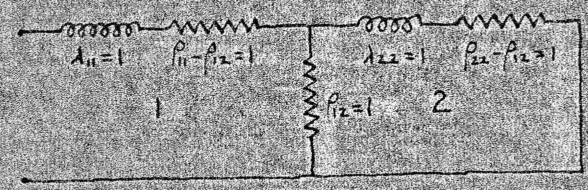


FIG. 29

But note again that this network is exactly the network shown in figure 27, which corresponds to the point B on the segment AB, except that the two branches in mesh 2 are interchanged.

In the same way we obtain for the parameters of the network corresponding to the end point D of the segment CD, which corresponds to the point A on the segment AB, the values

$$\lambda_{11} = \frac{5}{4}, \ \lambda_{22} = 1, \ \lambda_{12} = \frac{1}{2}; \ \rho_{11} = \frac{7}{2}, \ \rho_{22} = 2, \ \rho_{12} = 2$$

and the corresponding network is shown in figure 30.

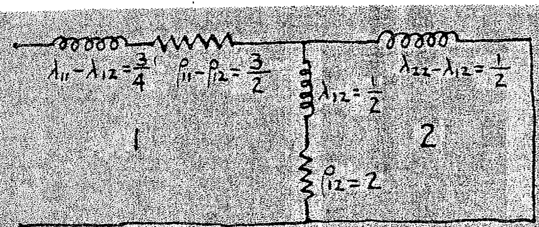


FIG. 30

Again note that this network is exactly that shown in figure 26, which corresponds to the end point A on the segment AB, except that the branches in mesh two are interchanged.

Thus it appears that segments AB and CD contain an infinite set of points whose coordinates are the mutual parameters of six-element and five-element networks, the end points giving the five-element networks. Furthermore, the segments AB and CD contain, at least from the electrical point of view, identical networks except that the branches in mesh 2 are interchanged. Thus segments AB and CD seem to be mirrors of each other about the point B or C if C were joined to B and CD became the extension of AB. The segment CD then reflects all the networks given by segment AB about the point B or C, and in doing so interchanges the branches in mesh two. Thus there is a continuous transformation of networks beginning with the end point A, which represents a five-element network, through the interior points of AB, representing six element networks to the end point B, again representing a five-element network. At B, we have a discontinuity, due to the change in sign of the intercept 1/2 in (115). But this mathematical discontinuity has a physical meaning. It means that it is the point about which reflection takes place, just as a plane mirror is a surface of discontinuity for light rays. Point C then represents a five-element network identical with that represented by B except that the branches of mesh 2 of the network are interchanged. Proceeding from C through the interior points, which represent six-element networks identical with those represented by the interior points of segment AB, except that the branches of mesh 2 are interchanged, we finally arrive at D, which is again a five-element network identical with that represented by point A, but with the branches in mesh 2 interchanged.

Thus far, we have obtained an infinite number of six-element networks having an impedance function given by (113), and but four networks (if we consider networks with the branches in mesh 2 interchanged as different networks) having only five elements. From our discussion in the introduction, we know that by partial fraction expansion and by continued fraction of $Z(p)$ and $1/Z(p)$, four-element networks can be obtained. Thus it almost appears as if (113) were not the impedance function of a four-element network. But let us see if that is so. The impedance function that we are interested in is

$$\frac{p^2+4p+3}{p+2} \qquad (113)$$

Certainly this expression is not changed if we multiply its numerator and denominator by another expression however complicated it may be since this expression will cancel out and we will have (113) again. Let us therefore, multiply the numerator and denominator of (113) by a real constant, which we shall represent by k^2. The reason for using k^2 instead of k will be apparent in the discussion. Multiplying (113) by it becomes

$$\frac{k^2p^2+4k^2p+3k^2}{k^2p+2k^2} \qquad (116)$$

As before all the coefficients are positive. Let us see if the resultant is negative. It is

$$\begin{vmatrix} k^2 & 4k^2 & 3k^2 \\ k^2 & 2k^2 & 0 \\ 0 & k^2 & 2k^2 \end{vmatrix}$$

This is equal to

$$k^6 \begin{vmatrix} 1 & 4 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= -k^6$$

which is of course negative and k^6 times the eliminant of (113). Hence we know that (116) represents the impedance function of some networks, which we knew before, since by removing k^2 (116) becomes (113).

However let us proceed as before to obtain networks having now (116) instead of (113) as their impedance function. It would appear that we should get the same networks as given by (113), but let us see.

The impedance function, of the form (116), of the most general network is of course

$$\frac{\Delta(\lambda)p^2 + \Delta_1(\lambda,p)p + \Delta(p)}{M_{11}(\lambda)p + M_{11}(p)} \qquad (102b)$$

Hence, proceeding as before, comparing (102b) with (116) we must have

$$\left.\begin{aligned} \Delta(\lambda) &= k^2 \\ \Delta_1(\lambda,p) &= 4k^2 \\ \Delta(p) &= 3k^2 \\ M_{11}(\lambda) &= k^2 \\ M_{11}(p) &= 2k^2 \end{aligned}\right\} \qquad (117)$$

Hence the parameters $\lambda_{11}, \lambda_{22}, \lambda_{12}$; $\rho_{11}, \rho_{22}, \rho_{12}$ of the networks having (116) as an impedance function must satisfy the following equations, writing (117) in open form

$$\lambda_{11}\lambda_{22} - \lambda_{12}^2 = k^2 \tag{118a}$$

$$\lambda_{11}\rho_{22} + \rho_{11}\lambda_{22} - 2\lambda_{12}\rho_{12} = 4k^2 \tag{118b}$$

$$\rho_{11}\rho_{22} - \rho_{12}^2 = 3k^2 \tag{118c}$$

$$\lambda_{22} = k^2 \tag{118d}$$

$$\rho_{22} = 2k^2 \tag{118e}$$

As before, page 78 substitute (118d) and (118e) in (118a) and (118c)

$$k^2\lambda_{11} - \lambda_{12}^2 = k^2$$

$$2k^2\rho_{11} - \rho_{12}^2 = 3k^2$$

$$\left.\begin{aligned} \lambda_{11} &= \frac{k^2 + \lambda_{12}^2}{k^2} \\ \rho_{11} &= \frac{3k^2 + \rho_{12}^2}{2k^2} \end{aligned}\right\} \tag{119}$$

Substituting (118d), (118e) and (119) in (118b), we have

$$\frac{k^2 + \lambda_{12}^2}{k^2} \cdot 2k^2 + \frac{3k^2 + \rho_{12}^2}{2k^2} \cdot k^2 - 2\lambda_{12}\rho_{12} = 4k^2$$

Clearing fractions

$$4k^2 + 4\lambda_{12}^2 + 3k^2 + \rho_{12}^2 - 4\lambda_{12}\rho_{12} = 8k^2$$

$$\therefore\ 4\lambda_{12}^2 - 4\lambda_{12}\rho_{12} + \rho_{12}^2 = k^2$$

and

$$(2\lambda_{12} - \rho_{12})^2 = k^2$$

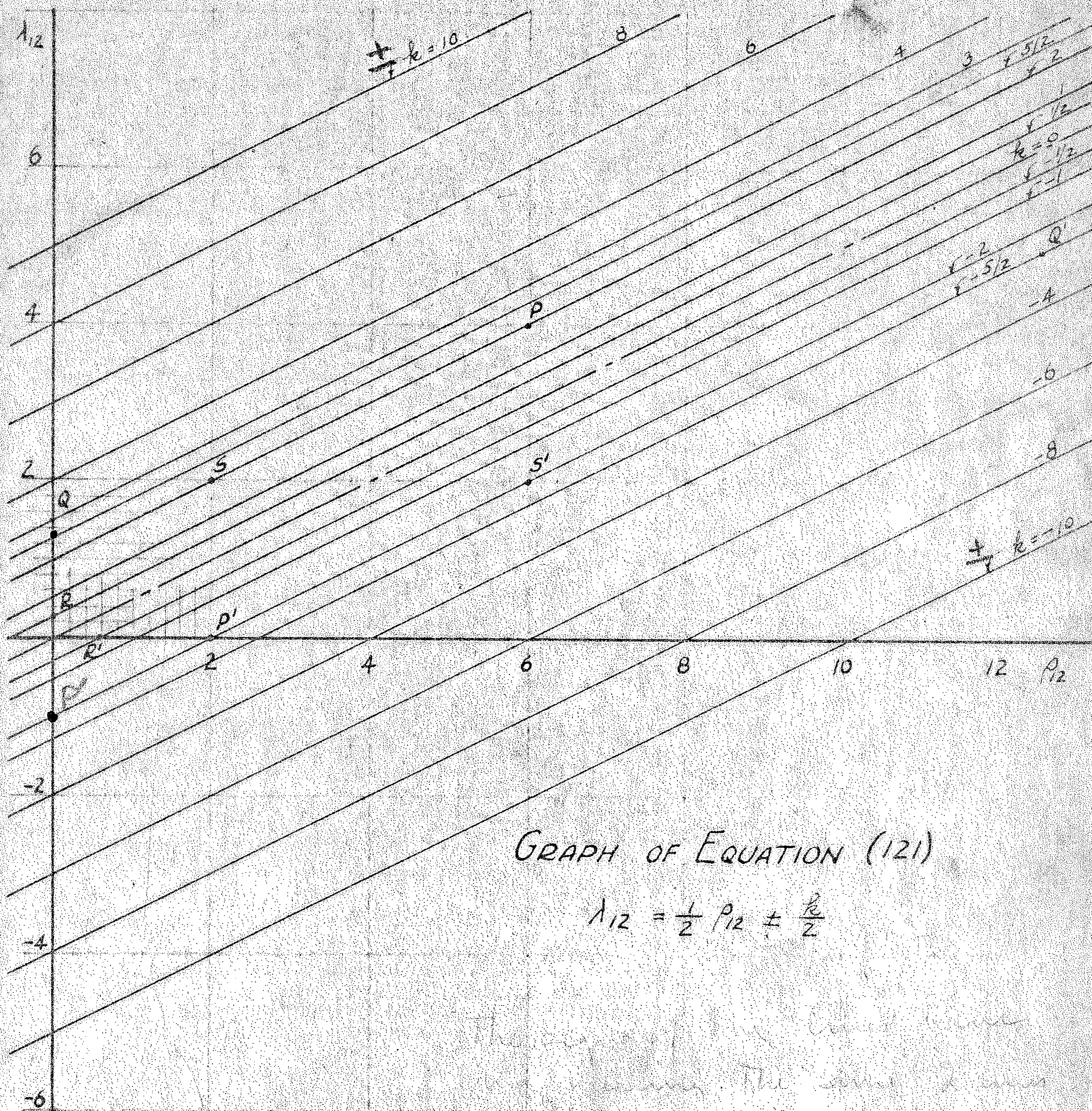
$$\therefore\ 2\lambda_{12} - \rho_{12} = \pm k \qquad (120)$$

Expressing this equation in slope intercept form, we have

$$\lambda_{12} = \frac{\rho_{12}}{2} \pm \frac{k}{2} \qquad (121)$$

which represents the equation of a family of straight lines having a slope 1/2 and λ_{12} intercept equal to $\pm \frac{k}{2}$ Equation (120) may be looked upon also as representing an hyperbola ? [illegible]! where $2\lambda_{12}$ and ρ_{12} represent the distances of any point on the hyperbola from its two foci, the difference between these two distances being k. Thus as k varies, we get a family of hyperbolas, and the mutual parameters of the network can be obtained by taking the distances from the hyperbola to the foci.

Let us consider equation (121) however, which represents a family of straight lines. This family of straight lines fills the entire plane, as k is made to assume values from zero to any real value however great. Figure 31 shows some of the lines of this family for $k=0,\ \pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6$ etc. Thus to every point in the plane there corresponds a triple of



GRAPH OF EQUATION (121)

$$\lambda_{12} = \frac{1}{2}\rho_{12} \pm \frac{k}{2}$$

FIG. 31

of values, namely λ_{12}, ρ_{12} and k, which are all simultaneously zero at the origin. All the points in the plane therefore represent networks having (116) for an impedance function, provided λ_{12} and ρ_{12} satisfy the conditions (104) and (112). Our example (113), page 81 was then a special case of (116), where the value of k is equal to unity.

Let us now see if we can obtain four-element networks, that is, the minimal forms, having (116) or (112) as an impedance function. In Chapter II we saw what these minimal forms were, and what two elements could be removed without changing the form of the impedance function. Figure 32 shows the most general two-mesh network having inductance and resistance elements.

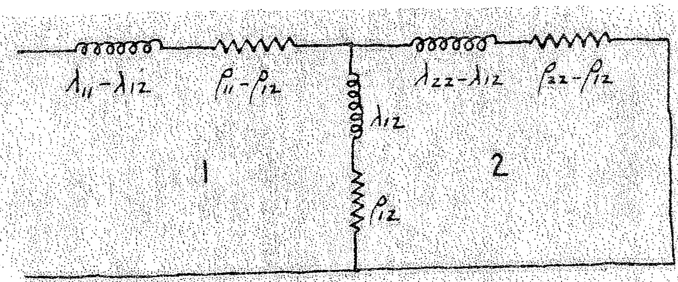
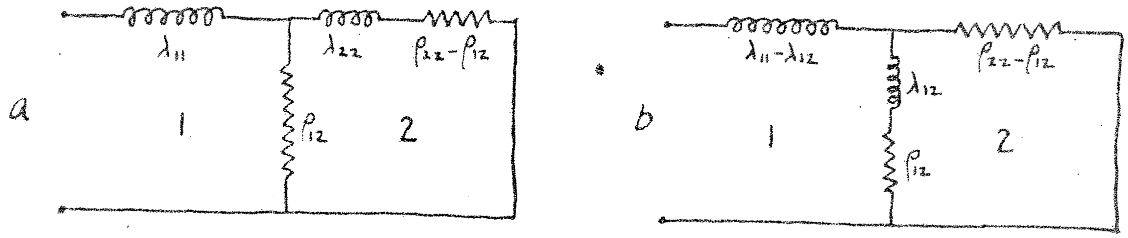


FIG. 32

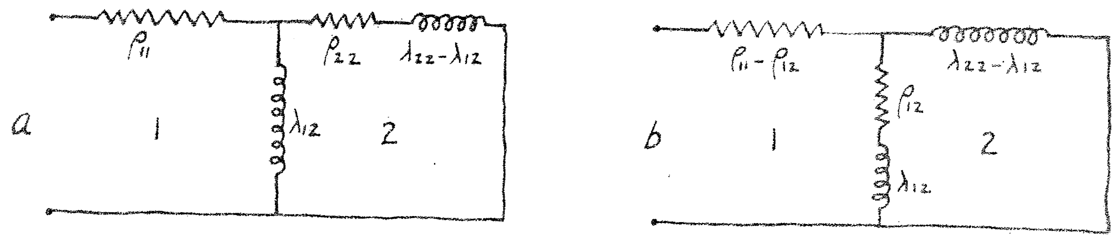
Now proceeding to remove two elements at a time from this general network, but limiting this removal to the conditions that the form of the impedance function be preserved, we obtain the following eight networks, four of which are images of the other, that is their corresponding branches in mesh 2 are interchanged. These networks coupled in pairs are shown in figures 33, 34, 35 and 36. The removal of the corresponding elements from the general network of figure 32 is indicated by equating the corresponding parameters to zero.



$\lambda_{12} = 0$
$\rho_{11} - \rho_{12} = 0$

$\lambda_{22} - \lambda_{12} = 0$
$\rho_{11} - \rho_{12} = 0$

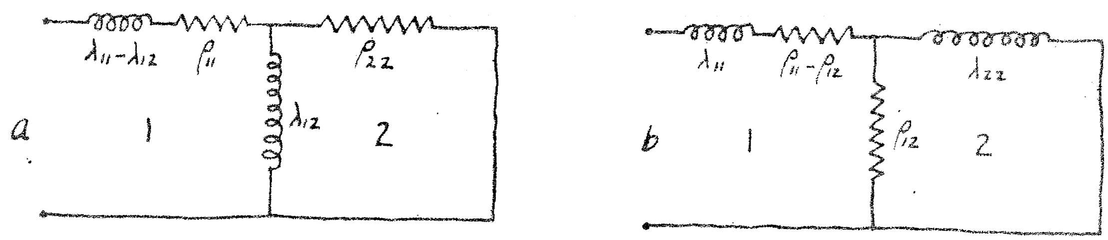
FIG. 33



$\lambda_{11} - \lambda_{12} = 0$
$\rho_{12} = 0$

$\lambda_{11} - \lambda_{12} = 0$
$\rho_{22} - \rho_{12} = 0$

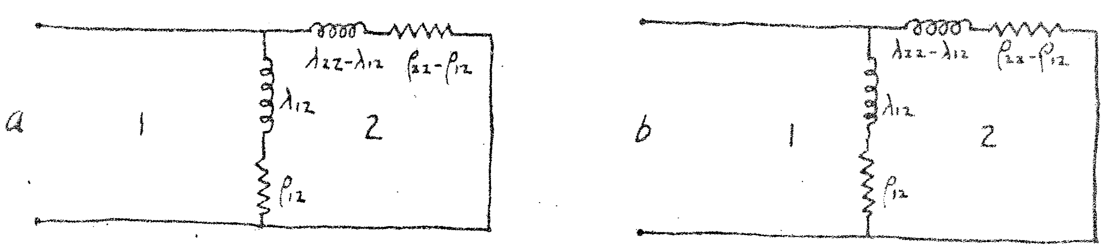
FIG. 34



$\lambda_{22} - \lambda_{12} = 0$
$\rho_{12} = 0$

$\lambda_{12} = 0$
$\rho_{22} - \rho_{12} = 0$

FIG. 35



$\lambda_{11} - \lambda_{12} = 0$
$\rho_{11} - \rho_{12} = 0$

$\lambda_{11} - \lambda_{12} = 0$
$\rho_{11} - \rho_{12} = 0$

FIG. 36

Figures 33 to 36 show all the possible four-element networks having (116) for an impedance function. Let us now obtain the actual values of the parameters of these networks.

Network a, figure 33 can be described by the two equations below it, that is

$$\lambda_{12} = 0$$

$$\rho_{11} - \rho_{12} = 0$$

Going back to (121), we find that

$$\lambda_{12} = \frac{\rho_{12}}{2} \pm \frac{k}{2} \tag{121}$$

Since $\lambda_{12} = 0$, we have from (121)

$$\rho_{12} = -(\pm k)$$

Since we exclude negative resistances from our discussion, the term-k is used, so that

$$\rho_{12} = k$$

From (118a) and (118c).

$$\lambda_{11} = \frac{k^2 + \lambda_{12}^2}{\lambda_{22}}$$

and

$$\rho_{11} = \frac{3k^2 + \rho_{12}^2}{\rho_{22}}$$

But from (118d) and (118e),

$$\lambda_{22} = k^2$$

and

$$\rho_{22} = 2k^2$$

Hence

$$\lambda_{11} = \frac{k^2}{k^2} = 1$$

and

$$\rho_{11} = \frac{3k^2 + k^2}{2k^2} = 2$$

But $\rho_{11} - \rho_{12} = 0$ for this network. Hence

$$\rho_{11} = k = 2$$

So that the parameters of network a figure 33 are

$$\lambda_{11} = 1, \quad \lambda_{22} = 4, \quad \lambda_{12} = 0; \quad \rho_{11} = 2, \quad \rho_{22} = 8, \quad \rho_{12} = 2$$

This network is shown in figure 33a.

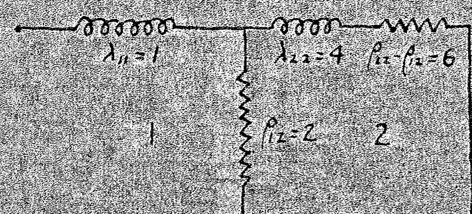


FIG. 33a

Let us check the impedance of this network by the usual method and by formula (102b), to see if (116) is obtained, k^2 being equal to 4. We have

$$Z(p) = p + \frac{2(4p+6)}{2+4p+6}$$

$$= p + \frac{8p+12}{4p+8}$$

$$= \frac{4p^2+16p+12}{4p+8} \qquad (116a)$$

But this is exactly (116), for $k=2$. Divide the numerator and denominator by 4, (113) is of course obtained. Let us now check this result by formula (102b). Thus

$$Z(p) = \frac{\begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix} p^2 + \left\{ \begin{vmatrix} 1 & 2 \\ 0 & 8 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} \right\} p + \begin{vmatrix} 2 & 2 \\ 2 & 8 \end{vmatrix}}{4p+8}$$

$$= \frac{4p^2+16p+12}{4p+8} \qquad (116a)$$

which checks the result above. The point $\lambda_{12}=0$, $\rho_{12}=2$, $k=-2$ that is point P' on figure 31, corresponds to the network shown in figure 33a. Note that this point lies on the line $k=-2$ which is what we should expect.

Network 33b is described by the equations

$$\lambda_{22} - \lambda_{12} = 0$$

$$\rho_{11} - \rho_{12} = 0$$

Let us, as above, obtain the actual values of the network elements in order that the impedance function of the network be given by (116).

Proceeding in exactly the same manner as before we have from (121)

$$\lambda_{12} = \frac{\rho_{12}}{2} \pm \frac{k}{2} \qquad (121)$$

But in this case $\lambda_{22} - \lambda_{12} = 0$ and $\lambda_{22} = k^2$

Hence $\lambda_{12} = k^2$ so that

$$k^2 = \frac{\rho_{12}}{2} \pm \frac{k}{2}$$

and $\rho_{12} = 2k^2 \mp k$

But $\rho_{11} - \rho_{12} = 0$, so that

$$\rho_{11} = 2k^2 \mp k$$

But from (118a)

$$\lambda_{11} = \frac{k^2 + \lambda_{12}^2}{\lambda_{22}}$$

$$= \frac{k^2 + k^4}{k^2} = 1 + k^2$$

And from (118c)

$$\rho_{22} = \frac{3k^2 + \rho_{12}^2}{\rho_{11}}$$

$$= \frac{3k^2 + (2k^2 \mp k)^2}{2k^2 \mp k}$$

But $\rho_{22} = 2k^2$

Hence

$$2k^2 = \frac{3k^2 + (2k^2 \mp k)^2}{2k^2 \mp k}$$

And

$$4k^4 \mp 2k^3 = 3k^2 + 4k^4 \mp 4k^3 + k^2$$

$$\therefore \pm 2k^3 - 4k^2 = 0$$

$$\therefore k^2 = 0$$

$$\pm 2k - 4 = 0$$

$$\therefore \pm 2k = 4$$

$$k = \pm 2$$

Using the value $\mp k = +2$ we have for the mutual parameters of the network $\lambda_{12} = 4$, $\rho_{12} = 6$ and

$$\lambda_{11} = 5, \ \lambda_{22} = 4, \ \lambda_{12} = 4 \ ; \ \rho_{11} = 6, \ \rho_{22} = 8, \ \rho_{12} = 6$$

The corresponding network is shown in figure 33b

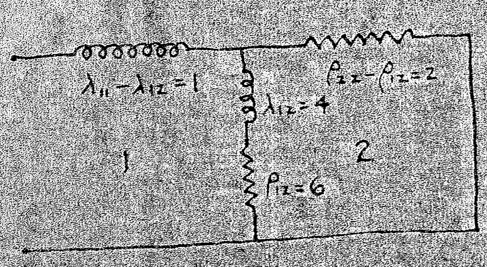


FIG. 33 b

This network is of course the same as the network in figure 33a, but with the branches in mesh 2 interchanged. Its impedance will of course be the same as the network figure 33a, and thus equal to (116), with $k^2 = 4$. The network just obtained, figure 33b, corresponds to the point $\lambda_{12} = 4$, $\rho_{12} = 6$, $k = \pm 2$ that is the point P, which, so to speak, is the image of point P'. The use of $\mp k = -2$ gives a network with a negative resistance in mesh 2.

Continuing in this way with the network, figure 34a, where

$$\lambda_{11} - \lambda_{12} = 0$$

$$\rho_{12} = 0$$

we have from (121)

$$\lambda_{12} = \frac{\rho_{12}}{2} \pm \frac{k}{2}$$

Since $\rho_{12} = 0$ $\qquad \lambda_{12} = \pm \frac{k}{2}$

For positive inductance, the plus value must be used.

From (118a)

$$\lambda_{11} = \frac{k^2 + \frac{k^2}{4}}{\lambda_{22}}$$

As before, $\lambda_{22} = k^2$ and $\rho_{22} = 2k^2$

Thus $$\lambda_{11} = \frac{\frac{5k^2}{4}}{k^2} = \frac{5}{4}$$

But $\lambda_{11} - \lambda_{12} = 0$ hence $\frac{k}{2} = \frac{5}{4}$

and

$$k = \frac{5}{2}$$

Then

$$\rho_{11} = \frac{3k^2+0}{2k^2} = \frac{3}{2}$$

Hence the network parameters are $\lambda_{11} = \frac{5}{4}$, $\lambda_{22} = \frac{25}{4}$, $\lambda_{12} = \frac{5}{4}$; $\rho_{11} = \frac{3}{2}$, $\rho_{22} = \frac{25}{2}$ and $\rho_{12} = 0$ and the resulting network is

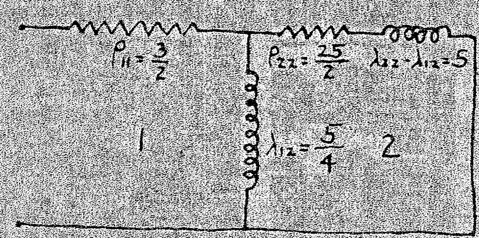


FIG 34 a

Computing the impedance of this network, we have

$$Z(p) = \frac{3}{2} + \frac{\frac{5}{4}p\left(5p + \frac{25}{2}\right)}{\frac{5}{4}p + \left(5p + \frac{25}{2}\right)}$$

$$= \frac{\frac{25}{4}p^2 + 25p + \frac{75}{4}}{\frac{25}{4}p + \frac{25}{2}}$$

$$= \frac{\left(\frac{5}{2}\right)^2 p^2 + 4\left(\frac{5}{2}\right)^2 p + 3\left(\frac{5}{2}\right)^2}{\left(\frac{5}{2}\right)^2 p + 2\left(\frac{5}{2}\right)} \qquad (116\,b)$$

Which is exactly (116) for $k = \frac{5}{2}$

By use of the impedance formula

$$Z(p) = \frac{\begin{vmatrix} \frac{5}{4} & \frac{5}{4} \\ \frac{5}{4} & \frac{25}{4} \end{vmatrix} p^2 + \left\{ \begin{vmatrix} \frac{5}{4} & 0 \\ \frac{5}{4} & \frac{25}{2} \end{vmatrix} + \begin{vmatrix} \frac{3}{2} & \frac{5}{4} \\ 0 & \frac{25}{4} \end{vmatrix} \right\} p + \begin{vmatrix} \frac{3}{2} & 0 \\ 0 & \frac{25}{2} \end{vmatrix}}{\left(\frac{5}{2}\right)^2 p \quad + 2\left(\frac{5}{2}\right)^2}$$

$$= \frac{\frac{25}{4} p^2 + 25 p + \frac{75}{4}}{\frac{25}{4} p + \frac{25}{2}}$$

which of course checks the above.

The network shown in figure 34a thus corresponds to the point $\lambda_{12} = \frac{5}{4}$, $\rho_{12} = 0$, $k = + \frac{5}{2}$ that is the point Q in figure 31.

For network 34b, we have

$$\lambda_{11} - \lambda_{12} = 0$$

$$\rho_{22} - \rho_{12} = 0$$

$$\lambda_{12} = \frac{\rho_{12}}{2} \pm \frac{k}{2}$$

But in this case, $\rho_{12} = \rho_{22}$ and $\rho_{22} = 2k^2$, Hence

$$\lambda_{12} = \frac{2k^2}{2} \pm \frac{k}{2}$$

$$= k^2 \pm \frac{k}{2}$$

$$\therefore \lambda_{11} = k^2 \pm \frac{k}{2}$$

And

$$\rho_{11} = \frac{3k^2 + \rho_{12}^2}{\rho_{22}}$$

$$\therefore \rho_{11} = \frac{3k^2 + 4k^4}{2k^2}$$

$$= \frac{3 + 4k^2}{2}$$

and

$$\lambda_{22} = \frac{k^2 + \lambda_{12}{}^2}{\lambda_{11}}$$

But

$$\lambda_{22} = k^2$$

$$\therefore k^2\left(k^2 \pm \frac{k}{2}\right) = k^2 + \left(k^2 \pm \frac{k}{2}\right)^2$$

And

$$k^2 \pm \frac{k}{2} = 1 + k^2 \pm k + \frac{1}{4}$$

$$\pm \frac{k}{2} \mp k = \frac{5}{4}$$

$$\pm \frac{k}{2} = \frac{5}{4}$$

$$k = \pm \frac{5}{2}$$

For positive inductance, as can readily be checked the negative value must be used and

$$\pm k = -\frac{5}{2}$$

Thus the parameters of the network are

$$\lambda_{11} = 5 \,, \quad \lambda_{22} = \frac{25}{4} \,, \quad \lambda_{12} = 5 \;; \quad \rho_{11} = 14 \,, \quad \rho_{22} = \frac{25}{2} \,, \quad \rho_{12} = \frac{25}{2}$$

and the network constructed from these parameters, shown in figure 34b, is seen to be exactly that shown in figure 34a, with the branches in mesh 2 interchanged.

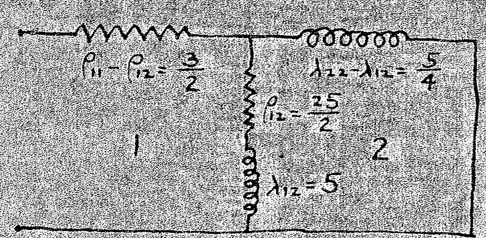


FIG 34 b

Computing the impedance of this network by the impedance formula, we have

$$Z(p) = \frac{\begin{vmatrix} 5 & 5 \\ 5 & \frac{25}{4} \end{vmatrix} p^2 + \left\{ \begin{vmatrix} 5 & \frac{25}{2} \\ 5 & \frac{25}{2} \end{vmatrix} + \begin{vmatrix} 14 & 5 \\ \frac{25}{2} & \frac{25}{4} \end{vmatrix} \right\} p + \begin{vmatrix} 14 & \frac{25}{2} \\ \frac{25}{2} & \frac{25}{2} \end{vmatrix}}{\frac{25}{4} p + \frac{25}{2}}$$

$$= \frac{\frac{25}{4} p^2 + 25 p + \frac{75}{4}}{\frac{25}{4} p + \frac{25}{2}}$$

which is exactly (116) for $k = -\frac{5}{2}$

The network shown in figure 34b corresponds to the point $\rho_{12} = \frac{25}{2}$, $\lambda_{12} = 5$ and $k = -\frac{5}{2}$, that is, the point Q' figure 31.

Continuing with the network shown in figure 35a, we have

$$\lambda_{22} - \lambda_{12} = 0$$

$$\rho_{12} = 0$$

From (121)

$$\lambda_{12} = \frac{\rho_{12}}{2} \pm \frac{k}{2}$$

Hence

$$\lambda_{12} = \pm \frac{k}{2}$$

For positive inductance, the plus sign must be used, so that

$$\lambda_{12} = \frac{k}{2}$$

As before, $\lambda_{22} = k^2$ and since $\lambda_{22} = \lambda_{12}$

$$\therefore k^2 = \frac{k}{2}$$

and $k = 0$, $k = \frac{1}{2}$

Using the value $k = \frac{1}{2}$, we have $\lambda_{12} = \frac{1}{4}$ and $\lambda_{22} = \frac{1}{4}$ and

$$\lambda_{11} = \frac{\frac{1}{4} + \frac{1}{16}}{\frac{1}{4}} = \frac{5}{4}$$

Also $\rho_{22} = 2k^2 = \frac{1}{2}$ and

$$\rho_{11} = \frac{3k^2 + 0}{\rho_{22}}$$

$$= \frac{3}{2}$$

Hence the parameters of our network are

$$\lambda_{11} = \frac{5}{4}, \quad \lambda_{22} = \frac{1}{4}, \quad \lambda_{12} = \frac{1}{4}; \quad \rho_{11} = \frac{3}{2}, \quad \rho_{22} = \frac{1}{2}, \quad \rho_{12} = 0$$

and the network is shown in figure 35a.

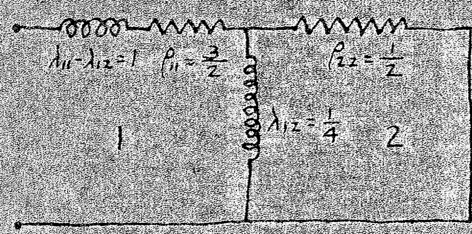


FIG. 35 a

The impedance of the network is then, by the impedance formula

$$Z(p) = \frac{\begin{vmatrix} \frac{5}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{vmatrix} p^2 + \left\{ \begin{vmatrix} \frac{5}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{vmatrix} + \begin{vmatrix} \frac{3}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} \end{vmatrix} \right\} p + \begin{vmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{vmatrix}}{\frac{1}{4} p + 1}$$

$$= \frac{\frac{1}{4} p^2 + p + \frac{3}{4}}{\frac{1}{4} p + \frac{1}{2}}$$

which may be written

$$Z(p) = \frac{\left(\frac{1}{2}\right)^2 p^2 + 4\left(\frac{1}{2}\right)^2 p + 3\left(\frac{1}{2}\right)^2}{\left(\frac{1}{2}\right)^2 p + 2\left(\frac{1}{2}\right)^2} \qquad (116\ c)$$

which is exactly (116) for $k = \frac{1}{2}$. If the value $k = 0$ were used, we should have all the parameters zero except

λ_{11} and ρ_{11}, which would equal respectively $\frac{5}{4}$ and $\frac{3}{2}$, which are the elements in mesh 1. The network shown in figure 35a corresponds to the point $\rho_{12} = 0,\ \lambda_{12} = \frac{1}{4},\ k = \frac{1}{2}$, that is point R, figure 31.

For network b, figure 35, we have

$$\lambda_{12} = 0$$

$$\rho_{22} - \rho_{12} = 0$$

and from (121)

$$\lambda_{12} = \frac{\rho_{12}}{2} \pm \frac{k}{2}$$

Hence

$$0 = \frac{\rho_{12}}{2} \pm \frac{k}{2}$$

and

$$\rho_{12} = -(\pm k)$$

For positive resistance, the -k is used. Then

$$\rho_{12} = k$$

and since

$$\rho_{22} = \rho_{12}$$

$$\therefore \rho_{22} = k$$

Also

$$\rho_{22} = 2k^2$$

hence

$$k = 2k^2$$

$$\therefore k = 0,\ k = \frac{1}{2}$$

Now

$$\rho_{11} = \frac{3(\frac{1}{4}) + (\frac{1}{2})^2}{2(\frac{1}{2})^2}$$

$$= 2.$$

Furthermore $\lambda_{22} = k^2 = \frac{1}{4}$

and $$\lambda_{11} = \frac{\frac{1}{4} + 0}{\frac{1}{4}} = 1$$

Hence the parameters of the corresponding network are

$$\lambda_{11} = 1, \quad \lambda_{22} = \frac{1}{4}, \quad \lambda_{12} = 0 \ ; \quad \rho_{11} = 2, \quad \rho_{22} = \frac{1}{2}, \quad \rho_{12} = \frac{1}{2}$$

and the network is shown in figure 35b

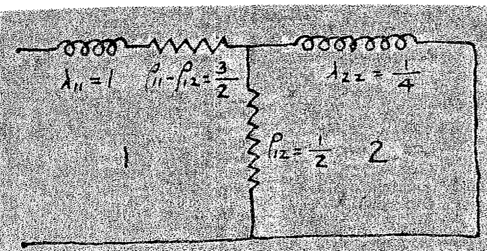


FIG. 35 b

But this is the network shown in figure 35a, with the branches in mesh 2 interchanged. This network corresponds to the point $\rho_{12} = \frac{1}{2}$, $\lambda_{12} = 0$, $k = \frac{1}{2}$, that is point R', figure 31.

Finally we have for network a, figure 36

$$\lambda_{11} - \lambda_{12} = 0$$
$$\rho_{11} - \rho_{12} = 0$$

As before, from (121), we have

$$\lambda_{12} = \frac{\rho_{12}}{2} \pm \frac{k}{2}$$

But $$\rho_{22} = \frac{3k^2 + \rho_{12}^2}{\rho_{11}}$$

$$\therefore \ 2k^2 \rho_{12} = 3k^2 + \rho_{12}^2$$

And $$\rho_{12}^2 - 2k^2 \rho_{12} + 3k^2 = 0$$

Solving for ρ_{12} we have

$$\rho_{12} = \frac{2k^2 \pm \sqrt{4k^2 - 12k^2}}{2}$$

$$= k\left(k \pm \sqrt{k^2 - 3}\right)$$

$$\therefore \lambda_{12} = \frac{1}{2}\left[k\left(k \pm \sqrt{k^2-3}\right) \pm k\right]$$

But

$$\lambda_{11}\lambda_{22} = k^2 + \lambda_{12}^2$$

$$\therefore \frac{1}{2}k^2\left[k\left(k \pm \sqrt{k^2-3}\right) \pm k\right] = k^2 + \frac{1}{4}\left[k\left(k \pm \sqrt{k^2-3}\right) \pm k\right]^2$$

And

$$\frac{1}{2}\left[k^3\left(k \pm \sqrt{k^2-3}\right) \pm k^3\right] = k^2 + \frac{1}{4}\left[k^2\left(k^2 \pm 2k\sqrt{k^2-3} + k^2 - 3\right) \pm 2k^2\left(k \pm \sqrt{k^2-3}\right) + k^2\right]$$

$$\therefore 2k^4 \pm 2k^3\sqrt{k^2-3} \pm 2k^3 = 4k^2 + k^4 \pm 2k^3\sqrt{k^2-3} + k^4 - 3k^2 \pm 2k^3 + 2k^2\sqrt{k^2-3} + k^2$$

$$\therefore 2k^2 + 2k^2\sqrt{k^2-3} = 0$$

And

$$1 + \sqrt{k^2-3} = 0$$

$$\sqrt{k^2-3} = -1$$

$$k^2 - 3 = 1$$

$$k = \pm 2$$

Using $k=+2$, we have for the parameters of the network

$$\lambda_{11}=2\ ,\ \lambda_{22}=4\ ,\ \lambda_{12}=2\ ;\ \rho_{11}=2,\ \rho_{22}=8\ ,\ \rho_{12}=2$$

and the corresponding network is shown in figure 36a.

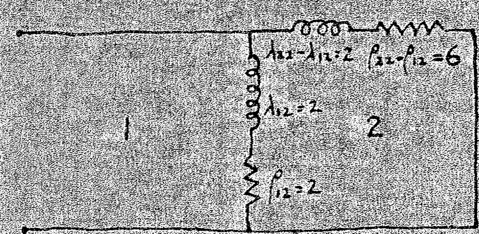


FIG. 36a

The impedance function for this network is

$$Z(p)=\frac{\begin{vmatrix}2&2\\2&4\end{vmatrix}p^2+\left\{\begin{vmatrix}2&6\\2&8\end{vmatrix}+\begin{vmatrix}6&2\\6&4\end{vmatrix}\right\}p+\begin{vmatrix}2&2\\2&8\end{vmatrix}}{4p+8}$$

$$=\frac{4p^2+16p+12}{4p+8} \qquad (116d)$$

which is exactly (116) for $k^2=4$. The network shown in figure 36a corresponds to the point $\rho_{12}=2,\ \lambda_{12}=2$ and $k=+2$ that is point S, figure 31.

Finally for $k=-2$, the network parameters are

$$\lambda_{11}=2,\ \lambda_{22}=4,\ \lambda_{12}=2\ ;\ \rho_{11}=6,\ \rho_{22}=8,\ \rho_{12}=6$$

and the corresponding network is shown in figure 36b.

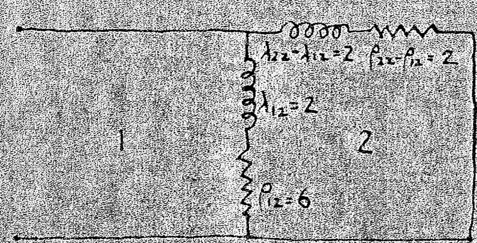


FIG. 36 b

This is of course the network shown in figure 36a with the branches in mesh 2 interchanged. The network figure 36b corresponds to the point $\rho_{12}=6,\ \lambda_{12}=2,\ k=-2$ that is point S', figure 31.

Summarizing the results thus far obtained we see that the impedance function is as absolute invariant to a certain transformation of the parameters of the network, and the coefficients of the impedance function are invariants to the same transformation of the network elements, but for a constant factor k^2, which is the same for all the coefficients. This invariant impedance function is represented by (102b), that is

$$Z(p)=\frac{\Delta(\lambda)p^2+\Delta_1(\lambda,\rho)p+\Delta(\rho)}{M_{11}(\lambda)p+M_{11}(\rho)} \qquad (102b)$$

which may be written

$$Z(p)=\frac{\begin{vmatrix}\lambda_{11} & \lambda_{12}\\ \lambda_{12} & \lambda_{22}\end{vmatrix}p^2+\left\{\begin{vmatrix}\lambda_{11} & \rho_{12}\\ \lambda_{12} & \rho_{22}\end{vmatrix}+\begin{vmatrix}\rho_{11} & \lambda_{12}\\ \rho_{12} & \lambda_{22}\end{vmatrix}\right\}p+\begin{vmatrix}\rho_{11} & \rho_{12}\\ \rho_{12} & \rho_{22}\end{vmatrix}}{\lambda_{22}p+\rho_{22}} \qquad (122)$$

For real parameters of the network, the resultant must be negative and the relation between the mutual parameters is expressed in terms of the resultant by (112). Now if negative inductances and resistances are permitted, and these may be realized physically, though not by coils and resistors respectively, the entire plane, with the exception of $k=0$ may represent networks having (122) as an impedance function. Equations (118) determine the values of the total paramaters. For positive inductance and resistance, there are regions in the plane that do no represent networks. Some points in the plane represent six-element networks, some five-element, and but eight points four element networks, four of which are the same respectively as the other four, but for the branches in mesh 2 interchanged. A similar study for five-element networks could be made as for four-element networks. In many cases it may be much better from the standpoint of performance and economy to use five-element instead of four-element networks. The graph, figure 31, is useful if it is desired to make use of certain standard coils or resistors, of which large quantities may be present. It would be instructive to explore the entire ρ_{12}, λ_{12} plane, outlining the regions corresponding to six-element, five-element and four-element networks with positive elements and the same with positive and negative elements and with negative elements.

It is hardly necessary to add that the above theory holds just as well for networks containing inductance and capacity elements and resistance and capacity elements, and can be extended, as we shall see later, to networks containing all three elememts.

Thus far we have considered the coefficients of the impedance function, rather than the question of its zeros and poles. We have seen in the introduction that Foster arrived at some interesting properties concerning the zeros and poles of the impedance function of a two-mesh network containing inductance and capacity elements, from the analogous dynamical problem treated by Routh.[14] The zeros and poles are respectively for Foster the resonant and anti-resonant frequences of the impedance. We have also seen that Cauer extended Foster's "Reactance Theorem" to networks containing any two kinds of elements.[15]

To fix ideas it will be instructive to consider the simple impedance function treated by Foster, namely

$$Z(p) = \frac{p^4 + 10p^2 + 9}{p(p^2 + 4)} \tag{123}$$

Note that this impedance function is exactly of the form (102c), page 72, which is to be expected, since Z(p) in (123) is the impedance function of a two-mesh network containing inductance and capacity elements. Let us proceed to examine its zeros and poles. Factoring the numerator we have

$$Z(p) = \frac{(p^2+1)(p^2+9)}{p(p^2+4)} \tag{123a}$$

$$= \frac{(p+j)(p-j)(p+3j)(p-3j)}{p(p+2j)(p-2j)} \tag{123b}$$

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14. R. M. Foster "A Reactance Theorem", Bell System Technical Journal, vol. 3, 1924, p. 259; and E.J.Rought, "Advanced Rigid Dynamics", 1884, p. 38.

15. Cauer "Die Verwirklichung von Wechselstromwiderstanden unsiv., "Archiv fur Eletrotechnik", Heft. 4, Band XVII.

Thus the zeros are pairs of pure imaginaries with opposite signs, and the poles, with the exception of the pole at zero are likewise pure imaginaries with opposite signs. These zeros and poles can be represented in the p- plane, whence it is readily seen, as it can also be seen from (123b), that the zeros and poles of (123b) separate each other. Figure 37 shows the zeros and poles represented on the p plane, the zeros by circles and the poles by dots on the imaginary axis. This separation property as shown in figure 37 is general for networks of any number of meshes containing two kinds of elements, except that for resistance and inductance elements, and resistance and capacity elements, the zeros and poles are negative reals.

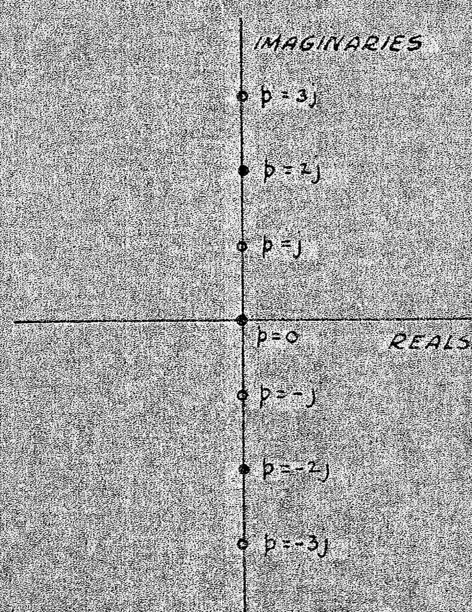

FIG. 37

Let us see how one may obtain the partial fraction expansions of Foster and hence the resulting minimal forms. To obtain the network of parallel resonant circuits, one expands $1/Z(p)$ in partial fractions. Thus

$$\frac{1}{Z(p)} = \frac{p(p^2+4)}{(p^2+1)(p^2+9)} \tag{124}$$

Expanding this in partial fractions, we have

$$\frac{p(p^2+4)}{(p^2+1)(p^2+9)} = \frac{Ap}{p^2+1} + \frac{Bp}{p^2+9} \tag{125}$$

$$\therefore p(p^2+4) = Ap(p^2+9) + Bp(p^2+1)$$

$$p^3+4p = (A+B)p^3 + (9A+B)p$$

$$\therefore A+B = 1$$

$$9A+B = 4$$

and

$$8A = 3$$

$$\therefore A = \frac{3}{8}$$

$$B = \frac{5}{8}$$

Hence the partial fractions are

$$\frac{\frac{3}{8}p}{p^2+1} + \frac{\frac{5}{8}p}{p^2+9} \tag{126}$$

But let us see what these two partial fractions mean. Note that a series circuit of inductance and capacity as shown in figure 38

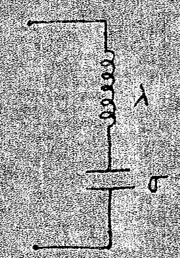

FIG.38

has an impedance

$$Z(p) = \lambda p + \frac{\sigma}{p} \tag{127}$$

$$= \frac{\lambda p^2 + \sigma}{p} \tag{127a}$$

And hence the admittance, which is $1/Z(p)$ is

$$A(p) = \frac{p}{\lambda p^2 + \sigma} \tag{128}$$

But note that the partial fractions in (126) are exactly of the form (128), and if we write (126) as follows

$$\frac{p}{\frac{8}{3}p^2 + \frac{8}{3}} + \frac{p}{\frac{8}{5}p^2 + \frac{72}{5}} \tag{126a}$$

then each term of the partial fraction expansion (126a) is exactly of the form (128) and hence each term represents an inductance and capacity element connected in series as shown in figure 38. Each term of (126a) represents the admittance of such a series circuit, and since admittances are added in parallel to give the total admittance, these series circuits represented by each term of (126a) are connected in parallel. The elements of these resonant circuits can be obtained by comparison with (128) and figure 38. Thus the first partial fraction of (126a) is the admittance of the resonant circuit

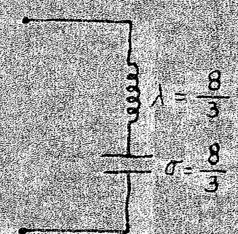

FIG. 39

shown in figure 39. The second partial fraction is the admittance of the resonant circuit shown in figure 40.

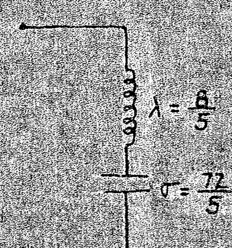

FIG. 40

Hence the total admittance, which is given by (126) is merely the sum of the admittance of the two resonant circuits shown respectively in figures 39 and 40. Hence a network consisting of two resonant circuits and having (124) for an admittance or (123) ~~and~~ for an impedance, is obtained by connecting the two circuits shown in figures 39 and 40 in parallel, as shown in figure 41.

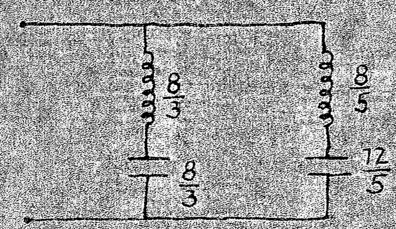

FIG. 41

The impedance of the network shown in figure 41 can be readily obtained and shown that it is exactly (123). This network is of course one of the minimal forms.

Now let us obtain, in a similar way, the partial expansion of the impedance function itself, and obtain another minimal form. Writing down the impedance itself, we have

$$Z(p) = \frac{p^4 + 10p^2 + 9}{p^3 + 4p}$$

Before expanding, it is necessary to divide the denominator of (123) into its numerator

$$\begin{array}{l|l} p^4 + 10p^2 + 9 & p^3 + 4p \\ \cline{2-2} p^4 + 4p^2 & p \\ \hline \quad 6p^2 + 9 & \end{array}$$

Hence

$$Z(p) = p + \frac{6p^2+9}{p^3+4p} \qquad (129)$$

Expanding the second term of (129), we have

$$\frac{6p^2+9}{p(p^2+4)} = \frac{A}{p} + \frac{Bp}{p^2+4}$$

$$\therefore \; 6p^2+9 = Ap^2+4A+Bp^2$$

$$\therefore \quad A+B=6$$

$$4A=9$$

And

$$A = \frac{9}{4}$$

$$B = \frac{15}{4}$$

Hence Z(p) in (129) may be written

$$Z(p) = p + \frac{\frac{9}{4}}{p} + \frac{\frac{15}{4}p}{p^2+4} \qquad (130)$$

But note that the first two terms of (130) represents the impedance of a series circuit of inductance and capacity elements as shown in figure 38, the impedance of which is given by (127) which is exactly of the form of the first two terms of (130). Thus the first two terms of (130) represents the circuit shown in figure 42

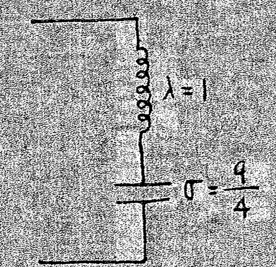

FIG. 42

Let us see however what the last term of (130) represents. Consider the impedance of the resonant circuit shown in figure 43, which consists of an inductance and capacity element in parallel

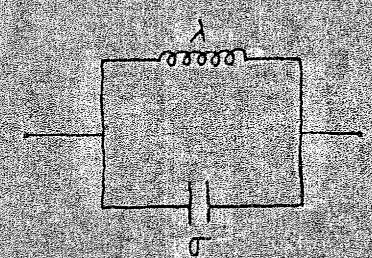

FIG. 43

The impedance of this circuit is

$$Z(p) = \frac{(\lambda p)\left(\frac{\sigma}{p}\right)}{\lambda p + \frac{\sigma}{p}} \qquad (131)$$

$$= \frac{\lambda \sigma p}{\lambda p^2 + \sigma}$$

$$= \frac{p}{\frac{1}{\sigma} p^2 + \frac{1}{\lambda}} \qquad (132)$$

But note that the last term of (130) is exactly of the form (132). Writing the last term of (130) in the form

$$\frac{p}{\frac{4}{15} p^2 + \frac{16}{15}} \qquad (133)$$

and comparing (133) with (132), we see that $\sigma = \frac{15}{4}$ and $\lambda = \frac{15}{16}$. The anti-resonant circuit represented by (133) is shown in figure 44.

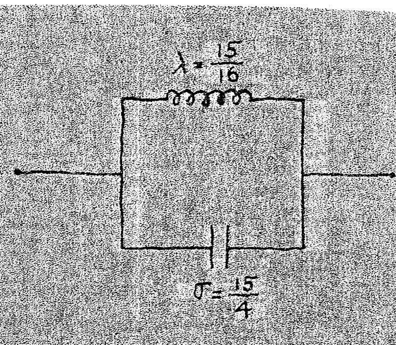

FIG. 44

The total impedance (130) is then the impedance of the circuits shown in figures 42 and 44 connected in series, resulting in a second minimal form, shown in figure 45.

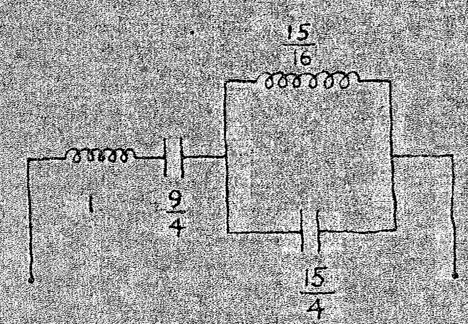

FIG.45

The impedance of the network of figure 45 can be readily obtained and seen to be exactly (123).

The networks shown in figure 41 and figure 45 are respectively the resonant and anti-resonant circuits that Foster obtains by his partial fraction expansion. As he states in his Reactance Theorem, page 262, these are the only two that his formulas will give. While Foster's method is illustrated here for a two-mesh network, the method is exactly the same for a network of any number of meshes.

At this point, Cauer proceeds by continued fraction expansion to obtain two more canonical forms having an impedance given by (123). To illustrate Cauer's method, let us first expand the admittance (124) into a continued fraction. Before doing this let us see how a ladder type network gives rise to a continued fraction. Thus the network shown in figure 46, which represents a most general ladder network, can have its impedance expressed by the continued fraction (134)

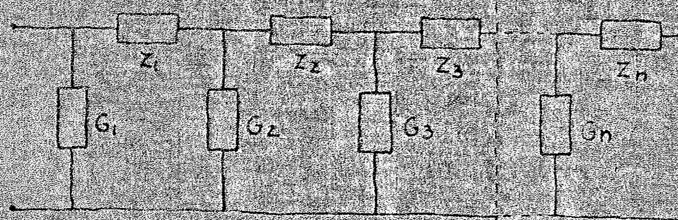

FIG. 46

$$Z(p) = \cfrac{1}{G_1 + \cfrac{1}{Z_1 + \cfrac{1}{G_2 + \cfrac{1}{Z_2 + \cfrac{1}{G_3 + \cfrac{1}{Z_3 + \cdots}}}}}} \qquad (134)$$

This is written more conveniently as follows[16]

$$Z(p) = \frac{1}{|G_1} + \frac{1|}{|Z_1} + \frac{1|}{|G_2} + \frac{1|}{|Z_2} + \frac{1|}{|G_3} + \frac{1|}{|Z_3} + \cdots \qquad (134a)$$

16. For a complete discussion of continued fractions see O. Perron, "Die Lehre von den Kettenbrüchen" 1913. See also E.B. Van Vleck, "Divergent Series and Continued Fractions" appearing in the Boston Colloquium Lectures on Mathematics, 1905. T. C. Fry has recently written an interesting paper on "The Use of Continued Fractions in the Design of Electrical Networks", Bulletin of the American Mathematical Society, vol. XXV, 1929, pp. 463-498, where he makes use of some important theorems of Stieltjes, which may be found in Stieltjes "Oeuvres Complètes", Tome II, 1918.

The admittance of the network shown in figure 46a can likewise be expressed readily by the continued fraction (135).

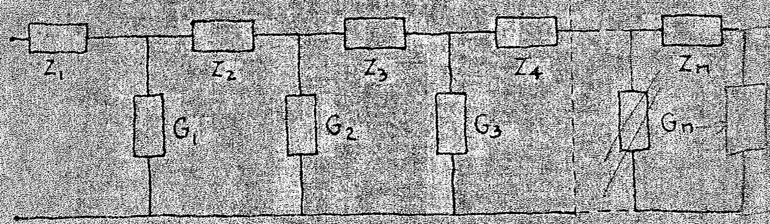

FIG. 46 a

$$G(p) = \cfrac{1}{Z_1 + \cfrac{1}{G_1 + \cfrac{1}{Z_2 + \cfrac{1}{G_2 + \cfrac{1}{Z_3 + \cfrac{1}{G_3 + ---}}}}}} \qquad (135)$$

or more conveniently written

$$G(p) = \frac{1}{Z_1} + \frac{1}{G_1} + \frac{1}{Z_2} + \frac{1}{G_2} + \frac{1}{Z_3} + \frac{1}{G_3} + ---- \qquad (135a)$$

In particular, when $Z_1, Z_2, Z_3 ------ Z_n$ are pure inductances and $G_1, G_2, G_3 ----- G_n$ are pure capacities, the admittance G(p) becomes

$$G(p) = \cfrac{1}{\lambda_1 p + \cfrac{1}{C_1 p + \cfrac{1}{\lambda_2 p + \cfrac{1}{C_2 p + \cfrac{1}{\lambda_3 p + \cfrac{1}{C_3 p + \cdots}}}}}} \qquad (136)$$

where λ_j and C_j are respectively the inductances and capacities of the various branches. Figure 47 shows a network having (136) for its admittance function.

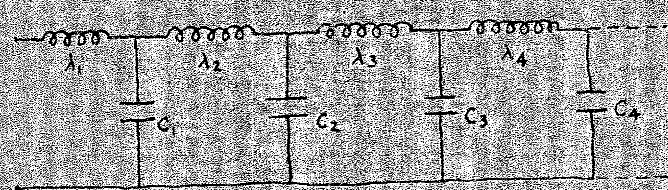

FIG. 47

Let us see if we can now expand the admittance function (124), which may be written

$$G(p) = \frac{p^3 + 4p}{p^4 + 10p^2 + 9} \qquad (124a)$$

into a continued fraction. If we can expand (124a) into a continued fraction of the form (136), it is a simple matter to obtain the values of the elements of the corresponding network, and so build the corresponding network. Expanding (124a) into a continued fraction, we have

$$G(p) = \frac{p^3 + 4p}{p^4 + 10p^2 + 9} \qquad (124a)$$

$$= \frac{1}{\dfrac{p^4 + 10p^2 + 9}{p^3 + 4p}} \qquad (137)$$

Dividing $p^4 + 10p^2 + 9$ by $p^3 + 4p$, we have

$$\begin{array}{l|l} p^4 + 10p^2 + 9 & p^3 + 4p \\ \cline{2-2} p^4 + 4p^2 & p \\ \cline{1-1} \quad 6p^2 + 9 & \end{array}$$

Hence $$\frac{p^4 + 10p^2 + 9}{p^3 + 4p} = p + \frac{6p^2 + 9}{p^3 + 4p}$$

Hence (137) becomes

$$G(p) = \frac{1}{p + \dfrac{6p^2 + 9}{p^3 + 4p}} \qquad (138)$$

$$= \frac{1}{p + \dfrac{1}{\dfrac{p^3 + 4p}{6p^2 + 9}}} \qquad (139)$$

Again dividing $p^3 + 4p$ by $6p^2 + 9$, we have

$$\frac{p^3 + 4p}{6p^2 + 9} = \frac{1}{6} p + \frac{\frac{5}{2} p}{6p^2 + 9}$$

Hence (139) becomes

$$G(p) = \cfrac{1}{p + \cfrac{1}{\frac{1}{6}p + \cfrac{\frac{5}{2}p}{6p^2+9}}} \qquad (140)$$

$$= \cfrac{1}{p + \cfrac{1}{\frac{1}{6}p + \cfrac{1}{\cfrac{6p^2+9}{\frac{5}{2}p}}}} \qquad (141)$$

Dividing $6p^2+9$ by $\frac{5}{2}p$, we have

$$\frac{6p^2+9}{\frac{5}{2}p} = \frac{12}{5}p + \frac{9}{\frac{5}{2}p}$$

and (141) becomes

$$G(p) = \cfrac{1}{p + \cfrac{1}{\frac{1}{6}p + \cfrac{1}{\frac{12}{5}p + \cfrac{9}{\frac{5}{2}p}}}}$$

$$= \cfrac{1}{p+ \cfrac{1}{\frac{1}{6}p + \cfrac{1}{\frac{12}{5}p + \cfrac{1}{\frac{5}{18}p}}}} \qquad (142)$$

But note that (142) is exactly of the form of (136). Hence the corresponding network is shown in figure 48, and the elements of the network are obtained directly from the elements of the continued fraction (142), on comparison with (136). This network is of course another minimal

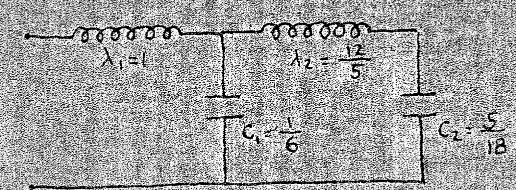

FIG. 48

form.

The c terms in (136) represent now capacity and not elastance.

While Cauer's method of continued fraction expansion has not been followed exactly in the above, sufficient has been given to exhibit the general idea of his method. It is possible to expand the impedance function or admittance function in various ways as a continued fraction, but I have used the above method to show how the admittance can easily be expanded as a continued fraction, without being compelled to go to various formulas, such as Cauer gives, for the expansion.

Finally, the network shown in figure 49 may be expressed as a continued fraction since it is a special case of the network shown in figure 46.

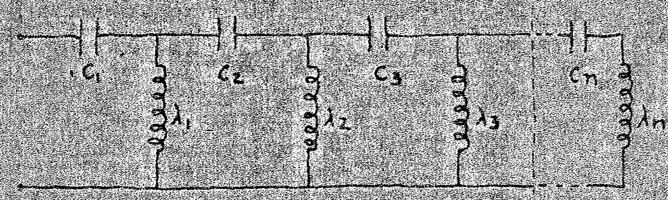

FIG. 49

For this network, if we let $\frac{1}{c_i p} = \sigma_i q$ and $\frac{1}{\lambda_i p} = \mu_i q$ then

$$G(q) = \cfrac{1}{\sigma_1 q + \cfrac{1}{\mu_1 q + \cfrac{1}{\sigma_2 q + \cfrac{1}{\mu_2 q + \cfrac{1}{\sigma_3 q + \cfrac{1}{\mu_3 q + \cdots}}}}}} \qquad (143)$$

Hence, conversely, by making a similar transformation from p to q in (124a) and proceeding as before, a two-mesh network of the form of network shown in figure 49 is obtained, the elements of the network being obtained from the elements of the continued fraction. This network, the fourth minimal form, is shown in figure 50.

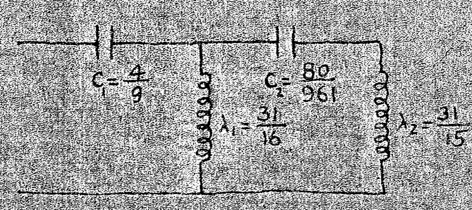

FIG. 50

While we have treated only the capacity and inductance network to illustrate the separation properties of the zeros and poles, and the partial fraction and continued fraction expansion of the network, the same methods apply to networks containing inductance and resistance elements and resistance and capacity elements. As we have said, in the latter two cases, the zeros and poles are negative reals and separate each other. The above methods also hold, with more complexity, to networks of any number of meshes having two kinds of elements present.

It will be useful here to state some of the important theorems of Stieltjes on continued fractions as given by Fry[17]. First, a Stieltjes fraction

$$f(z) = \cfrac{1}{a_1 z + \cfrac{1}{a_2 z + \cfrac{1}{a_3 z + \cdots}}}$$

converges for every value of the complex variable z, except perhaps negative real values, provided the a's are positive reals, and provided

$$\sum_1^\infty a_j$$

diverges.[18] Second, the function $f(z)$ defined by a convergent

17. T. C. Fry, loc. cit.pp. 466 and 467.

18. Stieltjes, Oeuvres Complètes, vol. II, 1918, p. 465 or Ann. Fac. Sci. Toulouse, 8,1894, J.1-122, 9, 1895, A.1-47.

Stieltjes fraction can always be expressed in the form

$$f(z) = \int_0^\infty \frac{d\phi(x)}{z+x}$$

where $\phi(x)$ is a monotonic non-decreasing real function of the real variable where $\phi(0)=0$ and $\phi(\infty)=\frac{1}{a_1}$ It is not necessary that $\phi(x)$ be continuous.[19]

Third, the function $f(z)$ defined by a convergent Stieltjes fraction is a regular analytic function, except at certain points of the negative real axis,[20] and is real for positive real values of Z. The same is true of its reciprocal. Fourth, the function $\phi(x)$ is related to $f(z)$ by the law

$$\phi(x) = \frac{1}{2\pi i}\int f(z)\,dz$$

the path of integration extending along a circle of radius x about the origin, beginning at $-x-i0$ and ending at $-x+i0$.[21]

Fifth, conversely, if $\phi(x)$ is any monotonic non-decreasing real function of x in the interval (o,b), and constant for $x \geqq b$ the function $f(z)$ defined by

$$f(z) = \int_0^\infty \frac{d\phi(x)}{z+x}$$

possesses a convergent Stieltjes expansion with positive real coefficients.[22] Finally, if $\phi(x)$ is any monotonic non-decreasing

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19. Stieltjes, Oeuvres Complètes, vol. II, 1918, pp. 491, 493.

20. Perron, loc. cit. p. 369.

21. Perron, loc. cit. p. 372.

22. Perron, loc. cit. p. 388.

function of x, such that the integrals

$$\int_0^\infty x^n \, d\phi(x)$$

all exist, the function $f(z)$ defined by

$$f(z) = \int_0^\infty \frac{d\phi(x)}{z+x}$$

possesses a formal Stieltjes expansion with positive real coefficients, which, if it converges at all, converges to the value $f(z)$ for all values of Z except those on the negative real axis.[23].

These theorems of Stieltjes have been given because they are of considerable importance in one type of network. Ladder-type networks form of course a sub-group in the complete infinite group of networks having the same impedance function (which is an absolute invariant of the group). The ladder-type network, the impedance or admittance of which may readily be written as a continued fraction, can be treated exactly as any network of the group, and its one, two or three matrices containing the coefficients of the fundamental quadratic forms of the network, may be written down at once from the elements of the network, or just as well, from the elements of the continued fraction.

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23. Stieltjes, loc. cit. p. 504.

The present mathematical theory of continued fractions has essentially to do with infinite continued fractions and their convergence.[24] A ladder-type network with an infinite number of meshes, that is an infinite number of degrees of freedom, is of course represented by an infinite continued fraction. The matrices of the coefficients of the corresponding quadratic forms will be matrices with an infinite number of elements. The convergence of the one, two or three infinite quadratic forms of the ladder network will determine the convergence of the infinite continued fraction.

Finite continued fractions have received little treatment, although they are important for electric circuit theory in that any finite ladder-type network can be represented, as we have seen, by a finite continued fraction. It would be very useful to be able to determine the elements of a continued fraction from its n^{th} convergent. Thus, for example, it would be nice in general to be able to obtain directly the elements of the continued fraction (142) from its convergent (124a), without having to go through the process of division, as we have done. However, as we have pointed out, finite ladder-type networks form a sub-group in our infinite

---

24. For example, the Stieltjes theorems given above and practically all of Perron, is concerned essentially with infinite continued fractions.

group of networks having the same impedance function, and so can be treated as any network, although the already existing body of knowledge of continued fractions may be of considerable aid in clarification and simplification.[25]

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25. E.B. Van Vleck, in "Divergent Series and Continued Fractions", loc. cit. p. 93 says back in 1903 "Unquestionably the instrument by which greatest progress has been made thus far is the integral. The first successes, however, were reached by Laguerre and Stieltjes through the use of continued fractions, and very possibly, in the end, the continued fraction will prove to be the best as it was the earliest tool".

CHAPTER IV

The Impedance Function for Network of n-Meshes with Two Kinds of Elements.

We have seen that the impedance function of a two mesh network containing two kinds of network elements could be expressed in very convenient form by means of our symbolic notation. In this form, the coefficients of the impedance function could be written down at once from an inspection of the network elements, thus saving considerable labor in the computation of the impedance function. Let us extend the above method of expressing the impedance function to networks of any number of meshes, containing but two kinds of elements. Later, extensions will be made to networks of any number of meshes containing all three elements.

The determinant of the three mesh network containing inductance and resistance elements is

$$D(p) = \begin{vmatrix} \lambda_{11}p+\rho_{11} & \lambda_{12}p+\rho_{12} & \lambda_{13}p+\rho_{13} \\ \lambda_{12}p+\rho_{12} & \lambda_{22}p+\rho_{22} & \lambda_{23}p+\rho_{23} \\ \lambda_{13}p+\rho_{13} & \lambda_{23}p+\rho_{23} & \lambda_{33}p+\rho_{33} \end{vmatrix} \qquad (144)$$

All the elements of D(p) are taken as positive, although the non-diagonal terms may be positive or negative depending upon the assumed direction of the mesh currents. Expanding D(p), we have

$$D(p) = \begin{vmatrix} \lambda_{11}p & \lambda_{12}p+\rho_{12} & \lambda_{13}p+\rho_{13} \\ \lambda_{12}p & \lambda_{22}p+\rho_{22} & \lambda_{23}p+\rho_{23} \\ \lambda_{13}p & \lambda_{23}p+\rho_{23} & \lambda_{33}p+\rho_{33} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12}p+\rho_{12} & \lambda_{13}p+\rho_{13} \\ \rho_{12} & \lambda_{22}p+\rho_{22} & \lambda_{23}p+\rho_{23} \\ \rho_{13} & \lambda_{23}p+\rho_{23} & \lambda_{33}p+\rho_{33} \end{vmatrix}$$

$$= \begin{vmatrix} \lambda_{11}p & \lambda_{12}p & \lambda_{13}p+\rho_{13} \\ \lambda_{12}p & \lambda_{22}p & \lambda_{23}p+\rho_{23} \\ \lambda_{13}p & \lambda_{23}p & \lambda_{33}p+\rho_{33} \end{vmatrix} + \begin{vmatrix} \lambda_{11}p & \rho_{12} & \lambda_{13}p+\rho_{13} \\ \lambda_{12}p & \rho_{22} & \lambda_{23}p+\rho_{23} \\ \lambda_{13}p & \rho_{23} & \lambda_{33}p+\rho_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} \rho_{11} & \lambda_{12}p & \lambda_{13}p+\rho_{13} \\ \rho_{12} & \lambda_{22}p & \lambda_{23}p+\rho_{23} \\ \rho_{13} & \lambda_{23}p & \lambda_{33}p+\rho_{33} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \rho_{12} & \lambda_{13}p+\rho_{13} \\ \rho_{12} & \rho_{22} & \lambda_{23}p+\rho_{23} \\ \rho_{13} & \rho_{23} & \lambda_{33}p+\rho_{33} \end{vmatrix}$$

$$= \begin{vmatrix} \lambda_{11}p & \lambda_{12}p & \lambda_{13}p \\ \lambda_{12}p & \lambda_{22}p & \lambda_{23}p \\ \lambda_{13}p & \lambda_{23}p & \lambda_{33}p \end{vmatrix} + \begin{vmatrix} \lambda_{11}p & \lambda_{12}p & \rho_{13} \\ \lambda_{12}p & \lambda_{22}p & \rho_{23} \\ \lambda_{13}p & \lambda_{23}p & \rho_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} \lambda_{11}p & \rho_{12} & \lambda_{13}p \\ \lambda_{12}p & \rho_{22} & \lambda_{23}p \\ \lambda_{13}p & \rho_{23} & \lambda_{33}p \end{vmatrix} + \begin{vmatrix} \lambda_{11}p & \rho_{12} & \rho_{13} \\ \lambda_{12}p & \rho_{13} & \rho_{23} \\ \lambda_{13}p & \rho_{23} & \rho_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} \rho_{11} & \lambda_{12}p & \lambda_{13}p \\ \rho_{12} & \lambda_{22}p & \lambda_{23}p \\ \rho_{13} & \lambda_{23}p & \lambda_{33}p \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12}p & \rho_{13} \\ \rho_{12} & \lambda_{22}p & \rho_{23} \\ \rho_{13} & \lambda_{23}p & \rho_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} \rho_{11} & \rho_{12} & \lambda_{13}p \\ \rho_{12} & \rho_{22} & \lambda_{23}p \\ \rho_{13} & \rho_{23} & \lambda_{33}p \end{vmatrix} + \begin{vmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{12} & \rho_{22} & \rho_{23} \\ \rho_{13} & \rho_{23} & \rho_{33} \end{vmatrix}$$

$$\left.\begin{aligned} &= \begin{vmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{vmatrix} p^3 + \left\{ \begin{vmatrix} \lambda_{11} & \lambda_{12} & \rho_{13} \\ \lambda_{12} & \lambda_{22} & \rho_{23} \\ \lambda_{13} & \lambda_{23} & \rho_{33} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \rho_{12} & \lambda_{13} \\ \lambda_{12} & \rho_{22} & \lambda_{23} \\ \lambda_{13} & \rho_{23} & \lambda_{33} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} & \lambda_{13} \\ \rho_{12} & \lambda_{22} & \lambda_{23} \\ \rho_{13} & \lambda_{23} & \lambda_{33} \end{vmatrix} \right\} p^2 \\ &+ \left\{ \begin{vmatrix} \rho_{11} & \rho_{12} & \lambda_{13} \\ \rho_{12} & \rho_{22} & \lambda_{23} \\ \rho_{13} & \rho_{23} & \lambda_{33} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} & \rho_{13} \\ \rho_{12} & \lambda_{22} & \rho_{23} \\ \rho_{13} & \lambda_{23} & \rho_{33} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \rho_{12} & \rho_{13} \\ \lambda_{12} & \rho_{22} & \rho_{23} \\ \lambda_{13} & \rho_{23} & \rho_{33} \end{vmatrix} \right\} p + \begin{vmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{12} & \rho_{22} & \rho_{23} \\ \rho_{13} & \rho_{23} & \rho_{33} \end{vmatrix} \end{aligned}\right\} \quad (45)$$

As in the two-mesh case, we can use our symbolic notation except that the Δ's will now be third order determinants and the M's will be second order ones. Thus let

$$\Delta(\lambda) = \begin{vmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{vmatrix} \qquad \Delta(\rho) = \begin{vmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{12} & \rho_{22} & \rho_{23} \\ \rho_{13} & \rho_{23} & \rho_{33} \end{vmatrix} \tag{146}$$

$$\Delta_1(\lambda,\rho) = \begin{vmatrix} \lambda_{11} & \lambda_{12} & \rho_{13} \\ \lambda_{12} & \lambda_{22} & \rho_{23} \\ \lambda_{13} & \lambda_{23} & \rho_{33} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \rho_{12} & \lambda_{13} \\ \lambda_{12} & \rho_{22} & \lambda_{23} \\ \lambda_{13} & \rho_{23} & \lambda_{33} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} & \lambda_{13} \\ \rho_{12} & \lambda_{22} & \lambda_{23} \\ \rho_{13} & \lambda_{23} & \lambda_{33} \end{vmatrix} \tag{147}$$

$$\Delta_1(\rho,\lambda) = \begin{vmatrix} \rho_{11} & \rho_{12} & \lambda_{13} \\ \rho_{12} & \rho_{22} & \lambda_{23} \\ \rho_{13} & \rho_{23} & \lambda_{33} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} & \rho_{13} \\ \rho_{12} & \lambda_{22} & \rho_{23} \\ \rho_{13} & \lambda_{23} & \rho_{33} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \rho_{12} & \rho_{13} \\ \lambda_{12} & \rho_{22} & \rho_{23} \\ \lambda_{13} & \rho_{23} & \rho_{33} \end{vmatrix} \tag{148}$$

It is to be noted that the coefficients of the determinant of the network can be formed from the matrices of the coefficients of the inductance and resistance quadratic forms, which matrices are for the three mesh case, respectively

$$\begin{Vmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{Vmatrix} \qquad \begin{Vmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{12} & \rho_{22} & \rho_{23} \\ \rho_{13} & \rho_{23} & \rho_{33} \end{Vmatrix}$$

$\Delta(\lambda)$ is thus the determinant of the λ matrix and $\Delta(\rho)$ is the determinant of the ρ matrix. $\Delta_1(\lambda,\rho)$ and $\Delta_1(\rho,\lambda)$ are readily seen to be formed from the two determinants in a manner similar to that of $\Delta_1(\lambda,\rho)$ in the two-mesh case. That is $\Delta_1(\lambda,\rho)$ is formed from $\Delta(\lambda)$ by replacing one column at a time the terms in $\Delta(\lambda)$ by corresponding ρ terms, and then adding the resulting three determinants formed. Thus,

$$\Delta(\lambda) = \begin{vmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{vmatrix}$$

Replacing the third column of the λ terms in $\Delta(\lambda)$ by ρ terms, we have

$$\begin{vmatrix} \lambda_{11} & \lambda_{12} & \rho_{13} \\ \lambda_{12} & \lambda_{22} & \rho_{23} \\ \lambda_{13} & \lambda_{23} & \rho_{33} \end{vmatrix}$$

Now replacing the second column of λ terms in $\Delta(\lambda)$ by ρ terms, we have

$$\begin{vmatrix} \lambda_{11} & \rho_{12} & \lambda_{13} \\ \lambda_{12} & \rho_{22} & \lambda_{23} \\ \lambda_{13} & \rho_{23} & \lambda_{33} \end{vmatrix}$$

Finally, replacing the first column of the λ terms in $\Delta(\lambda)$ by ρ terms, we have

$$\begin{vmatrix} \rho_{11} & \lambda_{12} & \lambda_{13} \\ \rho_{12} & \lambda_{22} & \lambda_{23} \\ \rho_{13} & \lambda_{23} & \lambda_{33} \end{vmatrix}$$

Adding these three mixed determinants we arrive at $\Delta_1(\lambda,\rho)$ thus

$$\Delta_1(\lambda,\rho) = \begin{vmatrix} \lambda_{11} & \lambda_{12} & \rho_{13} \\ \lambda_{12} & \lambda_{22} & \rho_{23} \\ \lambda_{13} & \lambda_{23} & \rho_{33} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \rho_{12} & \lambda_{13} \\ \lambda_{12} & \rho_{22} & \lambda_{23} \\ \lambda_{13} & \rho_{23} & \lambda_{33} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} & \lambda_{13} \\ \rho_{12} & \lambda_{22} & \lambda_{23} \\ \rho_{13} & \lambda_{23} & \lambda_{33} \end{vmatrix} \qquad (147)$$

The numeral to the right of Δ indicates that we are to replace the λ terms in $\Delta(\lambda)$ a column at a time. The λ term is placed first in the parenthesis of $\Delta_1(\lambda,\rho)$ to indicate that we are to begin with $\Delta(\lambda)$ and then replace the λ terms in $\Delta(\lambda)$ a column at a time by ρ terms, which replacement by ρ terms is indicated by the ρ term in the parenthesis of $\Delta_1(\lambda,\rho)$

The meaning of $\Delta_1(\rho,\lambda)$ is made clear from the above explanation. We begin now with $\Delta(\rho)$ and replace a column at a time, the ρ terms in $\Delta(\rho)$ by λ terms and add the resulting three mixed determinants formed. Unlike the two-mesh case

$$\Delta_1(\lambda,\rho) \neq \Delta_1(\rho,\lambda)$$

in general.

Using this symbolic notation, the determinant of the network (145) becomes[26]

$$D(p) = \Delta(\lambda)\, p^3 + \Delta_1(\lambda,\rho)\, p^2 + \Delta_1(\rho,\lambda)\, p + \Delta(\rho) \qquad (145a)$$

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26. Note the similarity of (145a) with (4) p. 164 in M. Bocher's Introduction to Higher Algebra, 1927, where he considers the theorem: If two conics intersect in four and only four distinct points, there exists a non-singular collineation which reduces their equations to the normal form. Note that his discriminant (3) is our D(p), equation (144) page 138 His a terms correspond to our λ terms and his b terms to our ρ terms, and his λ to our p. He uses the minus sign, we use the plus sign. Note also his Δ', Θ', Θ, and Δ in (4), correspond respectively to our $\Delta(\lambda)$, $\Delta_1(\lambda,\rho)$ $\Delta_1(\rho,\lambda)$ and $\Delta(\rho)$. He also shows (page 166) that his coefficients Θ, Θ', Δ and Δ' are invariants of weight 2. This we have seen to be the case in Chapter III for the two-mesh network with two kinds of network elements. It is also true for the first minor of D(p). Thus it is that the impedance function, which is the ratio of two relative invariants of weight 2 is itself an <u>absolute</u> invariant. Later we shall see that the above invariance properties hold just as well for the two-mesh network, and for the n-mesh network with all three kinds of elements present.

The minor of the element in the first row and first column of D(p) is obtained from (144) and is of course given by

$$M_{11}(p) = \begin{vmatrix} \lambda_{22}p+\rho_{22} & \lambda_{23}p+\rho_{23} \\ \lambda_{23}p+\rho_{23} & \lambda_{33}p+\rho_{33} \end{vmatrix} \qquad (149)$$

But (149) is exactly (87) page 64, which is the determinant of a two-mesh network containing inductance and resistance elements. Hence we can use (89), page 65 except that in place of Δ we use M_{11} since (149) is not now considered as the determinant of a two-mesh network, but rather the minor of the element in the first row and first column of a three mesh network. Thus, without going through the process of expanding, (149) may be written

$$M_{11}(p) = M_{11}(\lambda)p^2 + M_{11}^{(1)}(\lambda,\rho) + M_{11}(\rho) \qquad (149a)$$

The small index (1) is placed above and slight to the right of M_{11} has the same meaning as the index 1 in $\Delta_1(\lambda,\rho)$. It is placed thus $M_{11}^{(1)}(\lambda,\rho)$ in order to avoid confusion which would occur if it were placed as in $\Delta_1(\lambda,\rho)$.

The impedance function then becomes

$$Z(p) = \frac{D(p)}{M_{11}(p)}$$

$$= \frac{\Delta(\lambda)p^3 + \Delta_1(\lambda,\rho)p^2 + \Delta_1(\rho,\lambda)p + \Delta(\rho)}{M_{11}(\lambda)p^2 + M_{11}^{(1)}(\lambda,\rho)p + M_{11}(\rho)} \qquad (150)$$

Thus (150) is the impedance function of the most general 3 mesh network containing inductance and resistance elements.

Now let us see what the impedance function of the most general four-mesh network containing resistance and inductance elements. The determinant of such a network is

$$D(p) = \begin{vmatrix} \lambda_{11}p+\rho_{11} & \lambda_{12}p+\rho_{12} & \lambda_{13}p+\rho_{13} & \lambda_{14}p+\rho_{14} \\ \lambda_{12}p+\rho_{12} & \lambda_{22}p+\rho_{22} & \lambda_{23}p+\rho_{23} & \lambda_{24}p+\rho_{24} \\ \lambda_{13}p+\rho_{13} & \lambda_{23}p+\rho_{23} & \lambda_{33}p+\rho_{33} & \lambda_{34}p+\rho_{34} \\ \lambda_{14}p+\rho_{14} & \lambda_{24}p+\rho_{24} & \lambda_{34}p+\rho_{34} & \lambda_{44}p+\rho_{44} \end{vmatrix} \qquad (151)$$

It will be useful in expanding this determinant to use the more compact determinant notation. Thus let $|a_{11}\ a_{12}|$ represent the complete determinant

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}$$

and $\quad |a_{11} \;\; a_{12} \;\; a_{13}|$ $\quad$ the complete determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

With this notation D(p) may be written

$$D(p) = |\lambda_{11}p+\rho_{11} \quad \lambda_{12}p+\rho_{12} \quad \lambda_{13}p+\rho_{13} \quad \lambda_{14}p+\rho_{14}|$$

Expanding D(p) we have

$$D(p) = |\lambda_{11}p \quad \lambda_{12}p+\rho_{12} \quad \lambda_{13}p+\rho_{13} \quad \lambda_{14}p+\rho_{14}| + |\rho_{11} \quad \lambda_{12}p+\rho_{12} \quad \lambda_{13}p+\rho_{13} \quad \lambda_{14}p+\rho_{14}|$$

$$= |\lambda_{11}p \quad \lambda_{12}p \quad \lambda_{13}p+\rho_{13} \quad \lambda_{14}p+\rho_{14}| + |\lambda_{11}p \quad \rho_{12} \quad \lambda_{13}p+\rho_{13} \quad \lambda_{14}p+\rho_{14}|$$

$$+ |\rho_{11} \quad \lambda_{12}p \quad \lambda_{13}p+\rho_{13} \quad \lambda_{14}p+\rho_{14}| + |\rho_{11} \quad \rho_{12} \quad \lambda_{13}p+\rho_{13} \quad \lambda_{14}p+\rho_{14}|$$

$$= |\lambda_{11}p \quad \lambda_{12}p \quad \lambda_{13}p \quad \lambda_{14}p+\rho_{14}| + |\lambda_{11}p \quad \lambda_{12}p \quad \rho_{13} \quad \lambda_{14}p+\rho_{14}|$$

$$+ |\lambda_{11}p \quad \rho_{12} \quad \lambda_{13}p \quad \lambda_{14}p+\rho_{14}| + |\lambda_{11}p \quad \rho_{12} \quad \rho_{13} \quad \lambda_{14}p+\rho_{14}|$$

$$+ |\rho_{11} \quad \lambda_{12}p \quad \lambda_{13}p \quad \lambda_{14}p+\rho_{14}| + |\rho_{11} \quad \lambda_{12}p \quad \rho_{13} \quad \lambda_{14}p+\rho_{14}|$$

$$+ |\rho_{11} \quad \rho_{12} \quad \lambda_{13}p \quad \lambda_{14}p+\rho_{14}| + |\rho_{11} \quad \rho_{12} \quad \rho_{13} \quad \lambda_{14}p+\rho_{14}|$$

$$= |\lambda_{11}p \;\; \lambda_{12}p \;\; \lambda_{13}p \;\; \lambda_{14}p| + |\lambda_{11}p \;\; \lambda_{12}p \;\; \lambda_{13}p \;\; \rho_{14}|$$

$$+ |\lambda_{11}p \;\; \lambda_{12}p \;\; \rho_{13} \;\; \lambda_{14}p| + |\lambda_{11}p \;\; \lambda_{12}p \;\; \rho_{13} \;\; \rho_{14}|$$

$$+ |\lambda_{11}p \;\; \rho_{12} \;\; \lambda_{13}p \;\; \lambda_{14}p| + |\lambda_{11}p \;\; \rho_{12} \;\; \lambda_{13}p \;\; \rho_{14}|$$

$$+ |\lambda_{11}p \;\; \rho_{12} \;\; \rho_{13} \;\; \lambda_{14}p| + |\lambda_{11}p \;\; \rho_{12} \;\; \rho_{13} \;\; \rho_{14}|$$

$$+ |\rho_{11} \;\; \lambda_{12}p \;\; \lambda_{13}p \;\; \lambda_{14}p| + |\rho_{11} \;\; \lambda_{12}p \;\; \lambda_{13}p \;\; \rho_{14}|$$

$$+ |\rho_{11} \;\; \lambda_{12}p \;\; \rho_{13} \;\; \lambda_{14}p| + |\rho_{11} \;\; \lambda_{12}p \;\; \rho_{13} \;\; \rho_{14}|$$

$$+ |\rho_{11} \;\; \rho_{12} \;\; \lambda_{13}p \;\; \lambda_{14}p| + |\rho_{11} \;\; \rho_{12} \;\; \lambda_{13}p \;\; \rho_{14}|$$

$$+ |\rho_{11} \;\; \rho_{12} \;\; \rho_{13} \;\; \lambda_{14}p| + |\rho_{11} \;\; \rho_{12} \;\; \rho_{13} \;\; \rho_{14}|$$

$$= |\lambda_{11} \lambda_{12} \lambda_{13} \lambda_{14}|\, p^4 + \left\{ |\lambda_{11} \lambda_{12} \lambda_{13} \rho_{14}| + |\lambda_{11} \lambda_{12} \rho_{13} \lambda_{14}| + |\lambda_{11} \rho_{12} \lambda_{13} \lambda_{14}| + |\rho_{11} \lambda_{12} \lambda_{13} \lambda_{14}| \right\} p^3$$

$$+ \left\{ |\lambda_{11} \lambda_{12} \rho_{13} \rho_{14}| + |\lambda_{11} \rho_{12} \lambda_{13} \rho_{14}| + |\lambda_{11} \rho_{12} \rho_{13} \lambda_{14}| + |\rho_{11} \lambda_{12} \lambda_{13} \rho_{14}| + |\rho_{11} \lambda_{12} \rho_{13} \lambda_{14}| + |\rho_{11} \rho_{12} \lambda_{13} \lambda_{14}| \right\} p^2$$

$$+ \left\{ |\rho_{11} \rho_{12} \rho_{13} \lambda_{14}| + |\rho_{11} \rho_{12} \lambda_{13} \rho_{14}| + |\rho_{11} \lambda_{12} \rho_{13} \rho_{14}| + |\lambda_{11} \rho_{12} \rho_{13} \rho_{14}| \right\} p + |\rho_{11} \rho_{12} \rho_{13} \rho_{14}|$$

Writing this in the usual determinant notation, we have

$$D(p) = \begin{vmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} & \lambda_{34} \\ \lambda_{14} & \lambda_{24} & \lambda_{34} & \lambda_{44} \end{vmatrix} p^4 + \left\{ \begin{vmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \beta_{14} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} & \beta_{24} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} & \beta_{34} \\ \lambda_{14} & \lambda_{24} & \lambda_{34} & \beta_{44} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \lambda_{12} & \beta_{13} & \lambda_{14} \\ \lambda_{12} & \lambda_{22} & \beta_{23} & \lambda_{24} \\ \lambda_{13} & \lambda_{23} & \beta_{33} & \lambda_{34} \\ \lambda_{14} & \lambda_{24} & \beta_{34} & \lambda_{44} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \beta_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{12} & \beta_{22} & \lambda_{23} & \lambda_{24} \\ \lambda_{13} & \beta_{23} & \lambda_{33} & \lambda_{34} \\ \lambda_{14} & \beta_{24} & \lambda_{34} & \lambda_{44} \end{vmatrix} + \begin{vmatrix} \beta_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \beta_{12} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\ \beta_{13} & \lambda_{23} & \lambda_{33} & \lambda_{34} \\ \beta_{14} & \lambda_{24} & \lambda_{34} & \lambda_{44} \end{vmatrix} \right\} p^3$$

$$+ \left\{ \begin{vmatrix} \lambda_{11} & \lambda_{12} & \beta_{13} & \beta_{14} \\ \lambda_{12} & \lambda_{22} & \beta_{23} & \beta_{24} \\ \lambda_{13} & \lambda_{23} & \beta_{33} & \beta_{34} \\ \lambda_{14} & \lambda_{24} & \beta_{34} & \beta_{44} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \beta_{12} & \lambda_{13} & \beta_{14} \\ \lambda_{12} & \beta_{22} & \lambda_{23} & \beta_{24} \\ \lambda_{13} & \beta_{23} & \lambda_{33} & \beta_{34} \\ \lambda_{14} & \beta_{24} & \lambda_{34} & \beta_{44} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \beta_{12} & \beta_{13} & \lambda_{14} \\ \lambda_{12} & \beta_{22} & \beta_{23} & \lambda_{24} \\ \lambda_{13} & \beta_{23} & \beta_{33} & \lambda_{34} \\ \lambda_{14} & \beta_{24} & \beta_{34} & \lambda_{44} \end{vmatrix} + \begin{vmatrix} \beta_{11} & \lambda_{12} & \lambda_{13} & \beta_{14} \\ \beta_{12} & \lambda_{22} & \lambda_{23} & \beta_{24} \\ \beta_{13} & \lambda_{23} & \lambda_{33} & \beta_{34} \\ \beta_{14} & \lambda_{24} & \lambda_{34} & \beta_{44} \end{vmatrix} + \begin{vmatrix} \beta_{11} & \lambda_{12} & \beta_{13} & \lambda_{14} \\ \beta_{12} & \lambda_{22} & \beta_{23} & \lambda_{24} \\ \beta_{13} & \lambda_{23} & \beta_{33} & \lambda_{34} \\ \beta_{14} & \lambda_{24} & \beta_{34} & \lambda_{44} \end{vmatrix} + \begin{vmatrix} \beta_{11} & \beta_{12} & \lambda_{13} & \lambda_{14} \\ \beta_{12} & \beta_{22} & \lambda_{23} & \lambda_{24} \\ \beta_{13} & \beta_{23} & \lambda_{33} & \lambda_{34} \\ \beta_{14} & \beta_{24} & \lambda_{34} & \lambda_{44} \end{vmatrix} \right\} p^2$$

$$+ \left\{ \begin{vmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \lambda_{14} \\ \beta_{12} & \beta_{22} & \beta_{23} & \lambda_{24} \\ \beta_{13} & \beta_{23} & \beta_{33} & \lambda_{34} \\ \beta_{14} & \beta_{24} & \beta_{34} & \lambda_{44} \end{vmatrix} + \begin{vmatrix} \beta_{11} & \beta_{12} & \lambda_{13} & \beta_{14} \\ \beta_{12} & \beta_{22} & \lambda_{23} & \beta_{24} \\ \beta_{13} & \beta_{23} & \lambda_{33} & \beta_{34} \\ \beta_{14} & \beta_{24} & \lambda_{34} & \beta_{44} \end{vmatrix} + \begin{vmatrix} \beta_{11} & \lambda_{12} & \beta_{13} & \beta_{14} \\ \beta_{12} & \lambda_{22} & \beta_{23} & \beta_{24} \\ \beta_{13} & \lambda_{23} & \beta_{33} & \beta_{34} \\ \beta_{14} & \lambda_{24} & \beta_{34} & \beta_{44} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \lambda_{12} & \beta_{22} & \beta_{23} & \beta_{24} \\ \lambda_{13} & \beta_{23} & \beta_{33} & \beta_{34} \\ \lambda_{14} & \beta_{24} & \beta_{34} & \beta_{44} \end{vmatrix} \right\} p$$

$$+ \begin{vmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{12} & \beta_{22} & \beta_{23} & \beta_{24} \\ \beta_{13} & \beta_{23} & \beta_{33} & \beta_{34} \\ \beta_{14} & \beta_{24} & \beta_{34} & \beta_{44} \end{vmatrix}$$

As before, we may now use our symbolic notation, the Δ's now being fourth order determinants and the M's third order ones.

Thus let

$$\Delta(\lambda) = \begin{vmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} & \lambda_{34} \\ \lambda_{14} & \lambda_{24} & \lambda_{34} & \lambda_{44} \end{vmatrix} \qquad \Delta(\rho) = \begin{vmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{12} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{13} & \rho_{23} & \rho_{33} & \rho_{34} \\ \rho_{14} & \rho_{24} & \rho_{34} & \rho_{44} \end{vmatrix}$$

$$\Delta_1(\lambda,\rho) = \begin{vmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \rho_{14} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} & \rho_{24} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} & \rho_{34} \\ \lambda_{14} & \lambda_{24} & \lambda_{34} & \rho_{44} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \lambda_{12} & \rho_{13} & \lambda_{14} \\ \lambda_{12} & \lambda_{22} & \rho_{23} & \lambda_{24} \\ \lambda_{13} & \lambda_{23} & \rho_{33} & \lambda_{34} \\ \lambda_{14} & \lambda_{24} & \rho_{34} & \lambda_{44} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \rho_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{12} & \rho_{22} & \lambda_{23} & \lambda_{24} \\ \lambda_{13} & \rho_{23} & \lambda_{33} & \lambda_{34} \\ \lambda_{14} & \rho_{24} & \lambda_{34} & \lambda_{44} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \rho_{12} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\ \rho_{13} & \lambda_{23} & \lambda_{33} & \lambda_{34} \\ \rho_{14} & \lambda_{24} & \lambda_{34} & \lambda_{44} \end{vmatrix}$$

$$\Delta_1(\rho,\lambda) = \begin{vmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \lambda_{14} \\ \rho_{12} & \rho_{22} & \rho_{23} & \lambda_{24} \\ \rho_{13} & \rho_{23} & \rho_{33} & \lambda_{34} \\ \rho_{14} & \rho_{24} & \rho_{34} & \lambda_{44} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \rho_{12} & \lambda_{13} & \rho_{14} \\ \rho_{12} & \rho_{22} & \lambda_{23} & \rho_{24} \\ \rho_{13} & \rho_{23} & \lambda_{33} & \rho_{34} \\ \rho_{14} & \rho_{24} & \lambda_{34} & \rho_{44} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} & \rho_{13} & \rho_{14} \\ \rho_{12} & \lambda_{22} & \rho_{23} & \rho_{24} \\ \rho_{13} & \lambda_{23} & \rho_{33} & \rho_{34} \\ \rho_{14} & \lambda_{24} & \rho_{34} & \rho_{44} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \lambda_{12} & \rho_{22} & \rho_{23} & \rho_{24} \\ \lambda_{13} & \rho_{23} & \rho_{33} & \rho_{34} \\ \lambda_{14} & \rho_{24} & \rho_{34} & \rho_{44} \end{vmatrix}$$

$$\Delta_2(\lambda,\rho) = \begin{vmatrix} \lambda_{11} & \lambda_{12} & \rho_{13} & \rho_{14} \\ \lambda_{12} & \lambda_{22} & \rho_{23} & \rho_{24} \\ \lambda_{13} & \lambda_{23} & \rho_{33} & \rho_{34} \\ \lambda_{14} & \lambda_{24} & \rho_{34} & \rho_{44} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \rho_{12} & \lambda_{13} & \rho_{14} \\ \lambda_{12} & \rho_{22} & \lambda_{23} & \rho_{24} \\ \lambda_{13} & \rho_{23} & \lambda_{33} & \rho_{34} \\ \lambda_{14} & \rho_{24} & \lambda_{34} & \rho_{44} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \rho_{12} & \rho_{13} & \lambda_{14} \\ \lambda_{12} & \rho_{22} & \rho_{23} & \lambda_{24} \\ \lambda_{13} & \rho_{23} & \rho_{33} & \lambda_{34} \\ \lambda_{14} & \rho_{24} & \rho_{34} & \lambda_{44} \end{vmatrix}$$

$$+ \begin{vmatrix} \rho_{11} & \lambda_{12} & \lambda_{13} & \rho_{14} \\ \rho_{12} & \lambda_{22} & \lambda_{23} & \rho_{24} \\ \rho_{13} & \lambda_{23} & \lambda_{33} & \rho_{34} \\ \rho_{14} & \lambda_{24} & \lambda_{34} & \rho_{44} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} & \rho_{13} & \lambda_{14} \\ \rho_{12} & \lambda_{22} & \rho_{23} & \lambda_{24} \\ \rho_{13} & \lambda_{23} & \rho_{33} & \lambda_{34} \\ \rho_{14} & \lambda_{24} & \rho_{34} & \lambda_{44} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \rho_{12} & \lambda_{13} & \lambda_{14} \\ \rho_{12} & \rho_{22} & \lambda_{23} & \lambda_{24} \\ \rho_{13} & \rho_{23} & \lambda_{33} & \lambda_{34} \\ \rho_{14} & \rho_{24} & \lambda_{34} & \lambda_{44} \end{vmatrix}$$

Note again that the coefficients of the determinant of the network can be formed from the matrices of the coefficients of the inductance and resistance quadratic forms, which matrices are for the four mesh case, respectively

$$\begin{Vmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} & \lambda_{34} \\ \lambda_{14} & \lambda_{24} & \lambda_{34} & \lambda_{44} \end{Vmatrix} \qquad \begin{Vmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{12} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{13} & \rho_{23} & \rho_{33} & \rho_{34} \\ \rho_{14} & \rho_{24} & \rho_{34} & \rho_{44} \end{Vmatrix}$$

$\Delta(\lambda)$ is the determinant of the λ matrix and $\Delta(\rho)$ is the determinant of the ρ matrix. $\Delta_1(\lambda,\rho)$ and $\Delta_1(\rho\lambda)$ are seen to be formed from the two determinants in a manner similar to that of $\Delta_1(\lambda,\rho)$ in the two and three mesh case. That is $\Delta_1(\lambda\rho)$ is formed from $\Delta(\lambda)$ by replacing one column at a time the λ terms in $\Delta(\lambda)$ by corresponding ρ terms and then adding the resulting four mixed determinants formed. $\Delta_2(\lambda,\rho)$ is also formed from $\Delta(\lambda)$ but by replacing two columns at a time the λ terms in $\Delta(\lambda)$ by corresponding ρ terms and then adding the resulting six mixed determinants formed. The index 2 to the right of Δ in $\Delta_2(\lambda,\rho)$ indicates that the substitution or replacement of λ terms by ρ terms be done two columns at a time. From the above explanations it is readily seen that in general

$$\Delta_1(\lambda,\rho) \neq \Delta_1(\rho\lambda)$$

But that

$$\Delta_2(\lambda,\rho) = \Delta_2(\rho\lambda)$$

By means of the above symbolic notation, the determinant of the network (151) becomes

$$D(p) = \Delta(\lambda)p^4 + \Delta_1(\lambda,\rho)p^3 + \Delta_2(\lambda,\rho)p^2 + \Delta_1(\rho,\lambda)p + \Delta(\rho) \qquad (152)$$

The minor of the element in the first row and first column of D(p) is obtained from (151) and is

$$M_{11}(p) = \begin{vmatrix} \lambda_{22}p + \rho_{22} & \lambda_{23}p + \rho_{23} & \lambda_{24}p + \rho_{24} \\ \lambda_{23}p + \rho_{23} & \lambda_{33}p + \rho_{33} & \lambda_{34}p + \rho_{34} \\ \lambda_{24}p + \rho_{24} & \lambda_{34}p + \rho_{34} & \lambda_{44}p + \rho_{44} \end{vmatrix} \qquad (153)$$

But this is exactly (144) page 138 which is the determinant of a three mesh network containing inductance and resistance elements. Hence we can make use of (145a), except that in place of Δ we use M_{11} since (153) is not now considered as the determinant of a three mesh network but the minor of the element in the first row and first column of a four mesh network. Thus without going through the process of evaluating (153), we may write

$$M_{11}(p) = M_{11}(\lambda)p^3 + M_{11}^{(1)}(\lambda,\rho)p^2 + M_{11}^{(1)}(\rho,\lambda)p + M_{11}(\rho) \qquad (153a)$$

The impedance function then becomes

$$Z(p) = \frac{D(p)}{M_{11}(p)}$$

$$= \frac{\Delta(\lambda)p^4 + \Delta_1(\lambda,\rho)p^3 + \Delta_2(\lambda,\rho)p^2 + \Delta_1(\beta\lambda)p + \Delta(\rho)}{M_{11}(\lambda)p^3 + M_{11}^{(1)}(\lambda,\rho)p^2 + M_{11}^{(1)}(\beta\lambda)p + M_{11}(\rho)} \qquad (154)$$

It is a simple matter to see what the impedance function of the five-mesh network containing inductance and resistance elements will be. This will be

$$Z(p) = \frac{\Delta(\lambda)p^5 + \Delta_1(\lambda,\rho)p^4 + \Delta_2(\lambda,\rho)p^3 + \Delta_2(\beta\lambda)p^2 + \Delta_1(\beta\lambda)p + \Delta(\rho)}{M_{11}(\lambda)p^4 + M_{11}^{(1)}(\lambda,\rho)p^3 + M_{11}^{(2)}(\lambda,\rho)p^2 + M_{11}^{(1)}(\beta\lambda)p + M_{11}(\rho)} \qquad (155)$$

For a network of six-meshes containing inductance and resistance elements, the impedance function would be

$$Z(p) = \frac{\Delta(\lambda)p^6 + \Delta_1(\lambda,\rho)p^5 + \Delta_2(\lambda,\rho)p^4 + \Delta_3(\lambda,\rho)p^3 + \Delta_2(\beta\lambda)p^2 + \Delta_1(\beta\lambda)p + \Delta(\rho)}{M_{11}(\lambda)p^5 + M_{11}^{(1)}(\lambda,\rho)p^4 + M_{11}^{(2)}(\lambda,\rho)p^3 + M_{11}^{(2)}(\beta\lambda)p^2 + M_{11}^{(1)}(\beta\lambda)p + M_{11}(\rho)} \qquad (156)$$

Formulas (155) and (156) can be obtained by expanding as we have done above, the determinants of the five and six-mesh networks containing inductance and resistance elements. By induction, the impedance function for the n-mesh network containing inductance and resistance elements is

$$Z(p) = \frac{\Delta(\lambda)p^{n} + \Delta_1(\lambda,\rho)p^{n-1} + \Delta_2(\lambda,\rho)p^{n-2} + \cdots + \Delta_2(\rho\lambda)p^{2} + \Delta_1(\rho\lambda)p + \Delta(\rho)}{M_{11}(\lambda)p^{n-1} + M_{11}^{(1)}(\lambda,\rho)p^{n-2} + \cdots + M_{11}^{(1)}(\rho\lambda)p + M_{11}(\rho)}$$

Thus far, the impedance function has been obtained for networks containing only inductance and resistance elements. It will now be shown that the corresponding impedance functions can be readily obtained in the same way for networks containing resistance and capacity elements, and inductance and capacity elements.

In equation (102a), page 72 we have the impedance function for the two-mesh network containing resistance and capacity elements. This is

$$Z(p) = \frac{\Delta(\rho)p^{2} + \Delta_1(\rho\sigma)p + \Delta(\sigma)}{p\left[M_{11}(\rho)p + M_{11}(\sigma)\right]} \qquad (102a)$$

With the exception of the p term outside the brackets, and the replacement of ρ in (102a) by λ and σ in (102a) by ρ, Z(p) in (102a) is exactly Z(p) in (102b). This modification of the impedance function in this way for the two-mesh network with resistance and capacity elements holds as well for the impedance function of any number of meshes. Thus the impedance function of the three-mesh network containing resistance and capacity elements is

$$Z(p) = \frac{\Delta(\rho)p^{3} + \Delta_1(\rho\sigma)p^{2} + \Delta_1(\sigma,\rho)p + \Delta(\sigma)}{p\left[M_{11}(\rho)p^{2} + M_{11}^{(1)}(\rho\sigma)p + M_{11}(\sigma)\right]} \qquad (157)$$

and the impedance function for the four-mesh network is

$$Z(p) = \frac{\Delta(\rho)p^4 + \Delta_1(\rho,\sigma)p^3 + \Delta_2(\rho,\sigma)p^2 + \Delta_1(\sigma,\rho)p + \Delta(\sigma)}{p[M_{11}(\rho)p^3 + M_{11}^{(1)}(\rho,\sigma)p^2 + M_{11}^{(1)}(\sigma,\rho)p + M_{11}(\sigma)} \qquad (158)$$

Finally the impedance function of the n-mesh network containing resistance and capacity elements is

$$Z(p) = \frac{\Delta(\rho)p^n + \Delta_1(\rho,\sigma)p^{n-1} + \Delta_2(\rho,\sigma)p^{n-2} + \cdots + \Delta_2(\sigma,\rho)p^2 + \Delta_1(\sigma,\rho)p + \Delta(\sigma)}{p[M_{11}(\rho)p^{n-1} + M_{11}^{(1)}(\rho,\sigma)p^{n-2} + \cdots + M_{11}^{(1)}(\sigma,\rho)p + M_{11}(\sigma)]} \qquad (159)$$

In a similar manner, the impedance function of a network of any number of meshes containing inductance and capacity elements is obtained. The impedance function for the two-mesh network is given in (102c) and is

$$Z(p) = \frac{\Delta(\lambda)p^4 + \Delta_1(\lambda,\sigma)p^2 + \Delta(\sigma)}{p[M_{11}(\lambda)p^2 + M_{11}(\sigma)]} \qquad (102c)$$

The three-mesh network impedance function is then

$$Z(p) = \frac{\Delta(\lambda)p^6 + \Delta_1(\lambda,\sigma)p^4 + \Delta_1(\sigma,\lambda)p^2 + \Delta(\sigma)}{p[M_{11}(\lambda)p^4 + M_{11}^{(1)}(\lambda,\rho)p^2 + M_{11}(\sigma)]} \qquad (160)$$

And the n-mesh network impedance function is

$$Z(p) = \frac{\Delta(\lambda)p^{2n} + \Delta_1(\lambda,\sigma)p^{2n-2} + \cdots + \Delta_1(\sigma,\lambda)p^2 + \Delta(\sigma)}{p\left[M_{11}(\lambda)p^{2n-2} + M_{11}^{(1)}(\lambda,\sigma)p^{2n-4} + \cdots + M_{11}^{(1)}(\sigma,\lambda)p^2 + M_{11}(\sigma)\right]} \qquad (161)$$

It will be useful to tabulate these formulas for reference.

IMPEDANCE FUNCTIONS FOR THREE-MESH NETWORKS
WITH TWO KINDS OF ELEMENTS.

Network Elements	Impedance Function
Inductance and Resistance	$Z(p) = \frac{\Delta(\lambda)p^3 + \Delta_1(\lambda,\rho)p^2 + \Delta_1(\rho,\lambda)p + \Delta(\rho)}{M_{11}(\lambda)p^2 + M_{11}^{(1)}(\lambda,\rho)p + M_{11}(\rho)}$ (162a)
Resistance and Capacity	$Z(p) = \frac{\Delta(\rho)p^3 + \Delta_1(\rho,\sigma)p^2 + \Delta_1(\sigma,\rho)p + \Delta(\sigma)}{p\left[M_{11}(\rho)p^2 + M_{11}^{(1)}(\rho,\sigma)p + M_{11}(\sigma)\right]}$ (162b)
Inductance and Capacity	$Z(p) = \frac{\Delta(\lambda)p^6 + \Delta_1(\lambda,\sigma)p^4 + \Delta_1(\sigma,\lambda)p^2 + \Delta(\sigma)}{p\left[M_{11}(\lambda)p^4 + M_{11}^{(1)}(\lambda,\sigma)p^2 + M_{11}(\sigma)\right]}$ (162c)

IMPEDANCE FUNCTIONS FOR N-MESH NETWORKS
WITH TWO KINDS OF ELEMENTS.

Network Elements	Impedance Function
Inductance and Resistance	$Z(p) = \frac{\Delta(\lambda)p^n + \Delta_1(\lambda,\rho)p^{n-1} + \Delta_2(\lambda,\rho)p^{n-2} + \cdots + \Delta_1(\rho,\lambda)p + \Delta(\rho)}{M_{11}(\lambda)p^{n-1} + M_{11}^{(1)}(\lambda,\rho)p^{n-2} + \cdots + M_{11}^{(1)}(\rho,\lambda)p + M_{11}(\rho)}$ (163a)
Resistance and Capacity	$Z(p) = \frac{\Delta(\rho)p^n + \Delta_1(\rho,\sigma)p^{n-1} + \Delta_2(\rho,\sigma)p^{n-2} + \cdots + \Delta_2(\sigma,\rho)p^2 + \Delta_1(\sigma,\rho)p + \Delta(\sigma)}{p\left[M_{11}(\rho)p^{n-1} + M_{11}^{(1)}(\rho,\sigma)p^{n-2} + \cdots + M_{11}^{(1)}(\sigma,\rho)p + M_{11}(\sigma)\right]}$ (163b)
Inductance and Capacity	$Z(p) = \frac{\Delta(\lambda)p^{2n} + \Delta_1(\lambda,\sigma)p^{2n-2} + \Delta_2(\lambda,\sigma)p^{2n-4} + \cdots + \Delta_2(\sigma,\lambda)p^4 + \Delta_1(\sigma,\lambda)p^2 + \Delta(\sigma)}{p\left[M_{11}(\lambda)p^{2n-2} + M_{11}^{(1)}(\lambda,\sigma)p^{2n-4} + \cdots + M_{11}^{(1)}(\sigma,\lambda)p^2 + M_{11}(\sigma)\right]}$ (163c)

By means of the formulas (163) the impedance function for a network of any number of meshes, containing any two kinds of elements, inductance and resistance, resistance and capacity and inductance and capacity, may be set up directly from the network elements themselves. Formulas (163) represent the impedance function of the most general network containing two kinds of network elements. Similar formulas will be obtained later for networks containing all three kinds of network elements.

By an examination of these formulas, it is seen that the coefficients are formed essentially from the elements of the matrices containing the coefficients of the three fundamental forms, namely the inductance, resistance and elastance quadratic forms. These matrices are respectively

$$\begin{Vmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{12} & & & \vdots \\ \vdots & & & \vdots \\ \lambda_{1n} & \cdots & \cdots & \lambda_{nn} \end{Vmatrix} \qquad \begin{Vmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & & & \vdots \\ \vdots & & & \vdots \\ \rho_{1n} & \cdots & \cdots & \rho_{nn} \end{Vmatrix} \qquad \begin{Vmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & & & \vdots \\ \vdots & & & \vdots \\ \sigma_{1n} & \cdots & \cdots & \sigma_{nn} \end{Vmatrix} \qquad (164)$$

The matrices (164) play therefore a most important role in the formation of the impedance function. They furnish of course the coefficients for the three fundamental forms, that is, the instantaneous magnetic and electrostatic energy, and the power loss in the resistance. These forms are (32), (33), and (34) in Chapter I, page 34.

CHAPTER V.

The Equivalence Equations for n-Mesh Networks With Two Kinds of Elements

We have seen that the most general three-mesh network containing inductance and resistance elements has an impedance function given by (162a) page 156, namely

$$Z(p) = \frac{\Delta(\lambda)\,p^3 + \Delta_1(\lambda,\rho)\,p^2 + \Delta_1(\rho,\lambda)\,p + \Delta(\rho)}{M_{11}(\lambda)\,p^2 + M_{11}^{(1)}(\lambda,\rho)\,p + M_{11}(\rho)} \qquad (162a)$$

As in the two-mesh network, we may proceed to remove as many elements as we desire from the most general three-mesh network, subject to the condition that the form of the impedance function be preserved. This means that the coefficients of (162a) cannot vanish, and that the numerator and denominator of (162a) cannot have a common factor, that is that the eliminant of the numerator and denominator of (162a) cannot vanish. Mathematically, these conditions are that

$$\left.\begin{array}{llll} \Delta(\lambda)\neq 0, & \Delta_1(\lambda,\rho)\neq 0, & \Delta_1(\rho,\lambda)\neq 0, & \Delta(\rho)\neq 0 \\ M_{11}(\lambda)\neq 0 & M_{11}(\rho)\neq 0 & M_{11}^{(1)}(\lambda,\rho)\neq 0 & \end{array}\right\} \qquad (163)$$

and the resultant[26]

$$\begin{vmatrix} \Delta(\lambda) & \Delta_1(\lambda,\rho) & \Delta_1(\rho,\lambda) & \Delta(\rho) & 0 \\ 0 & \Delta(\lambda) & \Delta_1(\lambda,\rho) & \Delta_1(\rho,\lambda) & \Delta(\rho) \\ M_{11}(\lambda) & M_{11}^{(1)}(\lambda,\rho) & M_{11}(\rho) & 0 & 0 \\ 0 & M_{11}(\lambda) & M_{11}^{(1)}(\lambda,\rho) & M_{11}(\rho) & 0 \\ 0 & 0 & M_{11}(\lambda) & M_{11}^{(1)}(\lambda,\rho) & M_{11}(\rho) \end{vmatrix} \neq 0 \qquad (164)$$

As in the two-mesh case, let us see what the vanishing of the resultant really means. Thus, consider the most general three mesh ladder network (which is not, of course, the most general three mesh network),

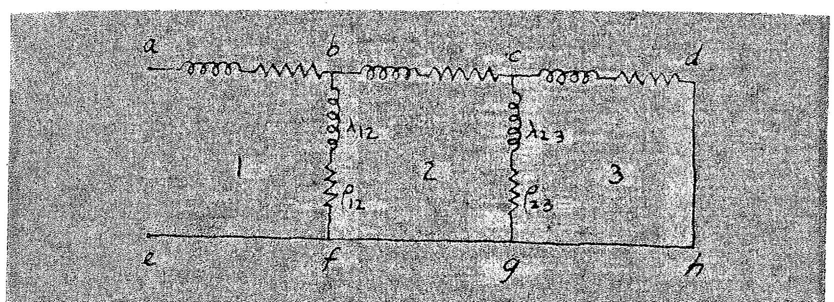

FIG 51

containing inductance and resistance elements. It is a simple matter to obtain the impedance function of this network, and it will be seen to have the form of (162a).

Now, as in the two-mesh case, let us proceed to remove as many of the elements in the network of figure 51 as we can,

26. See L. E. Dickson, Elementary Theory of Equations, 1914, page 154.

subject to the conditions (163) and (164). For example, suppose we remove the elements λ_{23} and ρ_{23}. Again we know that this means short circuiting the network between the points c and g, and the resulting impedance function would not be of the form (162a), but rather the impedance function of a two-mesh network. The impedance of this short-circuited network, shown in figure 52, can be computed by ordinary methods or by our formula (102b) and is

$$Z(p) = \frac{\Delta(\lambda)p^2 + \Delta_1(\lambda,\rho)p + \Delta(\rho)}{M_{11}(\lambda)p + M_{11}(\rho)} \qquad (102b)$$

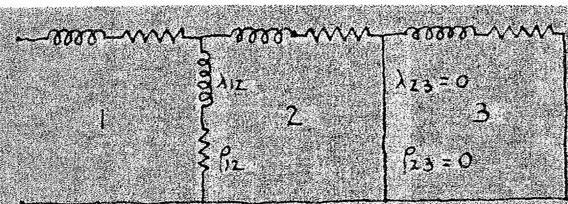

FIG. 52

However, let us proceed to obtain the impedance function of the network of figure 52, considered as a three-mesh network, using formula (162a). For the network of figure 52, we have then

$$\Delta(\lambda) = \begin{vmatrix} \lambda_{11} & \lambda_{12} & 0 \\ \lambda_{12} & \lambda_{22} & 0 \\ 0 & 0 & \lambda_{33} \end{vmatrix} = \lambda_{33}(\lambda_{11}\lambda_{22} - \lambda_{12}^2)$$

$$\Delta(\rho) = \begin{vmatrix} \rho_{11} & \rho_{12} & 0 \\ \rho_{12} & \rho_{22} & 0 \\ 0 & 0 & \rho_{33} \end{vmatrix} = \rho_{33}(\rho_{11}\rho_{22} - \rho_{12}^2)$$

$$\Delta_1(\lambda,\rho) = \begin{vmatrix} \lambda_{11} & \lambda_{12} & 0 \\ \lambda_{12} & \lambda_{22} & 0 \\ 0 & 0 & \rho_{33} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \rho_{12} & 0 \\ \lambda_{12} & \rho_{22} & 0 \\ 0 & 0 & \lambda_{33} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} & 0 \\ \rho_{12} & \lambda_{22} & 0 \\ 0 & 0 & \lambda_{33} \end{vmatrix}$$

$$= \rho_{33}\left(\lambda_{11}\lambda_{22} - \lambda_{12}^2\right) + \lambda_{33}\left(\lambda_{11}\rho_{22} - 2\lambda_{12}\rho_{12} + \rho_{11}\lambda_{22}\right)$$

$$\Delta_1(\rho,\lambda) = \begin{vmatrix} \rho_{11} & \rho_{12} & 0 \\ \rho_{12} & \rho_{22} & 0 \\ 0 & 0 & \lambda_{33} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} & 0 \\ \rho_{12} & \lambda_{22} & 0 \\ 0 & 0 & \rho_{33} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \rho_{12} & 0 \\ \lambda_{12} & \rho_{22} & 0 \\ 0 & 0 & \rho_{33} \end{vmatrix}$$

$$= \lambda_{33}\left(\rho_{11}\rho_{22} - \rho_{12}^2\right) + \rho_{33}\left(\lambda_{11}\rho_{22} - 2\lambda_{12}\rho_{12} + \rho_{11}\lambda_{22}\right)$$

$$M_{11}(\lambda) = \begin{vmatrix} \lambda_{22} & 0 \\ 0 & \lambda_{33} \end{vmatrix} = \lambda_{22}\lambda_{33}$$

$$M_{11}(\rho) = \begin{vmatrix} \rho_{22} & 0 \\ 0 & \rho_{33} \end{vmatrix} = \rho_{22}\rho_{33}$$

$$M_{11}(\lambda,\rho) = \begin{vmatrix} \lambda_{22} & 0 \\ 0 & \rho_{33} \end{vmatrix} + \begin{vmatrix} \rho_{22} & 0 \\ 0 & \lambda_{33} \end{vmatrix} = \lambda_{22}\rho_{33} + \rho_{22}\lambda_{33}$$

Substituting these values in (162a) we have for the impedance function of the network of figure 52

$$Z(p) = \frac{\left[\lambda_{33}\left(\lambda_{11}\lambda_{22}-\lambda_{12}^2\right)\right]p^3 + \left[\rho_{33}\left(\lambda_{11}\lambda_{22}-\lambda_{12}^2\right) + \lambda_{33}\left(\lambda_{11}\rho_{22} - 2\lambda_{12}\rho_{12} + \rho_{11}\lambda_{22}\right)\right]p^2 + \left[\lambda_{33}\left(\rho_{11}\rho_{22}-\rho_{12}^2\right) + \rho_{33}\left(\lambda_{11}\rho_{22} - 2\lambda_{12}\rho_{12} + \rho_{11}\lambda_{22}\right)\right]p + \left[\rho_{33}\left(\rho_{11}\rho_{22}-\rho_{12}^2\right)\right]}{\lambda_{22}\lambda_{33}\,p^2 + \left(\lambda_{22}\rho_{33} + \rho_{22}\lambda_{33}\right)p + \rho_{22}\rho_{33}} \tag{165}$$

This impedance function certainly seems to be of the form of (162a), but note that we can factor the numerator and denominator of (165), and that this factor is the same for both. Factoring the numerator and denominator for (165) we have

$$Z(p) = \frac{(\lambda_{11}\lambda_{22}-\lambda_{12}^2)p^2[\lambda_{33}p+\rho_{33}] + (\lambda_{11}\rho_{22}-2\lambda_{12}\rho_{12}+\rho_{11}\lambda_{22})p[\lambda_{33}p+\rho_{33}] + (\rho_{11}\rho_{22}-\rho_{12}^2)[\lambda_{33}p+\rho_{33}]}{(\lambda_{22}p+\rho_{22})(\lambda_{33}p+\rho_{33})} \tag{165a}$$

Cancelling the common factor in the numerator and denominator of (165a), we have

$$Z(p) = \frac{(\lambda_{11}\lambda_{22}-\lambda_{12}^2)p^2 + (\lambda_{11}\rho_{22}-2\lambda_{12}\rho_{12}+\rho_{11}\lambda_{22})p + (\rho_{11}\rho_{22}-\rho_{12}^2)}{\lambda_{22}p+\rho_{22}} \tag{166}$$

which is exactly (102b).

The fact that (165) thus had a common factor meant of course that the eliminant was zero, so that one of the conditions for the preservation of the form of our impedance function, namely (164) was violated.

Proceeding in the above manner, as we have for the two-mesh case, to remove as many of the network elements as we can from the most general three-mesh network containing inductance and resistance elements, subject to the condition that the form of the impedance function be preserved, namely that (163) and (164) hold, we finally arrive at the least general networks, that is, the canonical forms, shown in figure 53.

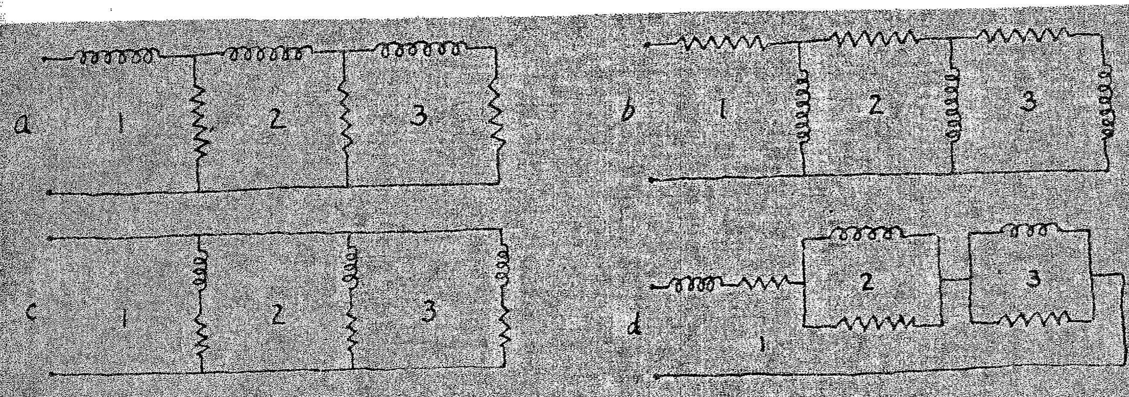

FIG.53

Note that networks a, b and c are ladder-type networks, but d is not. Networks a, b and c can be obtained from the more general network shown in figure 51, by a removal of elements, but network d is obtained by the removal of elements of a more general three-mesh network.

While we have limited ourselves here to three-mesh networks containing inductance and resistance elements, the above methods hold just as well for three-mesh networks containing inductance and capacity elements and, resistance and capacity elements. It also holds for networks of any number of meshes containing two kinds of elements, and also, as will be shown later, for networks having all three network elements present. 2
In this way, by a removal of network elements from the most general n-mesh network containing two kinds of elements, subject to the condition that the form of the impedance function be preserved, we arrive at the canonical forms shown in figures 5-13, in the introduction. These, as explained there, were obtained by Foster and Cauer by partial fraction and continued fraction expansion.

Let us consider in detail the impedance function of the most general three-mesh network containing inductance and resistance elements, namely

$$Z(p) = \frac{\Delta(\lambda)\,p^3 + \Delta_1(\lambda,\rho)\,p^2 + \Delta_1(\rho,\lambda)\,p + \Delta(\rho)}{M_{11}(\lambda)\,p^2 + M_{11}^{(1)}(\lambda,\rho)\,p + M_{11}(\rho)}$$

As we have seen the coefficients are determinants containing the actual network elements. That is

$$\Delta(\lambda) = \begin{vmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{vmatrix} \qquad \Delta(\rho) = \begin{vmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{12} & \rho_{22} & \rho_{23} \\ \rho_{13} & \rho_{23} & \rho_{33} \end{vmatrix}$$

$$\Delta_1(\lambda,\rho) = \begin{vmatrix} \lambda_{11} & \lambda_{12} & \rho_{13} \\ \lambda_{12} & \lambda_{22} & \rho_{23} \\ \lambda_{13} & \lambda_{23} & \rho_{33} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \rho_{12} & \lambda_{13} \\ \lambda_{12} & \rho_{22} & \lambda_{23} \\ \lambda_{13} & \rho_{23} & \lambda_{33} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} & \lambda_{13} \\ \rho_{12} & \lambda_{22} & \lambda_{23} \\ \rho_{13} & \lambda_{23} & \lambda_{33} \end{vmatrix}$$

$$\Delta_1(\rho,\lambda) = \begin{vmatrix} \rho_{11} & \rho_{12} & \lambda_{13} \\ \rho_{12} & \rho_{22} & \lambda_{23} \\ \rho_{13} & \rho_{23} & \lambda_{33} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} & \rho_{13} \\ \rho_{12} & \lambda_{22} & \rho_{23} \\ \rho_{13} & \lambda_{23} & \rho_{33} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \rho_{12} & \rho_{13} \\ \lambda_{12} & \rho_{22} & \rho_{23} \\ \lambda_{13} & \rho_{23} & \rho_{33} \end{vmatrix}$$

$$M_{11}(\lambda) = \begin{vmatrix} \lambda_{22} & \lambda_{23} \\ \lambda_{23} & \lambda_{33} \end{vmatrix} \qquad M_{11}(\rho) = \begin{vmatrix} \rho_{22} & \rho_{23} \\ \rho_{23} & \rho_{33} \end{vmatrix}$$

$$M_{11}(\lambda,\rho) = \begin{vmatrix} \lambda_{22} & \rho_{23} \\ \lambda_{23} & \rho_{33} \end{vmatrix} + \begin{vmatrix} \rho_{22} & \lambda_{23} \\ \rho_{23} & \lambda_{33} \end{vmatrix}$$

It is noted that the network parameters $\lambda_{11}, \lambda_{22}, \lambda_{33}, \lambda_{12}, \lambda_{13}, \lambda_{23}$; $\rho_{11}, \rho_{22}, \rho_{33}, \rho_{12}, \rho_{23}, \rho_{13}$ determine the nature of the coefficients of the impedance function. Now suppose we are given a definite impedance function of the form (162a), namely

$$Z(p) = \frac{a_0 p^3 + a_1 p^2 + a_2 p + a_3}{b_1 p^2 + b_2 p + b_3} \qquad (167)$$

As before, we can multiply the numerator and denominator of (167) by a positive constant k^2, so that (167) becomes

$$Z(p) = \frac{k^2 a_0 p^3 + k^2 a_1 p^2 + k^2 a_2 p + k^2 a_3}{k^2 b_1 p^2 + k^2 b_2 p + k^2 b_3} \qquad (167a)$$

Since (167a) is the impedance function of a definite three-mesh network containing inductance and resistance elements, and (162a) is the impedance function of the most general three-mesh network containing these elements, it follows that there exist real positive values of $\lambda_{11}, \lambda_{22}, \lambda_{33}, \lambda_{12}, \lambda_{13}, \lambda_{23}$; $\rho_{11}, \rho_{22}, \rho_{33}, \rho_{12}, \rho_{13}, \rho_{23}$ such that

$$\Delta(\lambda) = k^2 a_0 \qquad (168a)$$

$$\Delta_1(\lambda, \rho) = k^2 a_1 \qquad (168b)$$

$$\Delta_1(\rho, \lambda) = k^2 a_2 \qquad (168c)$$

$$\Delta(\rho) = k^2 a_3 \qquad (168d)$$

$$M_{11}(\lambda) = k^2 b_1 \qquad (168e)$$

$$M_{11}^{(1)}(\lambda, \rho) = k^2 b_2 \qquad (168f)$$

$$M_{11}(\rho) = k^2 b_3 \qquad (168g)$$

In open form this system of equations is

$$\begin{vmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{vmatrix} = k^2 a_0 \tag{169a}$$

$$\begin{vmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{12} & \rho_{22} & \rho_{23} \\ \rho_{13} & \rho_{23} & \rho_{33} \end{vmatrix} = k^2 a_3 \tag{169b}$$

$$\begin{vmatrix} \lambda_{11} & \lambda_{12} & \rho_{13} \\ \lambda_{12} & \lambda_{22} & \rho_{23} \\ \lambda_{13} & \lambda_{23} & \rho_{33} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \rho_{12} & \lambda_{13} \\ \lambda_{12} & \rho_{13} & \lambda_{23} \\ \lambda_{13} & \rho_{23} & \lambda_{33} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} & \lambda_{13} \\ \rho_{12} & \lambda_{22} & \lambda_{23} \\ \rho_{13} & \lambda_{23} & \lambda_{33} \end{vmatrix} = k^2 a_1 \tag{169c}$$

$$\begin{vmatrix} \rho_{11} & \rho_{12} & \lambda_{13} \\ \rho_{12} & \rho_{22} & \lambda_{23} \\ \rho_{13} & \rho_{23} & \lambda_{33} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} & \rho_{13} \\ \rho_{12} & \lambda_{22} & \rho_{23} \\ \rho_{13} & \lambda_{23} & \rho_{33} \end{vmatrix} + \begin{vmatrix} \lambda_{11} & \rho_{12} & \rho_{13} \\ \lambda_{12} & \rho_{22} & \rho_{23} \\ \lambda_{13} & \rho_{23} & \rho_{33} \end{vmatrix} = k^2 a_2 \tag{169d}$$

$$\begin{vmatrix} \lambda_{22} & \lambda_{23} \\ \lambda_{23} & \lambda_{33} \end{vmatrix} = k^2 b_1 \tag{169e}$$

$$\begin{vmatrix} \rho_{22} & \rho_{23} \\ \rho_{23} & \rho_{33} \end{vmatrix} = k^2 b_3 \tag{169f}$$

$$\begin{vmatrix} \lambda_{22} & \rho_{23} \\ \lambda_{23} & \rho_{33} \end{vmatrix} + \begin{vmatrix} \rho_{22} & \lambda_{23} \\ \rho_{23} & \lambda_{33} \end{vmatrix} = k^2 b_2 \tag{169g}$$

These equations may be called the equivalence equations of the network, since the complete infinite set of networks satisfying (169) are equivalent. Thus any set of real numbers may be a possible set of parameters of a physical network having (167) for an impedance function provided these parameters satisfy (169) and provided the mutual parameters are always less than the total parameters.

In the system of equations (169), consider all the mutual parameters as arbitrary. We have then, substituting 169e, f and g in 169 a, b and c.

$$k^2 b_1 \lambda_{11} - \lambda_{13}^2 \lambda_{22} - \lambda_{12}^2 \lambda_{33} = k^2 a_0 - 2\lambda_{12}\lambda_{13}\lambda_{23} \qquad (170a)$$

$$k^2 b_3 \rho_{11} - \rho_{13}^2 \rho_{22} - \rho_{12}^2 \rho_{33} = k^2 a_3 - 2\rho_{12}\rho_{13}\rho_{23} \qquad (170b)$$

$$k^2 b_1 \rho_{11} + k^2 b_2 \lambda_{11} - \lambda_{13}^2 \rho_{22} - \lambda_{12}^2 \rho_{33} - 2\rho_{12}\lambda_{12}\lambda_{33} - 2\rho_{13}\lambda_{13}\lambda_{22} = k^2 a_1 - 2\rho_{12}\lambda_{13}\lambda_{23} - 2\rho_{23}\lambda_{12}\lambda_{13} - 2\rho_{13}\lambda_{12}\lambda_{23} \qquad (170c)$$

$$k^2 b_3 \lambda_{11} + k^2 b_2 \rho_{11} - \rho_{13}^2 \lambda_{22} - \rho_{12}^2 \lambda_{33} - 2\lambda_{12}\rho_{12}\rho_{33} - 2\lambda_{13}\rho_{13}\rho_{22} = k^2 a_2 - 2\lambda_{12}\rho_{13}\rho_{23} - 2\lambda_{23}\rho_{12}\rho_{13} - 2\lambda_{13}\rho_{12}\rho_{23} \qquad (170d)$$

Now from 169e and f, we have

$$\lambda_{22} = \frac{k^2 b_1 + \lambda_{23}^2}{\lambda_{33}} \qquad (171a)$$

$$\rho_{22} = \frac{k^2 b_3 + \rho_{23}^2}{\rho_{33}} \qquad (171b)$$

Substituting 171a and b in 169g, we have

$$(k^2 b_1 + \lambda_{23}^2)\frac{\rho_{33}}{\lambda_{33}} + (k^2 b_3 + \rho_{23}^2)\frac{\lambda_{33}}{\rho_{33}} = k^2 b_2 + 2\lambda_{23}\rho_{23} \qquad (172)$$

Hence, letting $x = \frac{\rho_{33}}{\lambda_{33}}$, we have

$$(k^2 b_1 + \lambda_{23}^2)x^2 - (k^2 b_2 + 2\lambda_{23}\rho_{23})x + (k^2 b_3 + \rho_{23}^2) = 0$$

$$\therefore x = \frac{(k^2 b_2 + 2\lambda_{23}\rho_{23}) \pm \sqrt{(k^2 b_2 + 2\lambda_{23}\rho_{23})^2 - 4(k^2 b_1 + \lambda_{23}^2)(k^2 b_3 + \rho_{23}^2)}}{2(k^2 b_1 + \lambda_{23}^2)} \qquad (173)$$

Let m $= (k^2 b_2 + 2\lambda_{23}\rho_{23}) \pm \sqrt{(k^2 b_2 + 2\lambda_{23}\rho_{23})^2 - 4(k^2 b_1 + \lambda_{23}^2)(k^2 b_3 + \rho_{23}^2)}$

Then

$$\frac{\rho_{33}}{\lambda_{33}} = \frac{m}{2(k^2 b_1 + \lambda_{23}^2)} = \alpha$$

From 171a and b, we have

$$\frac{\lambda_{22}}{\rho_{22}} = \frac{m}{2(k^2 b_3 + \rho_{23}^2)} = \beta$$

Thus $\qquad \rho_{33} = \alpha\lambda_{33} \qquad (174a)$

And $\qquad \lambda_{22} = \beta\rho_{22} \qquad (174b)$

Substituting these values in (170), we have

$$k^2 b_1 \lambda_{11} - \beta \lambda_{13}^2 \rho_{22} - \lambda_{12}^2 \lambda_{33} = k^2 a_0 - 2\lambda_{12}\lambda_{13}\lambda_{23} \tag{175a}$$

$$k^2 b_3 \rho_{11} - \rho_{13}^2 \rho_{22} - \alpha \rho_{12}^2 \lambda_{33} = k^2 a_3 - 2\rho_{12}\rho_{13}\rho_{23} \tag{175b}$$

$$k^2 b_1 \rho_{11} + k^2 b_2 \lambda_{11} - \lambda_{13}^2 \rho_{22} - \alpha \lambda_{12}^2 \lambda_{33} - 2\rho_{12}\lambda_{12}\lambda_{33} - 2\beta \rho_{13}\lambda_{13}\rho_{22} = k^2 a_1 - 2\rho_{12}\lambda_{13}\lambda_{23} - 2\rho_{23}\lambda_{12}\lambda_{13} - 2\rho_{13}\lambda_{12}\lambda_{23} \tag{175c}$$

$$k^2 b_3 \lambda_{11} + k^2 b_2 \rho_{11} - \beta \rho_{13}^2 \rho_{22} - \rho_{12}^2 \lambda_{33} - 2\alpha \lambda_{12}\rho_{12}\lambda_{33} - 2\lambda_{13}\rho_{13}\rho_{22} = k^2 a_2 - 2\lambda_{12}\rho_{13}\rho_{23} - 2\lambda_{23}\rho_{12}\rho_{13} - 2\lambda_{13}\rho_{12}\rho_{23} \tag{175d}$$

Let the right hand sides of 175a, b, c and d be respectively set equal to r_1, r_2, r_3, r_4 and write (174) in the usual form. We have then

$$k^2 b_1 \lambda_{11} - \lambda_{12}^2 \lambda_{33} \qquad - \beta \lambda_{13}^2 \rho_{22} = r_1 \tag{176a}$$

$$- \alpha \rho_{12}^2 \lambda_{33} + k^2 b_3 \rho_{11} - \rho_{13}^2 \rho_{22} = r_2 \tag{176b}$$

$$k^2 b_2 \lambda_{11} - (\alpha \lambda_{12}^2 + 2\rho_{12}\lambda_{12})\lambda_{33} + k^2 b_1 \rho_{11} - (\lambda_{13}^2 + 2\beta \rho_{13}\lambda_{13})\rho_{22} = r_3 \tag{176c}$$

$$k^2 b_3 \lambda_{11} - (\rho_{12}^2 + 2\alpha \lambda_{12}\rho_{12})\lambda_{33} + k^2 b_2 \rho_{11} - (\beta \rho_{13}^2 + 2\lambda_{13}\rho_{13})\rho_{22} = r_4 \tag{176d}$$

The equations (176) are linear and can be solved by the usual method of determinants. Thus, for example, λ_{11} equals the determinant (177a) divided by the determinant (177b)

$$\begin{vmatrix} r_1 & -\lambda_{12}^2 & 0 & -\beta\lambda_{13}^2 \\ r_2 & -\alpha\rho_{12}^2 & k^2 b_3 & -\rho_{13}^2 \\ r_3 & -(\alpha\lambda_{12}^2 + 2\rho_{12}\lambda_{12}) & k^2 b_1 & -(\lambda_{13}^2 + 2\beta\rho_{13}\lambda_{13}) \\ r_4 & -(\rho_{12}^2 + 2\alpha\lambda_{12}\rho_{12}) & k^2 b_2 & -(\beta\rho_{13}^2 + 2\lambda_{13}\rho_{13}) \end{vmatrix} \tag{177a}$$

$$\begin{vmatrix} k^2b_1 & -\lambda_{12}^2 & 0 & -\beta\lambda_{13}^2 \\ 0 & -\alpha\rho_{12}^2 & k^2b_3 & -\rho_{13}^2 \\ k^2b_2 & -(\alpha\lambda_{12}^2+2\rho_{12}\lambda_{12}) & k^2b_1 & -(\lambda_{13}^2+2\beta\rho_{13}\lambda_{13}) \\ k^2b_3 & -(\rho_{12}^2+2\alpha\lambda_{12}\rho_{12}) & k^2b_2 & -(\beta\rho_{13}^2+2\lambda_{13}\rho_{13}) \end{vmatrix} \qquad (177b)$$

Thus the system of equations (176) allows the solution for the total parameters $\lambda_{11}, \lambda_{33}, \rho_{11}, \rho_{22}$ in terms of the mutual parameters. Equations (174) give the other two total parameters λ_{22} and ρ_{33}.

It is seen that although the system of equations (176) can be solved for the total parameters, the solutions are by no means simple. It may be possible, by means of a graphical method similar to the one in the two-mesh case, to obtain solutions in simpler and more elegant form. Perhaps a vector notation, which we have introduced for the two-mesh network with three kinds of network elements (next chapter), would help simplify matters. Or it may turn out that the solutions for the three-mesh network, and networks with more than three meshes, are inherently complicated.

It will be useful to write down the equivalence equations for the four-mesh network containing inductance and resistance elements. These can be obtained by induction from

(168 or from (154), page 153 . These equations are

$$\left.\begin{aligned}
\Delta(\lambda) &= k^2 a_0 \\
\Delta_1(\lambda,\rho) &= k^2 a_1 \\
\Delta_2(\lambda,\rho) &= k^2 a_2 \\
\Delta_1(\rho,\lambda) &= k^2 a_3 \\
\Delta(\rho) &= k^2 a_4 \\
M_{11}(\lambda) &= k^2 b_1 \\
M_{11}^{(1)}(\lambda,\rho) &= k^2 b_2 \\
M_{11}^{(1)}(\rho,\lambda) &= k^2 b_3 \\
M_{11}(\rho) &= k^2 b_4
\end{aligned}\right\} \qquad (178)$$

As before any system of real positive numbers satisfying (178) may become a possible set of mutual and total parameters of a network provided the numbers representing the mutual parameters are less than those representing the total parameters.

The equivalence equations for the five-mesh network containing inductance and resistance elements are

$$\left.\begin{aligned}
\Delta(\lambda) &= k^2 a_0 \\
\Delta_1(\lambda,\rho) &= k^2 a_1 \\
\Delta_2(\lambda,\rho) &= k^2 a_2 \\
\Delta_2(\rho,\lambda) &= k^2 a_3 \\
\Delta_1(\rho,\lambda) &= k^2 a_4 \\
\Delta(\rho) &= k^2 a_5 \\
M_{11}(\lambda) &= k^2 b_1 \\
M_{11}^{(1)}(\lambda,\rho) &= k^2 b_2 \\
M_{11}^{(2)}(\lambda,\rho) &= k^2 b_3 \\
M_{11}^{(1)}(\lambda,\rho) &= k^2 b_4 \\
M_{11}(\rho) &= k^2 b_5
\end{aligned}\right\} \qquad (179)$$

Finally the equivalence equations for the n-mesh network containing inductance and resistance elements are readily obtained by induction, and are given by

$$\left.\begin{array}{ll}
\Delta(\lambda) = k^2 a_0 & M_{11}(\lambda) = k^2 b_1 \\
\Delta_1(\lambda,\rho) = k^2 a_1 & M_{11}^{(1)}(\lambda,\rho) = k^2 b_2 \\
\Delta_2(\lambda,\rho) = k^2 a_2 & M_{11}^{(2)}(\lambda,\rho) = k^2 b_3 \\
\Delta_3(\lambda,\rho) = k^2 a_3 & M_{11}^{(3)}(\lambda,\rho) = k^2 b_4 \\
------- & ------- \\
\Delta_3(\rho,\lambda) = k^2 a_{n-3} & M_{11}^{(3)}(\rho,\lambda) = k^2 b_{n-3} \\
\Delta_2(\rho,\lambda) = k^2 a_{n-2} & M_{11}^{(2)}(\rho,\lambda) = k^2 b_{n-2} \\
\Delta_1(\rho,\lambda) = k^2 a_{n-1} & M_{11}^{(1)}(\rho,\lambda) = k^2 b_{n-1} \\
\Delta(\rho) = k^2 a_n & M_{11}(\rho) = k^2 b_n
\end{array}\right\} \qquad (180)$$

The solutions of this system of equations will give the parameters of the n-mesh network having an impedance given by

$$Z(p) = \frac{a_0 p^n + a_1 p^{n-1} + a_2 p^{n-2} + a_3 p^{n-3} + \cdots + a_{n-3} p^3 + a_{n-2} p^2 + a_{n-1} p + a_n}{b_1 p^{n-1} + b_2 p^{n-2} + b_3 p^{n-3} + \cdots + b_{n-2} p^2 + b_{n-1} p + b_n} \qquad (181)$$

While we have given the equivalence equations above for networks containing only inductance and resistance elements, they are exactly the same for networks containing inductance and capacity, and resistance and capacity elements. In the former, the ρ terms in (180) are replaced by σ terms, and in the latter, the λ terms in (180) are replaced by ρ terms, and the ρ terms in (181) are replaced by σ terms.

Thus far we have assumed that the expression

$$\frac{a_0 p^n + a_1 p^{n-1} + a_2 p^{n-2} + \cdots + a_{n-2} p^2 + a_{n-1} p + a_n}{b_1 p^{n-1} + b_2 p^{n-2} + \cdots + b_{n-2} p^2 + b_{n-1} p + b_n} \qquad (182)$$

was the impedance function of some definite physical network, and hence the system of equations (180) would give all the networks equivalent to the given network. Suppose however that we are given an expression like (182), where the a and b coefficients are real, what must the conditions on these coefficients be in order that (182) represent the impedance function of some physical network.

We have seen in the introduction that if the zeros and poles of (182) are negative reals, and if these zeros and poles separate each other, then (182) does represent the impedance function of some physical network. Hurwitz, as Cauer has shown, has expressed the conditions that the zeros and poles of (182) separate each other and that they be negative reals, in terms of the coefficients of (182).[27] These conditions are (1) that

27. A. Hurwitz, "Uber die Bedindungen, unter welchen eine Gleichung nur Wurzeln mit negativen reelen Teilen besitzt", Mathematical Annalen, Bd. 46, S. 273, 1895. See also W.Cauer, Die Verwirklichung usw. loc.cit. p. 371.

every second principal minor of the determinant

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \cdots \\ 0 & b_1 & b_2 & b_3 & b_4 & b_5 & \cdots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\ 0 & 0 & b_1 & b_2 & b_3 & b_4 & \cdots \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & 0 & b_1 & b_2 & b_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{183}$$

must be positive and (2) the n principal minors of the determinant

$$\begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 & \cdots \\ a_3 & a_2 & a_1 & a_0 & 0 & \cdots \\ a_5 & a_4 & a_3 & a_2 & a_1 & \cdots \\ a_7 & a_6 & a_5 & a_4 & a_3 & \cdots \\ a_9 & a_8 & a_7 & a_6 & a_5 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{184}$$

must be positive. Note that for $n=2$ (183) becomes

$$\begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & b_1 & b_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & b_1 & b_2 \end{vmatrix} \tag{185}$$

$$= a_0\left(a_1 b_1 b_2 - a_2 b_1^2 - a_0 b_2^2\right)$$

This is seen to be, except for a_0, the negative of the resultant obtained on page 80 . It is interesting to note that by interchanging the rows of (183), this determinant reduces, except for a multiplier, to the resultant of the numerator and denominator of (182).

The resultant of any two polynomials may also be expressed in terms of their roots. Thus if

$$f(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m \qquad (186)$$

and

$$g(x) = b_0 x^n + b_1 x^{n-1} + \cdots + b_n \qquad (187)$$

and if

$$\alpha_1, \alpha_2 \cdots\cdots \alpha_m$$

are the roots of $f(x)$, and

$$\beta_1, \beta_2 \cdots\cdots \beta_n$$

are the roots of g(x), then the resultant of f(x) and g(x) is

$$\begin{aligned} a_0^n b_0^m &(\alpha_1-\beta_1)(\alpha_1-\beta_2) \cdots (\alpha_1-\beta_n) \\ &(\alpha_2-\beta_1)(\alpha_2-\beta_2) \cdots (\alpha_2-\beta_n) \\ &\cdots\cdots\cdots\cdots \\ &(\alpha_m-\beta_1)(\alpha_m-\beta_2) \cdots (\alpha_m-\beta_n) \end{aligned}$$

Thus it is seen at once why the resultant vanishes when f(x) and g(x) have equal roots.[28]

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28. L. E. Dickson, loc. cit. p. 151 and Gordan Invariantheorie, Vol. I, 1885, p. 180.

CHAPTER VI.

Networks Containing Inductance, Resistance and Capacity Elements.

Thus far, we have considered networks having two kinds of elements, inductance and resistance, inductance and capacity and resistance and capacity. We have obtained expressions for the impedance functions directly from the elements of these networks, and the corresponding equivalence equations. Let us proceed to obtain similar results for network having all three kinds of elements - inductances, resistances and capacities.

The most general 2 mesh network containing inductance, resistance and capacity elements is shown in figure 54.

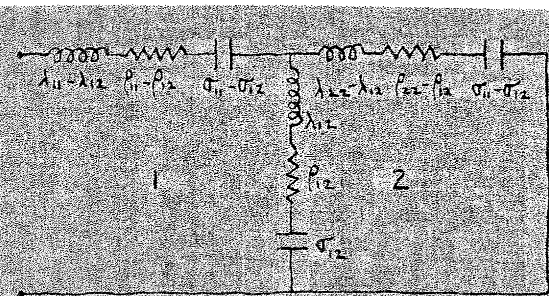


FIG. 54

The determinant D(p) of this network is

$$D(p) = \begin{vmatrix} \lambda_{11}p + \rho_{11} + \frac{\sigma_{11}}{p} & \lambda_{12}p + \rho_{12} + \frac{\sigma_{12}}{p} \\ \lambda_{12}p + \rho_{12} + \frac{\sigma_{12}}{p} & \lambda_{22}p + \rho_{22} + \frac{\sigma_{22}}{p} \end{vmatrix} \qquad (188)$$

$$= \frac{1}{p^2} \begin{vmatrix} \lambda_{11} p^2 + \rho_{11} p + \sigma_{11} & \lambda_{12} p^2 + \rho_{12} p + \sigma_{12} \\ \lambda_{12} p^2 + \rho_{12} p + \sigma_{12} & \lambda_{22} p^2 + \rho_{22} p + \sigma_{22} \end{vmatrix}$$

$$= \frac{1}{p^2} \left\{ \begin{vmatrix} \lambda_{11} p^2 & \lambda_{12} p^2 + \rho_{12} p + \sigma_{12} \\ \lambda_{12} p^2 & \lambda_{22} p^2 + \rho_{22} p + \sigma_{22} \end{vmatrix} + \begin{vmatrix} \rho_{11} p + \sigma_{11} & \lambda_{12} p^2 + \rho_{12} p + \sigma_{12} \\ \rho_{12} p + \sigma_{12} & \lambda_{22} p^2 + \rho_{22} p + \sigma_{22} \end{vmatrix} \right\}$$

$$= \frac{1}{p^2} \left\{ \begin{vmatrix} \lambda_{11} p^2 & \lambda_{12} p^2 \\ \lambda_{12} p^2 & \lambda_{22} p^2 \end{vmatrix} + \begin{vmatrix} \lambda_{11} p^2 & \rho_{12} p + \sigma_{12} \\ \lambda_{12} p^2 & \rho_{22} p + \sigma_{22} \end{vmatrix} + \begin{vmatrix} \rho_{11} p & \lambda_{12} p^2 + \rho_{12} p + \sigma_{12} \\ \rho_{12} p & \lambda_{22} p^2 + \rho_{22} p + \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \lambda_{12} p^2 + \rho_{12} p + \sigma_{12} \\ \sigma_{12} & \lambda_{22} p^2 + \rho_{22} p + \sigma_{22} \end{vmatrix} \right\}$$

$$= \frac{1}{p^2} \left\{ \begin{vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{vmatrix} p^4 + \begin{vmatrix} \lambda_{11} & \rho_{12} \\ \lambda_{12} & \rho_{22} \end{vmatrix} p^3 + \begin{vmatrix} \lambda_{11} & \sigma_{12} \\ \lambda_{12} & \sigma_{22} \end{vmatrix} p^2 + \begin{vmatrix} \rho_{11} & \lambda_{12} \\ \rho_{12} & \lambda_{22} \end{vmatrix} p^3 + \begin{vmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{vmatrix} p^2 \right.$$

$$\left. + \begin{vmatrix} \rho_{11} & \sigma_{12} \\ \rho_{12} & \sigma_{22} \end{vmatrix} p + \begin{vmatrix} \sigma_{11} & \lambda_{12} \\ \sigma_{12} & \lambda_{22} \end{vmatrix} p^2 + \begin{vmatrix} \sigma_{11} & \rho_{12} \\ \sigma_{12} & \rho_{22} \end{vmatrix} p + \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} \right\}$$

$$= \frac{1}{p^2} \left[\begin{vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{vmatrix} p^4 + \left\{ \begin{vmatrix} \lambda_{11} & \rho_{12} \\ \lambda_{12} & \rho_{22} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} \\ \rho_{12} & \lambda_{22} \end{vmatrix} \right\} p^3 + \left\{ \begin{vmatrix} \lambda_{11} & \sigma_{12} \\ \lambda_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \lambda_{12} \\ \sigma_{12} & \lambda_{22} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{vmatrix} \right\} p^2 \right.$$

$$\left. + \left\{ \begin{vmatrix} \rho_{11} & \sigma_{12} \\ \rho_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \rho_{12} \\ \sigma_{12} & \rho_{22} \end{vmatrix} \right\} p + \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} \right] \qquad (189)$$

Using our symbolic notation, we have

$$\Delta(\lambda) = \begin{vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{vmatrix} \qquad \Delta(\rho) = \begin{vmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{vmatrix} \qquad \Delta(\sigma) = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix}$$

$$\Delta_1(\lambda,\rho) = \begin{vmatrix} \lambda_{11} & \rho_{12} \\ \lambda_{12} & \rho_{22} \end{vmatrix} + \begin{vmatrix} \rho_{11} & \lambda_{12} \\ \rho_{12} & \lambda_{22} \end{vmatrix}$$

$$\Delta_1(\lambda,\sigma) = \begin{vmatrix} \lambda_{11} & \sigma_{12} \\ \lambda_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \lambda_{12} \\ \sigma_{12} & \lambda_{22} \end{vmatrix}$$

$$\underline{\Delta_1(\rho,\sigma)} = \begin{vmatrix} \rho_{11} & \sigma_{12} \\ \rho_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \rho_{12} \\ \sigma_{12} & \rho_{22} \end{vmatrix}$$

With this symbolic notation, (189) may be written

$$D(p) = \frac{1}{p^2}\left[\Delta(\lambda)p^4 + \Delta_1(\lambda,\rho)p^3 + \left\{\Delta_1(\lambda,\sigma) + \Delta(\rho)\right\}p^2 + \Delta_1(\sigma,\rho)p + \Delta(\sigma)\right] \qquad (190)$$

The minor of D(p) is given by (188) and is

$$M_{11}(p) = \lambda_{22}p + \rho_{22} + \frac{\sigma_{22}}{p}$$

$$= \frac{1}{p}\left(\lambda_{22}p^2 + \rho_{22}p + \sigma_{22}\right)$$

$$= \frac{1}{p}\left[M_{11}(\lambda)p^2 + M_{11}(\rho)p + M_{11}(\sigma)\right] \qquad (191)$$

Hence

$$Z(p) = \frac{D(p)}{M_{11}(p)}$$

$$= \frac{\Delta(\lambda)p^4 + \Delta_1(\lambda,\rho)p^3 + \{\Delta_1(\lambda,\sigma) + \Delta(\rho)\}p^2 + \Delta_1(\sigma,\rho)p + \Delta(\sigma)}{p\left[M_{11}(\lambda)p^2 + M_{11}(\rho)p + M_{11}(\sigma)\right]} \qquad (192)$$

Z(p) in (192) represents the most general two-mesh network containing inductance, resistance and capacity elements. As before the conditions that the form of the impedance function (192) be preserved are that neither the coefficients of (192) nor the eliminant of the numerator and denominator vanish. The removal of elements subject to these conditions will give networks with the least number of elements having (192) as an impedance function.

It is readily seen by expanding the determinants of (192) that all the coefficients of (192) are positive reals, provided we limit the network to positive elements only. Note also that (192) is formed at once from the matrices of the coefficients of the three fundamental quadratic forms of the electric circuit with inductance, resistance and capacity elements present. These matrices are of course

$$\left\|\begin{matrix}\lambda_{11} & \lambda_{12}\\ \lambda_{12} & \lambda_{22}\end{matrix}\right\|, \quad \left\|\begin{matrix}\rho_{11} & \rho_{12}\\ \rho_{12} & \rho_{22}\end{matrix}\right\|, \quad \left\|\begin{matrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{matrix}\right\|$$

and the rule of formation of the coefficients of (192) from these matrices is obvious.

Now let us suppose that we have an expression of the form of (192), such as

$$\frac{a_0 p^4 + a_1 p^3 + a_2 p^2 + a_3 p + a_4}{p(b_1 p^2 + b_2 p + b_3)} \tag{193}$$

where the a and b coefficients are any positive reals. Now let us ask ourselves this question: Does (193) represent the impedance function of a [illegible] ~~three~~-mesh network containing positive inductance, resistance and capacity elements? In general the answer is no. But, if the coefficients of (193) besides being positive reals, satisfy certain other conditions which will be given below, then we may say that (193) does represent the impedance function of a network with positive inductance, resistance and capacity elements. Without investigating (193) further, we do know from physical considerations that if (193) is to represent the impedance of a physical network, it is necessary that the zeros and poles of (193), which are in general, pairs of conjugate complex numbers, have negative reals and differ from each other, except for the pole at the origin. This follows from stability considerations in the network, since if the zeros and poles did not all have negative reals, and did not differ from each other, the current would become infinite with infinite time, which is impossible for a network with positive elements.

Let us see then what the conditions must be on the coefficients of (193), in order that it represent the impedance function of a physical network. In this we shall follow exactly the same method which we used in the two-mesh network with two kinds of network elements. This is of course the usual method used when it is desired to make the impedance of two networks equal at all frequencies. This is the method which Foster used in his paper, "Theorems Regarding the Driving-Point Impedance of Two-Mesh Circuits". In this case we desire to make the given expression (193) equal at all frequencies to the expression (192) for the most general network containing inductance, resistance and capacity elements.

Before proceeding to do this, we note that the expression (193) is unchanged if we multiply its numerator and denominator by a constant. As in the two-mesh case with two kinds of network elements, write (193) as follows

$$\frac{k^2 a_0 p^4 + k^2 a_1 p^3 + k^2 a_2 p^2 + k^2 a_3 p + k^2 a_4}{p(k^2 b_1 p^2 + k^2 b_2 p + k^2 b_3)} \qquad (194)$$

Comparing (194) with (192) we see that a_2 is formed from $\Delta_1(\lambda,\sigma)$ and $\Delta(\rho)$. Thus consider a_2 divided into two parts m and n, then we have

$$\Delta(\lambda) = k^2 a_0 \qquad (195a)$$

$$\Delta(\rho) = k^2 m \qquad (195b)$$

$$\Delta(\sigma) = k^2 a_4 \qquad (195c)$$

$$\Delta_1(\lambda,\rho) = k^2 a_1 \qquad (195d)$$

$$\Delta_1(\rho,\sigma) = k^2 a_3 \qquad (195e)$$

$$\Delta_1(\sigma,\lambda) = k^2 n \qquad (195f)$$

$$M_{11}(\lambda) = k^2 b_1 \qquad (195g)$$

$$M_{11}(\rho) = k^2 b_2 \qquad (195h)$$

$$M_{11}(\sigma) = k^2 b_3 \qquad (195i)$$

where

$$m + n = a_2$$

Note the similarity of this symmetrical system of equations with the system (108), page 77. It at once follows from this that the same methods of arriving at the conditions which the coefficients of (193) or (194) must satisfy in order that they represent the impedance function of a physical network, may be used in this case. Note for example that the equations (195a, b, d, g and h) are identical with the equations (108). Hence without going through a similar process to obtain an equation like (110), we may write at once

$$(k^2 b_2 \lambda_{12} - k^2 b_1 \rho_{12})^2 = k^2 a_1 \cdot k^2 b_1 \cdot k^2 b_2 - k^2 a_0 (k b_2)^2 - k^2 m (k b_1)^2$$

$$\therefore (b_2 \lambda_{12} - b_1 \rho_{12})^2 = k^2 (a_1 b_1 b_2 - a_0 b_2^2 - m b_1^2) \qquad (198)$$

Hence, since b_2, λ_{12}, b_1 and ρ_{12} are real, we must have

$$a_1 b_1 b_2 - a_0 b_2^2 - m b_1^2 \geqq 0$$

But the right hand side of (198) is -1 times the resultant of the numerator and denominator of

$$\frac{a_0 p^2 + a_1 p + m}{b_1 p + b_2}$$

Similarly from the symmetry of the equations (195) we have from (195b, c, e, h and i)

$$(b_3 \rho_{12} - b_2 \sigma_{12})^2 = k^2 (a_3 b_2 b_3 - m b_3^2 - a_4 b_2^2) \qquad (196)$$

Again, for real $b_3, \rho_{12}, b_2, \sigma_{12}$ we must have

$$a_3 b_2 b_3 - m b_3^2 - a_4 b_2^2 \geqq 0$$

Finally, from (195a, c, f, g and i) we have

$$(b_3 \lambda_{12} - b_1 \sigma_{12})^2 = k^2 (n b_1 b_3 - a_0 b_3^2 - a_4 b_1^2) \qquad (197)$$

Again for real $b_3, \lambda_{12}, b_1, \sigma_{12}$ it is necessary that

$$n b_1 b_3 - a_0 b_3^2 - a_4 b_1^2 \geqq 0$$

Note then that the right hand side of (196), (197) and (198) are respectively, except for a constant multiplier, the resultant of

$$\frac{mp^2 + a_3 p + a_4}{p(b_2 p + b_3)} \qquad (199a)$$

$$\frac{a_0 p^4 + n p^2 + a_4}{p(b_1 p + b_3)} \qquad (199b)$$

$$\frac{a_0 p^2 + a_1 p + m}{b_1 p + b_2} \qquad (199c)$$

The corresponding resultants are of course

$$\begin{vmatrix} m & a_3 & a_4 \\ b_2 & b_3 & 0 \\ 0 & b_2 & b_3 \end{vmatrix}, \quad \begin{vmatrix} a_0 & n & a_4 \\ b_1 & b_3 & 0 \\ 0 & b_1 & b_3 \end{vmatrix}, \quad \begin{vmatrix} a_0 & a_1 & m \\ b_1 & b_2 & 0 \\ 0 & b_1 & b_2 \end{vmatrix} \qquad (200)$$

Call these resultants respectively R_1, R_2 and R_3. Then the expressions on the **right** hand side of (196), (197) and (198) are respectively $\frac{1}{k^2}R_1$, $\frac{1}{k^2}R_2$, $\frac{1}{k^2}R_3$.

Thus, one of the conditions that (193) or (194) represent the impedance function of a physical network is that the three resultants (200) be respectively negative. This condition really means that the zeros and poles of each of the expressions (199) are negative reals and the pole separates the zeros. Extracting the square roots in (196), (197) and (198), we have

$$(b_3 \rho_{12} - b_2 \sigma_{12}) = \pm k \sqrt{-R_1} \qquad (201a)$$

$$(b_1 \sigma_{12} - b_3 \lambda_{12}) = \pm k \sqrt{-R_2} \qquad (201b)$$

$$(b_2 \lambda_{12} - b_1 \rho_{12}) = \pm k \sqrt{-R_3} \qquad (201c)$$

Note however that

$$(b_3 \rho_{12} - b_2 \sigma_{12})\, b_1 + (b_1 \sigma_{12} - b_3 \lambda_{12})\, b_2 + (b_2 \lambda_{12} - b_1 \rho_{12})\, b_3 = 0 \qquad (202)$$

Hence

$$\pm b_1 \sqrt{-R_1} \;\pm b_2 \sqrt{-R_2} \;\pm b_3 \sqrt{-R_3} = 0 \qquad (203)$$

The above results can perhaps be obtained more readily by means of a vector notation. It is well known that if $\alpha_1, \alpha_2, \alpha_3$ are components of a vector A, and $\beta_1, \beta_2, \beta_3$ are the components of a vector B, then the components of a vector C perpendicular to both the vectors A and B are respectively ~~to~~ :

$$(\alpha_2 \beta_3 - \alpha_3 \beta_2)\,, \quad (\alpha_3 \beta_1 - \alpha_1 \beta_3)\,, \quad (\alpha_1 \beta_2 - \alpha_2 \beta_1)$$

In our case, we may consider that $\lambda_{12}, \rho_{12}, \sigma_{12}$ represent the components of a vector P, and $\lambda_{22}, \rho_{22}, \sigma_{22}$ the components of a vector Q, then the components of a vector perpendicular to P and Q is the vector R having the components

$$(\rho_{12} \sigma_{22} - \sigma_{12} \rho_{22})\,; \quad (\sigma_{12} \lambda_{22} - \lambda_{12} \sigma_{22})\,; \quad (\lambda_{12} \rho_{22} - \rho_{12} \lambda_{22})$$

It follows that

$$P \cdot R = 0 \qquad (204a)$$

$$Q \cdot R = 0 \qquad (204b)$$

since R is the vector perpendicular to P and Q. Equation (202) is of course the same as (204b). Equation (204a) is of course

$$\lambda_{12}(\rho_{12}\sigma_{22}-\sigma_{12}\rho_{22})+\rho_{12}(\sigma_{12}\lambda_{22}-\lambda_{12}\sigma_{22})+\sigma_{12}(\lambda_{12}\rho_{22}-\rho_{12}\lambda_{22})=0$$

Thus in equation (203) if b_1, b_2 and b_3 represent the components of a vector then

$$\pm\sqrt{-R_1}\,,\quad \pm\sqrt{-R_2}\,,\quad \pm\sqrt{-R_3}$$

represent the components of a vector R perpendicular to the vector (b_1, b_2, b_3), so that

$$b\cdot\vec{R}=0$$

which is of course equation (203). Thus we can state the conditions which Foster has given that the expression

$$\frac{a_0p^4+a_1p^3+a_2p^2+a_3p+a_4}{p(b_1p^2+b_2p+b_3)} \qquad (193)$$

represent the impedance of a physical network. First, we divide a_2 into two parts m and n, such that $m+n=a_2$. Then form from (193) the three two-mesh impedances with two kinds of elements

$$\frac{mp^2+a_3p+a_4}{p(b_2p+b_3)} \qquad (205a)$$

$$\frac{a_0p^4+np^2+a_4}{p(b_1p+b_3)} \qquad (205b)$$

$$\frac{a_0p^2+a_1p+m}{b_1p+b_2} \qquad (205c)$$

which are respectively impedances of networks with capacity and resistance, inductance and capacity, and inductance and resistance. Then the conditions on the coefficients of (192) that it represent the impedance function of a physical network are that the coefficients of (193) be all real and positive, and that each of the resultants of (205) be negative, and if R_1, R_2 and R_3 are the corresponding resultants, then

$$\pm b_1\sqrt{-R_1} \pm b_2\sqrt{-R_2} \pm b_3\sqrt{-R_3} = 0 \qquad (203)$$

Let us see how to apply these conditions to a numerical problem. Thus consider the two-mesh network shown in figure 55.

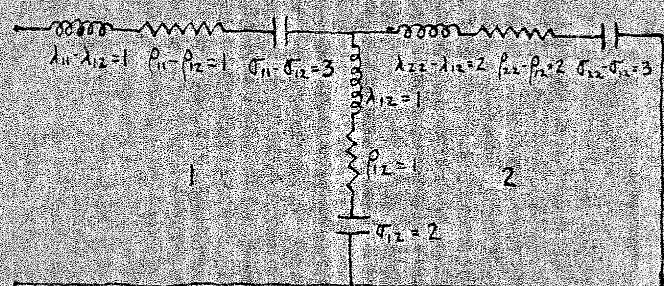


FIG. 55

Let us compute the impedance function of this network by the usual method. Thus

$$Z(p) = p+1+\frac{3}{p} + \frac{\left(p+1+\frac{2}{p}\right)\left(2p+2+\frac{3}{p}\right)}{\left(p+1+\frac{2}{p}\right)+\left(2p+2+\frac{3}{p}\right)}$$

$$= \frac{3p^4+3p^3+5p^2+3p^3+3p^2+5p+9p^2+9p+15+2p^4+4p^3+9p^2+7p+6}{p(3p^2+4p+5)}$$

$$\therefore \quad Z(p) = \frac{5p^4 + 10p^3 + 26p^2 + 21p + 21}{p\,(3p^2 + 3p + 5)} \qquad (206)$$

Let us calculate Z(p) by formula (192). We have for the parameters the following values

$$\lambda_{11} = 2,\ \lambda_{22} = 3,\ \lambda_{12} = 1\ ;\ \rho_{11} = 2,\ \rho_{22} = 3,\ \rho_{12} = 1\ ;\ \sigma_{11} = 5,\ \sigma_{22} = 5,\ \sigma_{12} = 2$$

Then

$$Z(p) = \frac{\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} p^4 + \left\{ \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \right\} p^3 + \left\{ \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 5 \end{vmatrix} + \begin{vmatrix} 5 & 1 \\ 2 & 3 \end{vmatrix} \right\} p^2 + \left\{ \begin{vmatrix} 2 & 2 \\ 1 & 5 \end{vmatrix} + \begin{vmatrix} 5 & 1 \\ 2 & 3 \end{vmatrix} \right\} p + \begin{vmatrix} 5 & 2 \\ 2 & 5 \end{vmatrix}}{p\,(3p^2 + 3p + 5)}$$

$$= \frac{5p^4 + 10p^3 + (5 + 21)p^2 + 21p + 21}{p\,(3p^2 + 3p + 5)} \qquad (206a)$$

Now we know that the expression (206a) represents the impedance function of a definite two-mesh network consisting of positive inductance, resistance and capacity elements. Hence we know that it must satisfy the conditions that (206a) represent the impedance function of a physical network. First, let us divide 26 into two real numbers m and n, such that $m + n = 26$. Now form the impedance functions

$$\frac{mp^2 + 21p + 21}{p\,(3p + 5)} \qquad (207a)$$

$$\frac{5p^4 + np^2 + 21}{p\,(3p + 5)} \qquad (207b)$$

$$\frac{5p^2 + 10p + m}{3p + 3} \qquad (207c)$$

Now form the corresponding resultants of (207).

These are

$$\begin{vmatrix} m & 21 & 21 \\ 3 & 5 & 0 \\ 0 & 3 & 5 \end{vmatrix} = 25m-126$$

In order that this be negative, it is necessary that

$$m \leqq \frac{126}{25} \qquad (208a)$$

$$\begin{vmatrix} 5 & n & 21 \\ 3 & 5 & 0 \\ 0 & 3 & 5 \end{vmatrix} = 314 - 15n$$

In order that this may be negative it is necessary that

$$n \geqq \frac{314}{15} \qquad (208b)$$

Finally

$$\begin{vmatrix} 5 & 10 & m \\ 3 & 3 & 0 \\ 0 & 3 & 3 \end{vmatrix} = -45+9m$$

Hence for this to be negative it is necessary that

$$m \leqq \frac{45}{9} = 5 \qquad (208c)$$

Comparing (208c) and (208a), we see that

$$m \leqq 5$$

From (208b)

$$n \geqq \frac{314}{15} = 20.866---$$

But

$$m + n = 26$$

Hence m may take on all real values between zero and 5 and n make take on all values between 21 ~~20.866---~~ and 26. This is expressed by the inequalities

$$\left.\begin{array}{l} 0 \leqq m \leqq 5 \\ 21\ \cancel{\frac{314}{15}} \leqq n \leqq 26 \end{array}\right\} \qquad (209)$$

From (203) we must have

$$\pm 3\sqrt{126 - 5m} \pm 3\sqrt{15n - 314} \pm 5\sqrt{45 - 9m} = 0 \qquad (210)$$

But $m + n = 26$

$\therefore \quad n = 26 - m$, and (210) becomes

$$\pm 3\sqrt{126-5m} \pm 3\sqrt{76-15m} \pm 15\sqrt{5-m} = 0 \qquad (210a)$$

In our example it is readily seen that $m = 5$, which is a solution of (210a). In general, it would be necessary to solve (210a) for m, which is not easy to do, since it is necessary to square the radicals, so that (210a) becomes in general a quadratic equation. Note that the solution for m can be obtained by allowing the resultant in (208c) to vanish.

Thus we see that the impedance function (206a) satisfied the condition that it represent the impedance of a physical network, which was to be expected.

Let us see now if (206a) represents the impedance of any other network for $m = 5$ than that shown in figure 55. Let us evaluate (201a,b,c). We have

$$\begin{aligned} R_1 &= 25m - 126 \\ &= 25(5) - 126 \\ &= -1 \end{aligned}$$

$$\begin{aligned} R_2 &= 314 - 15n \\ &= 314 - 15(21) \\ &= -1 \end{aligned}$$

$$\begin{aligned} R_3 &= -45 + 9m \\ &= -45 + 9(5) \\ &= 0 \end{aligned}$$

Hence the equations (201) become, since for our example, $b_1 = 3$, $b_2 = 3$, $b_3 = 5$

$$5\rho_{12} - 3\sigma_{12} = \pm k \qquad (211a)$$

$$3\sigma_{12} - 5\lambda_{12} = \pm k \qquad (211b)$$

and

$$3\lambda_{12} - 3\rho_{12} = 0 \qquad (211c)$$

These three equations represent straight lines. Expressing them in slope intercept form we have

$$\rho_{12} = \frac{3}{5}\sigma_{12} \pm \frac{k}{5} \qquad (212a)$$

$$\sigma_{12} = \frac{5}{3}\lambda_{12} \pm \frac{k}{3} \qquad (212b)$$

$$\lambda_{12} = \rho_{12} \qquad (212c)$$

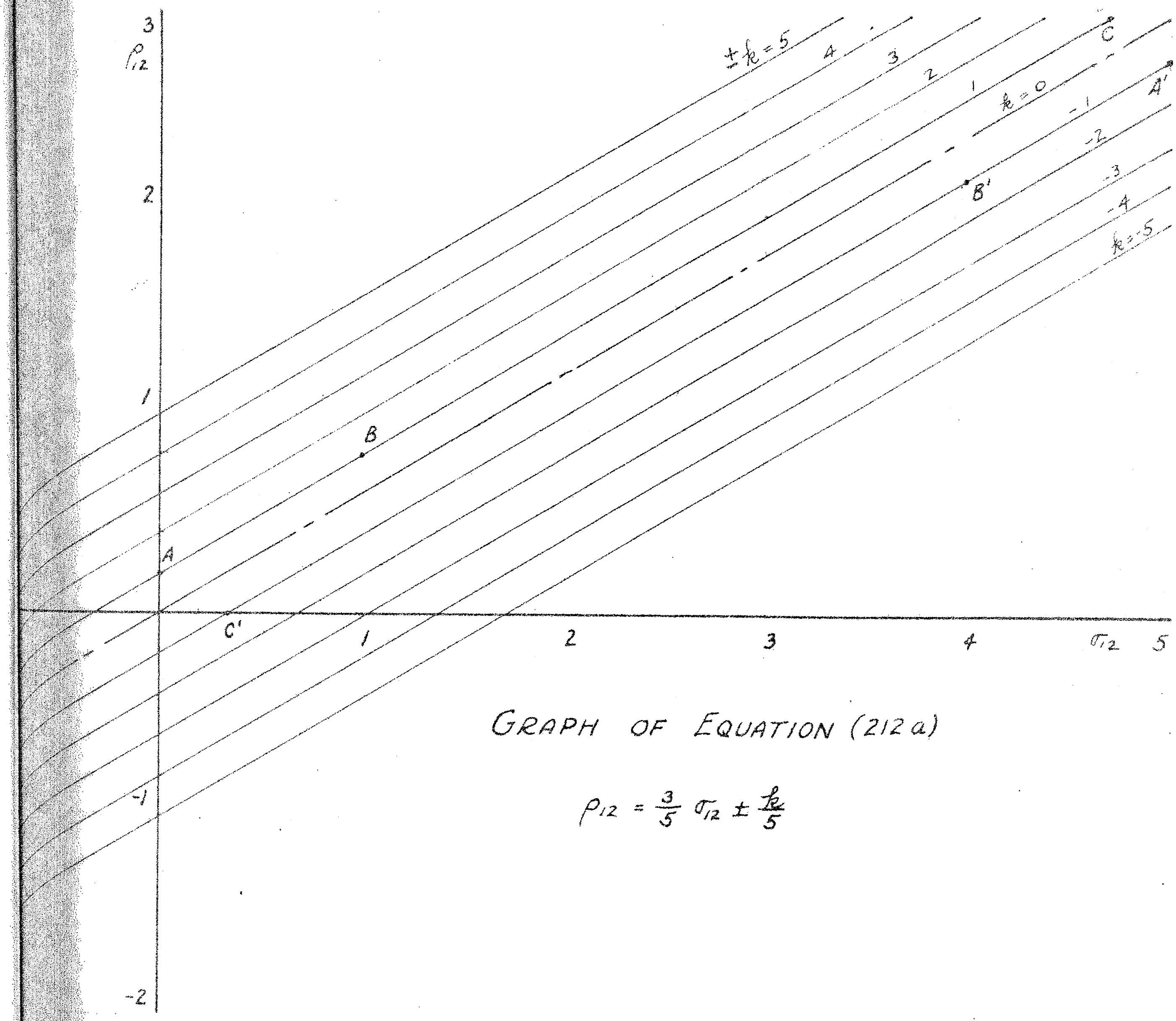
Substitute (212c) in (212b), and we have

$$\sigma_{12} = \frac{5}{3}\rho_{12} \pm \frac{k}{3}$$

And (212a) is

$$\sigma_{12} = \frac{5}{3}\rho_{12} \pm \frac{k}{3}$$

Hence only equation (212a) need be used to obtain all the networks equivalent to that shown in figure 55. Figure 56 shows the graph of this equation.



GRAPH OF EQUATION (212a)

$$\rho_{12} = \frac{3}{5}\sigma_{12} \pm \frac{k}{5}$$

FIG. 56

Thus considering the equation (212a)

$$\rho_{12} = \frac{3}{5} \sigma_{12} \pm \frac{k}{5} \qquad (212a)$$

suppose we take k = 1, then

$$\rho_{12} = \frac{3}{5} \sigma_{12} \pm \frac{1}{5}$$

Suppose σ_{12} is taken equal to zero, then $\rho_{12} = \frac{1}{5}$, and hence The known parameters are then

$$\sigma_{12} = 0, \quad \sigma_{22} = 5, \quad \rho_{12} = \frac{1}{5}, \quad \rho_{22} = 3, \quad \lambda_{12} = \frac{1}{5}, \quad \lambda_{22} = 3$$

Using (195) we proceed to find the other parameters. Thus

$$\lambda_{11} = \frac{5 + \left(\frac{1}{5}\right)^2}{3} = \frac{126}{75}$$

$$\rho_{11} = \frac{5 + \left(\frac{1}{5}\right)^2}{3} = \frac{126}{75}$$

$$\sigma_{11} = \frac{21 + 0}{5} = \frac{21}{5}$$

Hence the parameters of the network are

$$\lambda_{11} = \frac{126}{75}, \lambda_{22} = 3, \lambda_{12} = \frac{1}{5}; \rho_{11} = \frac{126}{75}, \rho_{22} = 3, \rho_{12} = \frac{1}{5}; \sigma_{11} = \frac{21}{5}, \sigma_{22} = 5, \sigma_{12} = 0$$

Hence the corresponding network is shown in figure 57

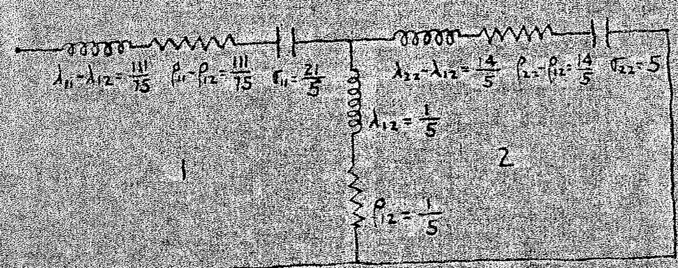


FIG. 57

Let us calculate the impedance of the network shown in figure 57. It is

$$Z(p) = \frac{\begin{vmatrix} \frac{126}{75} & \frac{1}{5} \\ \frac{1}{5} & 3 \end{vmatrix} p^4 + \left\{ \begin{vmatrix} \frac{126}{75} & \frac{1}{5} \\ \frac{1}{5} & 3 \end{vmatrix} + \begin{vmatrix} \frac{126}{75} & \frac{1}{5} \\ \frac{1}{5} & 3 \end{vmatrix} \right\} p^3 + \left\{ \begin{vmatrix} \frac{126}{75} & \frac{1}{5} \\ \frac{1}{5} & 3 \end{vmatrix} + \begin{vmatrix} \frac{126}{75} & 0 \\ \frac{1}{5} & 5 \end{vmatrix} + \begin{vmatrix} \frac{21}{5} & \frac{1}{5} \\ 0 & 3 \end{vmatrix} \right\} p^2 + \left\{ \begin{vmatrix} \frac{21}{5} & \frac{1}{5} \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} \frac{126}{75} & 0 \\ \frac{1}{5} & 5 \end{vmatrix} \right\} p + \begin{vmatrix} \frac{21}{5} & 0 \\ 0 & 5 \end{vmatrix}}{p(3p^2+3p+5)}$$

$$= \frac{5p^4 + 10p^3 + 26p^2 + 21p + 21}{p(3p^2+3p+5)} \qquad (206\,b)$$

which is exactly the impedance of the network shown in figure 55. Note that the network shown in figure 57 has one less element than that shown in figure 55. This network (figure 57) is represented by the point $\sigma_{12} = 0$, $\rho_{12} = \frac{1}{5}$, that is point A in figure 56.

Taking $k = -1$, we have

$$\rho_{12} = \frac{3}{5} \cdot \sigma_{12} - \frac{1}{5}$$

The image of the network in figure 57 is readily obtained by making $\sigma_{12} = 5$, in which case $\rho_{12} = \frac{14}{5}$ and hence $\lambda_{12} = \frac{14}{5}$

$$\therefore \lambda_{11} = \frac{5 + \left(\frac{14}{5}\right)^2}{3} = \frac{321}{75}$$

$$\therefore \rho_{11} = \frac{321}{75}$$

$$\sigma_{11} = \frac{21 + (5)^2}{5} = \frac{46}{5}$$

The parameters of the network are

$\lambda_{11} = \frac{321}{75}$, $\lambda_{22} = 3$, $\lambda_{12} = \frac{14}{5}$; $\rho_{11} = \frac{321}{75}$, $\rho_{22} = 3$, $\rho_{12} = \frac{14}{5}$; $\sigma_{11} = \frac{46}{5}$, $\sigma_{22} = 5$, $\sigma_{12} = 5$

The corresponding network is exactly that shown in figure 57 with the branches in mesh 2 interchanged. Point A' in figure 56 shows this network, the image of point A.

Thus we see, as in the two-mesh case with only two kinds of elements, that there is a one to one correspondence between the points in the mutual parameter plane and two-mesh electrical networks, provided we consider networks with the branches in mesh two interchanged as different networks and so corresponding to different points in the plane, which points may be considered images of each other. If all the points in the plane are included, networks with negative parameters may be obtained, suggesting as before, the possibility of making use of networks with negative parameters, which would of course have to be realized in ways other than by coils, resistors and condensers.

Let us for example let $\sigma_{12} = 1$ keeping k still equal to 1. Then

$$\rho_{12} = \frac{3}{5}\sigma_{12} + \frac{1}{5}$$

$$= \frac{4}{5}$$

Then from (212c) $\lambda_{12} = \frac{4}{5}$. Using (195) we proceed to find the other parameters. Thus

$$\lambda_{11} = \frac{5 + \left(\frac{4}{5}\right)^2}{3} = \frac{141}{75}$$

$$\rho_{11} = \frac{5 + \left(\frac{4}{5}\right)^2}{3} = \frac{141}{75}$$

$$\sigma_{11} = \frac{21+1}{5} = \frac{22}{5}$$

The parameters of the network are then

$\lambda_{11} = \frac{141}{75}$, $\lambda_{22} = 3$, $\lambda_{12} = \frac{4}{5}$; $\rho_{11} = \frac{141}{75}$, $\rho_{22} = 3$, $\rho_{12} = \frac{4}{5}$; $\sigma_{11} = \frac{26}{5}$, $\sigma_{22} = 5$, $\sigma_{12} = 1$

and the corresponding network is shown in figure 58.

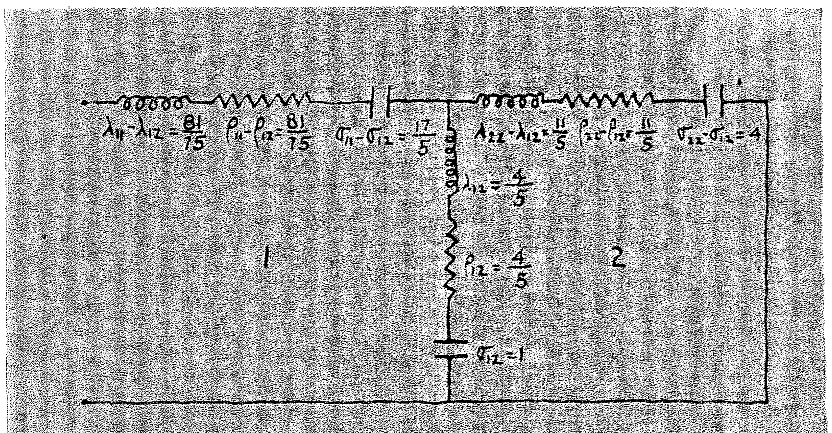


FIG. 58

The impedance function is seen to be

$$Z(p) = \frac{\begin{vmatrix} \frac{141}{75} & \frac{4}{5} \\ \frac{4}{5} & 3 \end{vmatrix} p^4 + \left\{ \begin{vmatrix} \frac{141}{75} & \frac{4}{5} \\ \frac{4}{5} & 3 \end{vmatrix} + \begin{vmatrix} \frac{141}{75} & \frac{4}{5} \\ \frac{4}{5} & 3 \end{vmatrix} \right\} p^3 + \left\{ \begin{vmatrix} \frac{141}{75} & \frac{4}{5} \\ \frac{4}{5} & 3 \end{vmatrix} + \begin{vmatrix} \frac{141}{75} & 1 \\ \frac{4}{5} & 5 \end{vmatrix} + \begin{vmatrix} \frac{22}{5} & \frac{4}{5} \\ 1 & 3 \end{vmatrix} \right\} p^2 + \left\{ \begin{vmatrix} \frac{141}{75} & 1 \\ \frac{4}{5} & 5 \end{vmatrix} + \begin{vmatrix} \frac{22}{5} & \frac{4}{5} \\ 1 & 3 \end{vmatrix} \right\} p + \begin{vmatrix} \frac{22}{5} & 1 \\ 1 & 5 \end{vmatrix}}{p(3p^2+3p+5)}$$

$$= \frac{5p^4 + 10p^3 + 26p^2 + 21p + 21}{p(3p^2 + 3p + 5)} \qquad (206\,c)$$

The network shown in figure 58 is seen to contain all nine of the network elements. This network corresponds to the point $\sigma_{12} = 1$, $\rho_{12} = \frac{4}{5}$, that is point B in the mutual parameter plane. Its image can be readily computed and found to be the point $\sigma_{12} = 4$, $\rho_{12} = \frac{11}{5}$ with $k = -1$, that is the point B'.

The maximum value that ρ_{12} may have is of course 3, if we are to consider positive elements only for our network. Hence from

$$\rho_{12} = \frac{3}{5}\sigma_{12} + \frac{1}{5}$$

we have that for $\rho_{12} = 3$, $\sigma_{12} = \frac{14}{3}$. Then $\lambda_{12} = 3$, and

$$\lambda_{11} = \frac{5+3^2}{3} = \frac{14}{3}$$

$$\rho_{11} = \frac{14}{3}$$

$$\sigma_{11} = \frac{21 + \left(\frac{14}{3}\right)^2}{5} = \frac{77}{9}$$

Then the parameters of the network are $\lambda_{11} = \frac{14}{3}$, $\lambda_{22} = 3$, $\lambda_{12} = 3$; $\rho_{11} = \frac{14}{3}$, $\rho_{22} = 3$, $\rho_{12} = 3$; $\sigma_{11} = \frac{77}{9}$, $\sigma_{22} = 5$, $\sigma_{12} = \frac{14}{3}$, and the network is shown in figure 59.

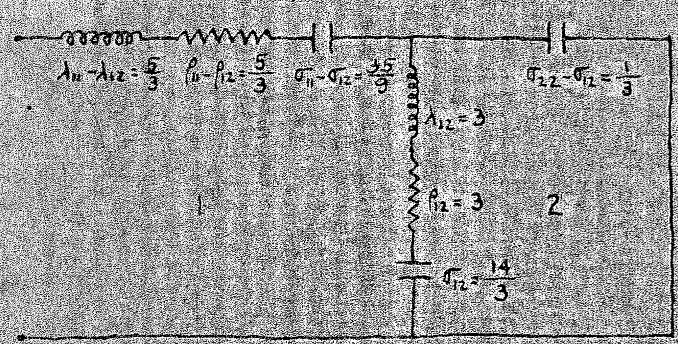


FIG. 59

Let us compute the impedance of this network. It is

$$Z(p) = \frac{\begin{vmatrix} \frac{14}{3} & 3 \\ 3 & 3 \end{vmatrix} p^4 + \left\{ \begin{vmatrix} \frac{14}{3} & 3 \\ 3 & 3 \end{vmatrix} + \begin{vmatrix} \frac{14}{3} & 3 \\ 3 & 3 \end{vmatrix} \right\} p^3 + \left\{ \begin{vmatrix} \frac{14}{3} & 3 \\ 3 & 3 \end{vmatrix} + \begin{vmatrix} \frac{14}{3} & \frac{14}{3} \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} \frac{77}{9} & 3 \\ \frac{14}{3} & 3 \end{vmatrix} \right\} p^2 + \left\{ \begin{vmatrix} \frac{14}{3} & \frac{14}{3} \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} \frac{77}{9} & 3 \\ \frac{14}{3} & 3 \end{vmatrix} \right\} p + \begin{vmatrix} \frac{77}{9} & \frac{14}{3} \\ \frac{14}{3} & 5 \end{vmatrix}}{p(3p^2+3p+5)}$$

$$= \frac{5p^4+10p^3+26p^2+21p+21}{p(3p^2+3p+5)}$$

The network in figure 59 is seen to be a network with but seven elements instead of nine or eight. It turns out to be a seven element network instead of an eight-element one because we happened to take the same values for the inductances and resistances to start with. This network corresponds to the point $\sigma_{12} = \frac{14}{3}$, $\rho_{12} = 3$ in the mutual parameter plane, that is, point C. The image point is readily obtained and is the point $\sigma_{12} = \frac{1}{3}$, $\rho_{12} = 0$, that is point C'.

It is a simple matter, as in the two-mesh case with two kinds of network elements to proceed to obtain the networks with the least number of elements. We shall proceed to do this with a network whose inductances are not equal to the resistances as in the previous example. It should be pointed out, however, that when an impedance function is obtained from a given network, it is unnecessary to divide the coefficient of p^2 into two parts m and n, and apply the conditions that it be the impedance of a physical network. We already know this to be the case, and furthermore, our method of constructing the impedance function, by means of our symbolic formula, automatically divides the coefficient of p^2 into two parts. Thus the problem of obtaining all the networks equivalent to a given network is much simpler than that of seeing if a given function which has the form of the impedance function of a two-mesh network, is in fact the impedance of a physical network with positive elements.

Thus for example, consider the network shown in figure 60.

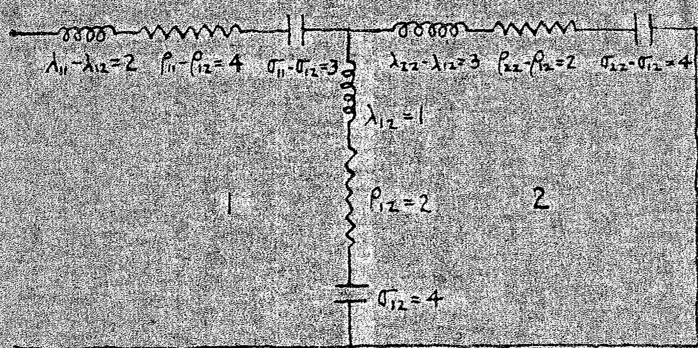


FIG. 60

The parameters of this network are $\lambda_{11} = 3$, $\lambda_{22} = 4$, $\lambda_{12} = 1$; $\rho_{11} = 6$, $\rho_{22} = 4$, $\rho_{12} = 2$; $\sigma_{11} = 7$, $\sigma_{22} = 8$; $\sigma_{12} = 4$, and the impedance function is

$$Z(p) = \frac{\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} p^4 + \left\{ \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 1 \\ 2 & 4 \end{vmatrix} \right\} p^3 + \left\{ \begin{vmatrix} 6 & 2 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 1 & 8 \end{vmatrix} + \begin{vmatrix} 7 & 1 \\ 4 & 4 \end{vmatrix} \right\} p^2 + \left\{ \begin{vmatrix} 6 & 4 \\ 2 & 8 \end{vmatrix} + \begin{vmatrix} 7 & 2 \\ 4 & 4 \end{vmatrix} \right\} p + \begin{vmatrix} 7 & 4 \\ 4 & 8 \end{vmatrix}}{p(4p^2 + 4p + 8)}$$

$$= \frac{11p^4 + 32p^3 + (20 + 44)p^2 + 60p + 40}{p(4p^2 + 4p + 8)} \qquad (207')$$

Note that the coefficient of the p^2 term automatically divides itself into the two numbers m and n, so that

$$m = 20$$
$$n = 44$$

Instead of proceeding as before to break up (207) into three impedances like (199a,b and c) and obtaining their resultants, we can simplify matters by writing the three determinants at once

$$R_1 = \begin{vmatrix} m & a_3 & a_4 \\ b_2 & b_3 & 0 \\ 0 & b_2 & b_3 \end{vmatrix} \qquad R_2 = \begin{vmatrix} a_0 & n & a_4 \\ b_1 & b_3 & 0 \\ 0 & b_1 & b_3 \end{vmatrix} \qquad R_3 = \begin{vmatrix} a_0 & a_1 & m \\ b_1 & b_2 & 0 \\ 0 & b_1 & b_2 \end{vmatrix}$$

In our example

$$R_1 = \begin{vmatrix} 20 & 60 & 40 \\ 4 & 8 & 0 \\ 0 & 4 & 8 \end{vmatrix} = 0$$

$$R_2 = \begin{vmatrix} 11 & 44 & 40 \\ 4 & 8 & 0 \\ 0 & 4 & 8 \end{vmatrix} = -64$$

$$R_3 = \begin{vmatrix} 11 & 32 & 20 \\ 4 & 4 & 0 \\ 0 & 4 & 4 \end{vmatrix} = -16$$

Note that R_1, R_2 and R_3 are all perfect squares, which will always be the case when we begin with a network having rational network elements. Note also that R_1, R_2 or R_3 are all negative or zeros, which we should expect, since these are the conditions for a physical network. Now obtain

$$V_1 = \sqrt{-R_1} \qquad V_2 = \sqrt{-R_2} \qquad V_3 = \sqrt{-R_3}$$

which in our case is

$$V_1 = 0 \qquad V_2 = \pm 8 \qquad V_3 = \pm 4$$

Note that $b \cdot V$ must be zero. In our case this is

$$4 \cdot 0 + 4(\pm 8) + 8(\pm 4) = 0$$

provided the signs are so chosen that V_2 and V_3 have opposite signs.

Having V_1, V_2 and V_3, we may at once write the equations (201a,b and c) which are

$$b_3 \rho_{12} - b_2 \sigma_{12} = \pm k V_1 \qquad (208a')$$

$$b_1 \sigma_{12} - b_3 \lambda_{12} = \pm k V_2 \qquad (208b')$$

$$b_2 \lambda_{12} - b_1 \rho_{12} = \pm k V_3 \qquad (208c')$$

These formulas are easily remembered if we regard $\lambda_{12}, \rho_{12}, \sigma_{12}$ as the components of a vector A and b_1, b_2, b_3 as the components of a vector B, and kV_1, kV_2, kV_3 as the components of a vector kV perpendicular to the two vectors A and B. The components of a vector perpendicular to the vector A and B are of course ($\rho_{12} b_3 - \sigma_{12} b_2$), ($\sigma_{12} b_1 - \lambda_{12} b_3$) and ($\lambda_{12} b_2 - \rho_{12} b_1$), and these components must of course be equal to the components of the vector kV, except for sign. This conception explains at once why $b \cdot R = 0$. It seems that this vector conception of the network can be generalized for any number of meshes, in which case we shall deal with vectors in n-dimensional space.

In our example then, the equations (208') are

$$8\rho_{12} - 4\sigma_{12} = 0 \qquad (209a')$$

$$4\sigma_{12} - 8\lambda_{12} = \pm k8 \qquad (209b')$$

$$4\lambda_{12} - 4\rho_{12} = \pm k4 \qquad (209c')$$

From (209a')

$$\rho_{12} = \frac{\sigma_{12}}{2} \qquad (210')$$

Substituting in (209c'), we have

$$4\lambda_{12} - 2\sigma_{12} = \pm k4$$

which is exactly (209b'). Thus (209b') may be used for our graph. This equation simplified is

$$\sigma_{12} - 2\lambda_{12} = \pm 2k \qquad (211')$$

This equation expressed in the slope-intercept form is

$$\sigma_{12} = 2\lambda_{12} \pm 2k \qquad (212')$$

This equation is of course a family of straight lines with k as a parameter, and its graph is shown in figure 61.

As we have seen, every point in this mutual parameter plane represents a two-mesh network with inductance, resistance and capacity elements, which however may be positive or negative. Certain regions in the plane represent networks having only positive elements. To every point in the plane there is an image point which represents a network with the branches in mesh 2 interchanged.

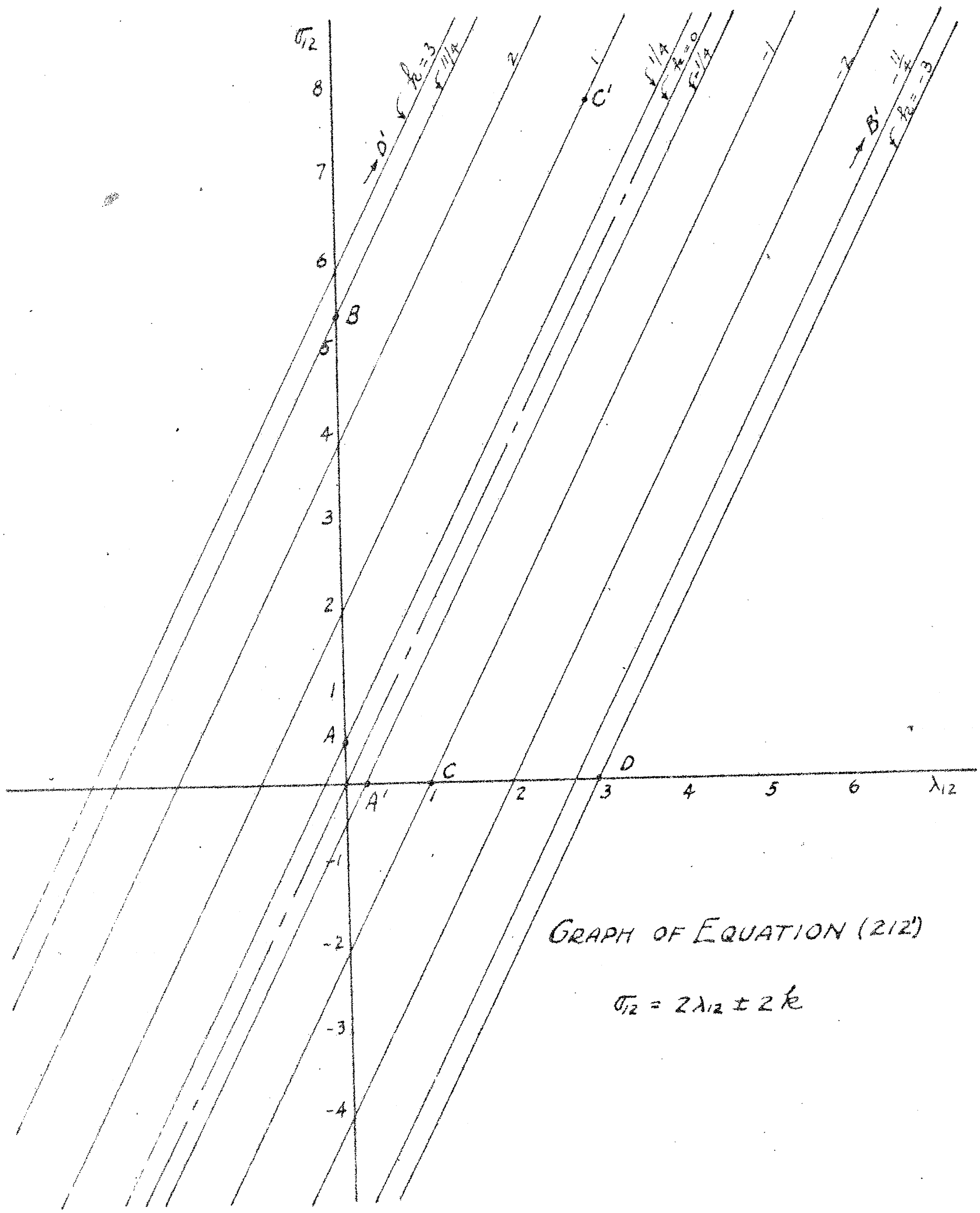

σ_{12}
λ_{12}
$k=3$
$11/4$
2
1
$1/4$
$k=0$
$-1/4$
-1
-2
$-11/4$
$k=-3$
A
B
C
D
A'
B'
C'
D'
GRAPH OF EQUATION (212')
$\sigma_{12} = 2\lambda_{12} \pm 2k$

FIG. 61

It is a simple matter by means of (212') and the system of equations (195) page 183, to obtain networks for any mutual parameter or we please, within certain regions in the plane. Let us however proceed to find the networks with the least number of elements having (207') as an impedance function. It is readily seen from (212') and (195), that in general we can reduce the most general network containing nine elements, to networks having but seven elements, by making one of the elements zero, and choosing k so that another element becomes zero. Let us proceed to do this. Thus, for example, make $\lambda_{12} = 0$ in (212'). Then

$$\sigma_{12} = \pm 2k$$

From (210')

$$\rho_{12} = \pm k$$

Now let us use the equations (195) to obtain all the other parameters. From (207') and (195g,h and i) we have that $\lambda_{22} = 4k^2$ $\rho_{22} = 4k^2$ and $\sigma_{22} = 8k^2$.

From (195a)

$$\lambda_{22}\lambda_{11} - \lambda_{12}^2 = 11k^2$$

$$\therefore \quad \lambda_{11} = \frac{11k^2 + \lambda_{12}^2}{\lambda_{22}}$$

$$= \frac{11k^2}{4k^2} = \frac{11}{4}$$

Now from (195b)

$$\rho_{11} = \frac{20k^2 + k^2}{4k^2} = \frac{21}{4}$$

And from (195c)

$$\sigma_{11} = \frac{40k^2 + 4k^2}{8k^2} = \frac{11}{2}$$

Suppose now that we take $\rho_{12} = \rho_{22}$

Then

$$\pm k = 4k^2$$

and

$$k = \pm \frac{1}{4}$$

Hence the parameters of the network are

$$\lambda_{11} = \frac{11}{4}\ ,\ \lambda_{22} = \frac{1}{4},\ \lambda_{12} = 0\ ;\ \rho_{11} = \frac{21}{4}\ ,\ \rho_{22} = \frac{1}{4}\ ,\ \rho_{12} = \frac{1}{4}\ ;\ \sigma_{11} = \frac{11}{2}\ ,\ \sigma_{22} = \frac{1}{2}\ ,\ \sigma_{12} = \frac{1}{2}$$

and the corresponding network is

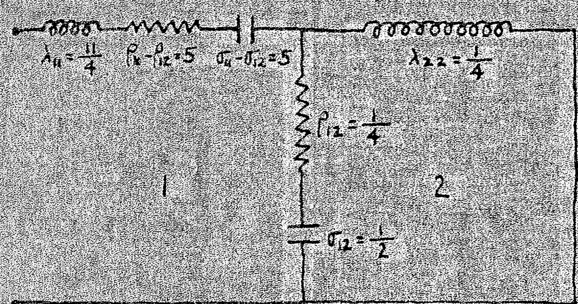


FIG. 61'

Let us compute the impedance of this network. It is

$$Z(p) = \frac{\begin{vmatrix} \frac{11}{4} & 0 \\ 0 & \frac{1}{4} \end{vmatrix} p^4 + \left\{ \begin{vmatrix} \frac{11}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} \end{vmatrix} + \begin{vmatrix} \frac{21}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} \end{vmatrix} \right\} p^3 + \left\{ \begin{vmatrix} \frac{21}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{vmatrix} + \begin{vmatrix} \frac{11}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{vmatrix} + \begin{vmatrix} \frac{11}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} \end{vmatrix} \right\} p^2 + \left\{ \begin{vmatrix} \frac{21}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{vmatrix} + \begin{vmatrix} \frac{11}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \end{vmatrix} \right\} p + \begin{vmatrix} \frac{11}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}}{p\left(\frac{1}{4}p^2 + \frac{1}{4}p + \frac{1}{2}\right)}$$

$$= \frac{\frac{11}{16}p^4 + \frac{32}{16}p^3 + \left(\frac{20}{16} + \frac{44}{16}\right)p^2 + \frac{60}{16}p + \frac{40}{16}}{p\left(\frac{4}{16}p^2 + \frac{4}{16}p + \frac{8}{16}\right)} \qquad (207a')$$

which is exactly (207')

Thus we have a network equivalent to the network shown in figure 61 with but six elements. We have six elements here instead of seven, because making $\rho_{22}-\rho_{12}=0$ also made $\sigma_{22}-\sigma_{12}=0$.

This network corresponds to the point $\lambda_{12}=0,\ \sigma_{12}=\frac{1}{2}$ with $k=\frac{1}{4}$, that is point A, figure 61. The image network is readily obtained and corresponds to the point $\lambda_{12}=\frac{1}{4},\ \sigma_{12}=0,$ $k=-\frac{1}{4}$, that is point A', figure 61.

Let us how take $\sigma_{12}=\sigma_{11}$

Then $$\pm 2k=\frac{11}{2}$$

and $$k=\pm\frac{11}{4}$$

The parameters of the network are then

$$\lambda_{11}=\frac{11}{4},\ \lambda_{22}=\frac{121}{4},\ \lambda_{12}=0;\ \rho_{11}=\frac{21}{4},\ \rho_{22}=\frac{121}{4},\ \rho_{12}=\frac{11}{4};\ \sigma_{11}=\frac{11}{2},\ \sigma_{22}=\frac{121}{2},\ \sigma_{12}=\frac{11}{2}$$

and the corresponding network is shown in figure 62.

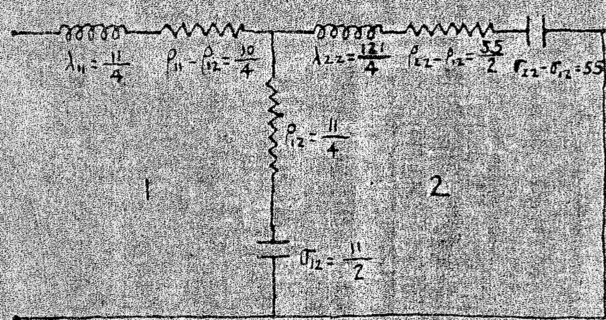


FIG. 62

The impedance of the network is seen to be

$$Z(p) = \frac{\begin{vmatrix} \frac{11}{4} & 0 \\ 0 & \frac{121}{4} \end{vmatrix} p^4 + \left\{ \begin{vmatrix} \frac{11}{4} & \frac{11}{4} \\ 0 & \frac{121}{4} \end{vmatrix} + \begin{vmatrix} \frac{21}{4} & 0 \\ \frac{11}{4} & \frac{121}{4} \end{vmatrix} \right\} p^3 + \left\{ \begin{vmatrix} \frac{21}{4} & \frac{11}{4} \\ \frac{11}{4} & \frac{121}{4} \end{vmatrix} + \begin{vmatrix} \frac{11}{4} & \frac{11}{2} \\ 0 & \frac{121}{2} \end{vmatrix} + \begin{vmatrix} \frac{11}{2} & 0 \\ \frac{11}{2} & \frac{121}{4} \end{vmatrix} \right\} p^2 + \left\{ \begin{vmatrix} \frac{21}{4} & \frac{11}{2} \\ \frac{11}{4} & \frac{121}{2} \end{vmatrix} + \begin{vmatrix} \frac{11}{2} & \frac{11}{4} \\ \frac{11}{2} & \frac{121}{4} \end{vmatrix} \right\} p + \begin{vmatrix} \frac{11}{2} & \frac{11}{2} \\ \frac{11}{2} & \frac{121}{2} \end{vmatrix}}{p\left(\frac{121}{4} p^2 + \frac{121}{4} p + \frac{121}{2} \right)}$$

$$= \frac{11\left(\frac{11}{4}\right)^2 p^4 + 32\left(\frac{11}{4}\right)^2 p^3 + \left[20\left(\frac{11}{4}\right)^2 + 44\left(\frac{11}{4}\right)^2\right] p^2 + 60\left(\frac{11}{4}\right)^2 p + 40\left(\frac{11}{4}\right)^2}{p\left[4\left(\frac{11}{4}\right)^2 p^2 + 4\left(\frac{11}{4}\right)^2 p + 8\left(\frac{11}{4}\right)^2\right]} \qquad (207b')$$

which is exactly (207').

The point in the mutual parameter plane corresponding to this network is the point $\lambda_{12} = 0$, $\sigma_{12} = \frac{11}{2}$, $k = \frac{11}{4}$ that is the point B, figure 61. The image point will be the point $\lambda_{12} = \frac{121}{4}$, $\sigma_{12} = 55$, $k = -\frac{11}{4}$, which is off the graph, and so is represented by the arrow B', to indicate that the image point is on the line $k = -\frac{11}{4}$ but off the graph.

Now let us make $\sigma_{22} - \sigma_{12} = 0$

$$\therefore 8k^2 = \pm 2k$$

and $\qquad k = 0$

or $\qquad k = \pm \frac{1}{4}$

The parameters of the network are then the same as those for figure 61', since making $\sigma_{22} - \sigma_{12} = 0$, also makes $\rho_{22} - \rho_{12} = 0$

We can proceed now to make, for example, $\sigma_{12} = 0$

then from (212')

$$\lambda_{12} = \pm k$$

and from (210')

$$\rho_{12} = 0$$

As before

$$\lambda_{11} = \frac{11k^2 + k^2}{4k^2} = 3$$

$$\rho_{11} = \frac{20k^2}{4k^2} = 5$$

$$\sigma_{11} = \frac{40k^2}{8k^2} = 5$$

The parameters of the network are then

$$\lambda_{11} = 3,\ \lambda_{22} = 4k^2;\ \lambda_{12} = \pm k;\ \rho_{11} = 5,\ \rho_{22} = 4k^2;\ \rho_{12} = 0;\ \sigma_{11} = 5,\ \sigma_{22} = 8k^2;\ \sigma_{12} = 0$$

Here we have parameters of seven element networks where k may take on any desired value resulting in positive elements. Thus, suppose we take $k = -1$. The parameters then are

$$\lambda_{11} = 3,\ \lambda_{22} = 4,\ \lambda_{12} = 1;\ \rho_{11} = 5,\ \rho_{22} = 4,\ \rho_{12} = 0;\ \sigma_{11} = 5,\ \sigma_{22} = 8,\ \sigma_{12} = 0$$

and the corresponding network is shown in figure 63

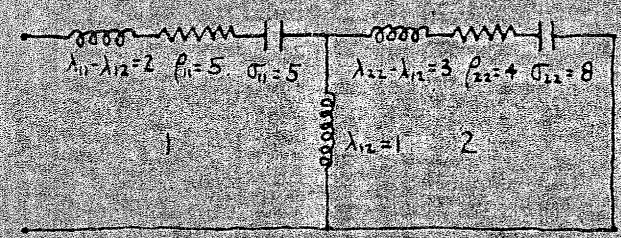


FIG 63

The impedance function of the network is

$$Z(p) = \frac{\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} p^4 + \left\{ \begin{vmatrix} 3 & 0 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 5 & 1 \\ 0 & 4 \end{vmatrix} \right\} p^3 + \left\{ \begin{vmatrix} 5 & 0 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 3 & 0 \\ 1 & 8 \end{vmatrix} + \begin{vmatrix} 5 & 1 \\ 0 & 4 \end{vmatrix} \right\} p^2 + \left\{ \begin{vmatrix} 5 & 0 \\ 0 & 8 \end{vmatrix} + \begin{vmatrix} 5 & 0 \\ 0 & 4 \end{vmatrix} \right\} p + \begin{vmatrix} 5 & 0 \\ 0 & 8 \end{vmatrix}}{p(4)(p^2+4p+8)}$$

$$= \frac{11p^4 + 32p^3 + (20+44)p^2 + 60p + 40}{p(4p^2+4p+8)} \qquad (207c')$$

which is exactly (207').

The point in the mutual parameter plane corresponding to this network is the point $\lambda_{12} = 1$, $\sigma_{12} = 0$, $k = -1$, that is, point C, figure 61. This image point will be $\lambda_{12} = 3$, $\sigma_{12} = 8$, $k = 1$ that is the point C', figure 61.

Let us proceed however to see if we can still eliminate one more element by properly choosing a value of k. Thus, suppose we make

$$\lambda_{11} - \lambda_{12} = 0$$

Then
$$\pm k = 3$$

and
$$k = \pm 3$$

The parameters of the network then are

$\lambda_{11} = 3$, $\lambda_{22} = 36$, $\lambda_{12} = 3$; $\rho_{11} = 5$, $\rho_{22} = 36$, $\rho_{12} = 0$; $\sigma_{11} = 5$, $\sigma_{22} = 72$, $\sigma_{12} = 0$

and the corresponding network is shown in figure 64.

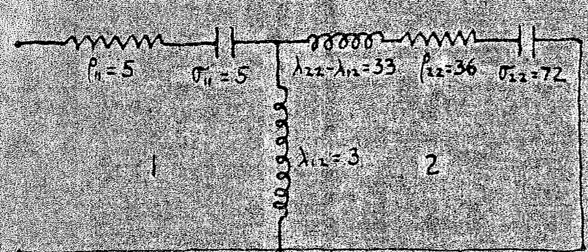


FIG 64

The impedance function of the network is

$$Z(p) = \frac{\begin{vmatrix} 3 & 3 \\ 3 & 36 \end{vmatrix} p^4 + \left\{ \begin{vmatrix} 3 & 0 \\ 3 & 36 \end{vmatrix} + \begin{vmatrix} 5 & 3 \\ 0 & 36 \end{vmatrix} \right\} p^3 + \left\{ \begin{vmatrix} 5 & 0 \\ 0 & 36 \end{vmatrix} + \begin{vmatrix} 3 & 0 \\ 3 & 72 \end{vmatrix} + \begin{vmatrix} 5 & 3 \\ 0 & 36 \end{vmatrix} \right\} p^2 + \left\{ \begin{vmatrix} 5 & 0 \\ 0 & 72 \end{vmatrix} + \begin{vmatrix} 5 & 0 \\ 0 & 36 \end{vmatrix} \right\} p + \begin{vmatrix} 5 & 0 \\ 0 & 72 \end{vmatrix}}{p(36p^2 + 36p + 72)}$$

$$= \frac{11(3^2)p^4 + 32(3^2)p^3 + [20(3^2) + 44(3^2)]p^2 + 60(3^2)p + 40(3^2)}{p[4(3^2)p^2 + 4(3^2)p + 8(3^2)]}$$

which is exactly (207').

The point in the mutual parameter plane is the point $\lambda_{12} = 3, \ \sigma_{12} = 0, \ k = -3$ that is point D, figure 61. The image point is $\lambda_{12} = 33, \ \sigma_{12} = 72, \ k = 3,$ which is off the graph, and so is represented by the arrow D', indicating that the point is on the line $k = 3$ but off the graph.

Now let us make

$$\lambda_{22} - \lambda_{12} = 0$$

Then

$$4k^2 = \pm k$$

And

$$k = 0$$

or

$$k = \pm \frac{1}{4}$$

The parameters of the network are then

$\lambda_{11} = 3, \ \lambda_{22} = \frac{1}{4}, \ \lambda_{12} = \frac{1}{4}; \ \rho_{11} = 5, \ \rho_{22} = \frac{1}{4}, \ \rho_{12} = 0; \ \sigma_{11} = 5, \ \sigma_{22} = \frac{1}{2}, \ \sigma_{12} = 0$

and the corresponding network is shown in figure 65.

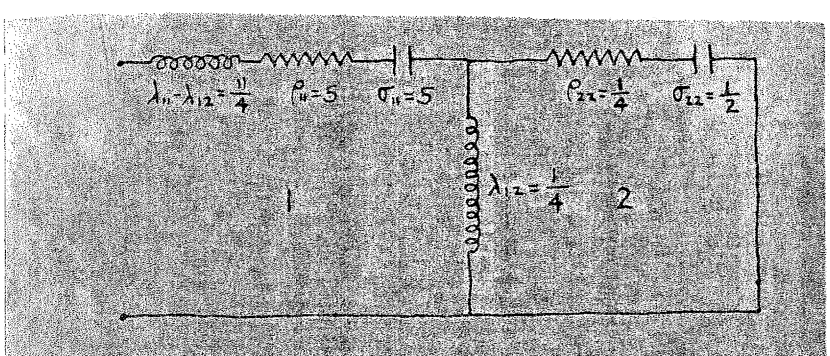


FIG.65

The impedance function of the network is

$$Z(p)=\frac{\begin{vmatrix}3&\frac{1}{4}\\ \frac{1}{4}&\frac{1}{4}\end{vmatrix}p^4+\left\{\begin{vmatrix}3&0\\ \frac{1}{4}&\frac{1}{4}\end{vmatrix}+\begin{vmatrix}5&\frac{1}{4}\\ 0&\frac{1}{4}\end{vmatrix}\right\}p^3+\left\{\begin{vmatrix}5&0\\ 0&\frac{1}{4}\end{vmatrix}+\begin{vmatrix}3&0\\ \frac{1}{4}&2\end{vmatrix}+\begin{vmatrix}5&\frac{1}{4}\\ 0&\frac{1}{4}\end{vmatrix}\right\}p^2+\left\{\begin{vmatrix}5&0\\ 0&\frac{1}{2}\end{vmatrix}+\begin{vmatrix}5&0\\ 0&\frac{1}{4}\end{vmatrix}\right\}p+\begin{vmatrix}5&0\\ 0&\frac{1}{2}\end{vmatrix}}{p\left(\frac{1}{4}p^2+\frac{1}{4}p+\frac{1}{2}\right)}$$

$$=\frac{11\left(\frac{1}{4}\right)^2p^4+32\left(\frac{1}{4}\right)^2p^3+\left[20\left(\frac{1}{4}\right)^2+44\left(\frac{1}{4}\right)^2\right]p^2+60\left(\frac{1}{4}\right)^2+40\left(\frac{1}{4}\right)^2}{p\left[4\left(\frac{1}{4}\right)^2+4\left(\frac{1}{4}\right)^2p+8\left(\frac{1}{4}\right)^2\right]}$$

which is exactly (207').

This network corresponds to the point $\lambda_{12}=\frac{1}{4}$, $\sigma_{12}=0$, $k=-1/24$ which is exactly the image of the network shown in figure 61, and is the point A', the image of A.

We can proceed then in the manner similar to the two-mesh network containing only two kinds of network elements, and obtain all the networks of the least number of elements equivalent to a given network, or having a given impedance function for an impedance. We make use of the mutual parameter plane, in the same way, to obtain all the equivalent networks, including those with the least number of elements. The entire group of electrical networks having a given impedance function is represented by points in the mutual parameter plane. If negative elements are not excluded, every point in the mutual parameter plane represents a network of the group. As in the two-mesh network with two kinds of network elements, we can remove elements of the most general network subject to the conditions that the form of the impedance function be preserved, and obtain the networks containing the least number of network elements. The values of the elements are readily obtained from the system of equations 195 and the equations (208').

It is to be noted that our method of arriving at the impedance function gives at once the values of m and n, which must remain invariant, except for a constant multiplier, as we change the network elements. Since in practice we always arrive ? at an impedance function by means of a physical network, it is at once seen that it is a simple matter to arrive at the complete set of equivalent networks. It is , of course, unnecessary to see if the coefficients of the impedance function satisfy the conditions for an impedance function. This simplifies the work considerably. The use of the three determinants R_1, R_2 and R_3 is even simpler than using the resultants, and the procedure is straight forward. It is only necessary to point to Foster's paper, to see the simplicity obtained by our method. Furthermore, our method is in a form which allows for a generalization to n-meshes.

The complete exploration of the mutual parameter plane, which we have not done, will reveal regions representing networks having only positive elements, networks having both positive and negative elements, and finally networks having all negative elements. The corresponding image regions will appear, and points representing the networks with the least number of network elements will be readily obtained, with the corresponding image points.

Let us proceed now to obtain the impedance function for the three-mesh network containing inductance, resistance and capacity elements. The determinant D(p) of such a network is

$$D(p) = \begin{vmatrix} \lambda_{11}p+\rho_{11}+\frac{\sigma_{11}}{p} & \lambda_{12}p+\rho_{12}+\frac{\sigma_{12}}{p} & \lambda_{13}p+\rho_{13}+\frac{\sigma_{13}}{p} \\ \lambda_{12}p+\rho_{12}+\frac{\sigma_{12}}{p} & \lambda_{22}p+\rho_{22}+\frac{\sigma_{22}}{p} & \lambda_{23}p+\rho_{23}+\frac{\sigma_{23}}{p} \\ \lambda_{13}p+\rho_{13}+\frac{\sigma_{13}}{p} & \lambda_{23}p+\rho_{23}+\frac{\sigma_{23}}{p} & \lambda_{33}p+\rho_{33}+\frac{\sigma_{33}}{p} \end{vmatrix} \tag{213}$$

$$= \frac{1}{p}\begin{vmatrix} \lambda_{11}p^2+\rho_{11}p+\sigma_{11} & \lambda_{12}p+\rho_{12}+\frac{\sigma_{12}}{p} & \lambda_{13}p+\rho_{13}+\frac{\sigma_{13}}{p} \\ \lambda_{12}p+\rho_{12}+\frac{\sigma_{12}}{p} & \lambda_{22}p+\rho_{22}+\frac{\sigma_{22}}{p} & \lambda_{23}p+\rho_{23}+\frac{\sigma_{23}}{p} \\ \lambda_{13}p+\rho_{13}+\frac{\sigma_{13}}{p} & \lambda_{23}p+\rho_{23}+\frac{\sigma_{23}}{p} & \lambda_{33}p+\rho_{33}+\frac{\sigma_{33}}{p} \end{vmatrix}$$

Using the abreviated determinant notation, we have

$$\left|\lambda_{11}p^2+\rho_{11}p+\sigma_{11} \quad \lambda_{12}p^2+\rho_{12}p+\sigma_{12} \quad \lambda_{13}p^2+\rho_{13}p+\sigma_{13}\right|$$

$$= \left|\lambda_{11}p^2 \quad \lambda_{12}p^2+\rho_{12}p+\sigma_{12} \quad \lambda_{13}p^2+\rho_{13}p+\sigma_{13}\right|$$

$$+ \left|\rho_{11}p \quad \lambda_{12}p^2+\rho_{12}p+\sigma_{12} \quad \lambda_{13}p^2+\rho_{13}p+\sigma_{13}\right|$$

$$+ \left|\sigma_{11} \quad \lambda_{12}p^2+\rho_{12}p+\sigma_{12} \quad \lambda_{13}p^2+\rho_{13}p+\sigma_{13}\right|$$

$$
\begin{aligned}
= &|\lambda_{11}p^2\ \lambda_{12}p^2\ \lambda_{13}p^2| + |\lambda_{11}p^2\ \lambda_{12}p^2\ \rho_{13}p| + |\lambda_{11}p^2\ \lambda_{12}p^2\ \sigma_{13}| \\
+ &|\lambda_{11}p^2\ \rho_{12}p\ \lambda_{13}p^2| + |\lambda_{11}p^2\ \rho_{12}p\ \rho_{13}p| + |\lambda_{11}p^2\ \rho_{12}p\ \sigma_{13}| \\
+ &|\lambda_{11}p^2\ \sigma_{12}\ \lambda_{13}p^2| + |\lambda_{11}p^2\ \sigma_{12}\ \rho_{13}p| + |\lambda_{11}p^2\ \sigma_{12}\ \sigma_{13}| \\
+ &|\rho_{11}p\ \lambda_{12}p^2\ \lambda_{13}p^2| + |\rho_{11}p\ \lambda_{12}p^2\ \rho_{13}p| + |\rho_{11}p\ \lambda_{12}p^2\ \sigma_{13}| \\
+ &|\rho_{11}p\ \rho_{12}p\ \lambda_{13}p^2| + |\rho_{11}p\ \rho_{12}p\ \rho_{13}p| + |\rho_{11}p\ \rho_{12}p\ \sigma_{13}| \\
+ &|\rho_{11}p\ \sigma_{12}\ \lambda_{13}p^2| + |\rho_{11}p\ \sigma_{12}\ \rho_{13}p| + |\rho_{11}p\ \sigma_{12}\ \sigma_{13}| \\
+ &|\sigma_{11}\ \lambda_{12}p^2\ \lambda_{13}p^2| + |\sigma_{11}\ \lambda_{12}p^2\ \rho_{13}p| + |\sigma_{11}\ \lambda_{12}p^2\ \sigma_{13}| \\
+ &|\sigma_{11}\ \rho_{12}p\ \lambda_{13}p^2| + |\sigma_{11}\ \rho_{12}p\ \rho_{13}p| + |\sigma_{11}\ \rho_{12}p\ \sigma_{13}| \\
+ &|\sigma_{11}\ \sigma_{12}\ \lambda_{13}p^2| + |\sigma_{11}\ \sigma_{12}\ \rho_{13}p| + |\sigma_{11}\ \sigma_{12}\ \sigma_{13}|
\end{aligned}
$$

$$
\begin{aligned}
= &|\lambda_{11}\ \lambda_{12}\ \lambda_{13}|\,p^6 + \{|\lambda_{11}\ \lambda_{12}\ \rho_{13}| + |\lambda_{11}\ \rho_{12}\ \lambda_{13}| + |\rho_{11}\ \lambda_{12}\ \lambda_{13}|\}\,p^5 \\
+ &\Big[\{|\lambda_{11}\ \lambda_{12}\ \sigma_{13}| + |\lambda_{11}\ \sigma_{12}\ \lambda_{13}| + |\sigma_{11}\ \lambda_{12}\ \lambda_{13}|\} + \{|\lambda_{11}\ \rho_{12}\ \rho_{13}| + |\rho_{11}\ \lambda_{12}\ \rho_{13}| + |\rho_{11}\ \rho_{12}\ \lambda_{13}|\}\Big]\,p^4 \\
+ &\Big[\{|\lambda_{11}\ \rho_{12}\ \sigma_{13}| + |\lambda_{11}\ \sigma_{12}\ \rho_{13}| + |\rho_{11}\ \lambda_{12}\ \sigma_{13}| + |\rho_{11}\ \sigma_{12}\ \lambda_{13}| + |\sigma_{11}\ \lambda_{12}\ \rho_{13}| + |\sigma_{11}\ \rho_{12}\ \lambda_{13}|\} + |\rho_{11}\ \rho_{12}\ \rho_{13}|\Big]\,p^3 \\
+ &\Big[\{|\lambda_{11}\ \sigma_{12}\ \sigma_{13}| + |\sigma_{11}\ \lambda_{12}\ \sigma_{13}| + |\sigma_{11}\ \sigma_{12}\ \lambda_{13}|\} + \{|\rho_{11}\ \rho_{12}\ \sigma_{13}| + |\rho_{11}\ \sigma_{12}\ \rho_{13}| + |\sigma_{11}\ \rho_{12}\ \rho_{13}|\}\Big]\,p^2 \\
+ &\{|\sigma_{11}\ \sigma_{12}\ \rho_{13}| + |\sigma_{11}\ \rho_{12}\ \sigma_{13}|\}\,p + |\sigma_{11}\ \sigma_{12}\ \sigma_{13}| \qquad (214)
\end{aligned}
$$

Using our symbolic notation, we have

$$D(p) = \frac{1}{p^3}\Big[\Delta(\lambda)p^6 + \Delta_1(\lambda,\rho)p^5 + \{\Delta_1(\rho,\lambda) + \Delta_1(\lambda,\sigma)\}p^4 + \{\Delta_1(\lambda,\rho,\sigma) + \Delta(\rho)\}p^3$$
$$+ \{\Delta_1(\sigma,\lambda) + \Delta_1(\rho,\sigma)\}p^2 + \Delta_1(\sigma,\rho)p + \Delta(\sigma)\Big] \qquad (215)$$

where

$$\Delta(\lambda) = |\lambda_{11}\,\lambda_{12}\,\lambda_{13}|$$
$$\Delta(\rho) = |\rho_{11}\,\rho_{12}\,\rho_{13}|$$
$$\Delta(\sigma) = |\sigma_{11}\,\sigma_{12}\,\sigma_{13}|$$
$$\Delta_1(\lambda,\rho) = |\lambda_{11}\,\lambda_{12}\,\rho_{13}| + |\lambda_{11}\,\rho_{12}\,\lambda_{13}| + |\rho_{11}\,\lambda_{12}\,\lambda_{13}|$$
$$\Delta_1(\rho,\lambda) = |\rho_{11}\,\rho_{12}\,\lambda_{13}| + |\rho_{11}\,\lambda_{12}\,\rho_{13}| + |\lambda_{11}\,\rho_{12}\,\rho_{13}|$$
$$\Delta_1(\lambda,\sigma) = |\lambda_{11}\,\lambda_{12}\,\sigma_{13}| + |\lambda_{11}\,\sigma_{12}\,\lambda_{13}| + |\sigma_{11}\,\lambda_{12}\,\lambda_{13}|$$
$$\Delta_1(\sigma,\lambda) = |\sigma_{11}\,\sigma_{12}\,\lambda_{13}| + |\sigma_{11}\,\lambda_{12}\,\sigma_{13}| + |\lambda_{11}\,\sigma_{12}\,\sigma_{13}|$$
$$\Delta_1(\rho,\sigma) = |\rho_{11}\,\rho_{12}\,\sigma_{13}| + |\rho_{11}\,\sigma_{12}\,\rho_{13}| + |\sigma_{11}\,\rho_{12}\,\rho_{13}|$$
$$\Delta_1(\sigma,\rho) = |\sigma_{11}\,\sigma_{12}\,\rho_{13}| + |\sigma_{11}\,\rho_{12}\,\sigma_{13}| + |\rho_{11}\,\sigma_{12}\,\sigma_{13}|$$
$$\Delta_1(\lambda,\rho,\sigma) = |\lambda_{11}\,\rho_{12}\,\sigma_{13}| + |\lambda_{11}\,\sigma_{12}\,\rho_{13}| + |\rho_{11}\,\lambda_{12}\,\sigma_{13}| + |\rho_{11}\,\sigma_{12}\,\lambda_{13}| + |\sigma_{11}\,\lambda_{12}\,\rho_{13}| + |\sigma_{11}\,\rho_{12}\,\lambda_{13}|$$

The minor of D(p) is given by (213) and is

$$M_{11}(p) = \begin{vmatrix} \lambda_{22}p + \rho_{22} + \frac{\sigma_{22}}{p} & \lambda_{23}p + \rho_{23} + \frac{\sigma_{23}}{p} \\ \lambda_{23}p + \rho_{23} + \frac{\sigma_{23}}{p} & \lambda_{33}p + \rho_{33} + \frac{\sigma_{33}}{p} \end{vmatrix} \qquad (216)$$

But this determinant is exactly of the form of (188), which is the determinant of the two-mesh network. Hence, without going through the process of expansion, we may write at once the expression (190), except that the Δ's in this expression are replaced by M's, to indicate that we have now to do with the minor of the element in the first row and first column of the determinant of a three-mesh network. Thus

$$M_{11}(p) = \frac{1}{p^2}\left[M_{11}(\lambda)p^4 + M_{11}^{(1)}(\lambda,\rho)p^3 + \left\{M_{11}^{(1)}(\lambda,\sigma) + M_{11}(\rho)\right\}p^2 + M_{11}^{(1)}(\sigma,\rho)p + M_{11}(\sigma)\right] \quad (217)$$

Hence we have

$$Z(p) = \frac{D(p)}{M_{11}(p)}$$

$$= \frac{\Delta(\lambda)p^6 + \Delta_1(\lambda,\rho)p^5 + \left\{\Delta_1(\rho,\lambda) + \Delta_1(\lambda,\sigma)\right\}p^4 + \left\{\Delta_1(\lambda,\rho,\sigma) + \Delta(\rho)\right\}p^3 + \left\{\Delta_1(\sigma,\lambda) + \Delta_1(\rho,\sigma)\right\}p^2 + \Delta_1(\sigma,\rho)p + \Delta(\sigma)}{p\left[M_{11}(\lambda)p^4 + M_{11}^{(1)}(\lambda,\rho)p^3 + \left\{M_{11}^{(1)}(\lambda,\sigma) + M_{11}(\rho)\right\}p^2 + M_{11}^{(1)}(\sigma,\rho)p + M_{11}(\sigma)\right]} \quad (218)$$

This is the impedance function of the most general three-mesh network containing inductance, resistance and capacity elements.

The network with the least number of elements may in general be obtained as usual by eliminating elements subject to the condition that the form of the impedance function be preserved.

For the purpose of obtaining network equivalent to a given network, it is unnecessary to make conditions upon the coefficients of the impedance function of the given network. A given network will always have an impedance the coefficients, or the zeros and poles of which, will always, of course, satisfy the conditions that a function be the impedance function of a physical network. The equivalent networks are obtained, as in the two-mesh case from the equivalence equations. Thus the impedance function of a physical network will be of the following form, as seen from (218)

$$Z(p) = \frac{a_0 p^6 + a_1 p^5 + (m_1+n_1)p^4 + (m_2+n_2)p^3 + (m_3+n_3)p^2 + a_5 p + a_6}{p(b_1 p^4 + b_2 p^3 + (r_1+s_1)p^2 + b_4 p + b_5)} \tag{219}$$

Note that for this impedance function

$$a_2 = m_1 + n_1$$
$$a_3 = m_2 + n_2$$
$$a_4 = m_3 + n_3$$
$$b_3 = r_1 + s_1$$

and in the computation of the impedance function by our method, the coefficients a_2, a_3, a_4 and b_3 are divided automatically into two parts.

All the networks equivalent to the given network having (219) for an impedance function are obtained by means of the equivalence equations, as in the two-mesh case. These equations are readily obtained, after multiplying the numerator and denominator of (219) by k^2 and comparing it with (218). The equivalence equations of the three-mesh network are then

$$\Delta(\lambda) = k^2 a_0 \qquad (220\,a)$$

$$\Delta(\rho) = k^2 m_2 \qquad (220\,b)$$

$$\Delta(\sigma) = k^2 a_6 \qquad (220\,c)$$

$$\Delta_1(\lambda,\rho) = k^2 a_1 \qquad (220\,d)$$

$$\Delta_1(\rho,\sigma) = k^2 m_3 \qquad (220\,e)$$

$$\Delta_1(\sigma,\lambda) = k^2 n_3 \qquad (220\,f)$$

$$\Delta_1(\rho,\lambda) = k^2 m_1 \qquad (220\,g)$$

$$\Delta_1(\sigma,\rho) = k^2 a_5 \qquad (220\,h)$$

$$\Delta_1(\lambda,\sigma) = k^2 n_1 \qquad (220\,i)$$

$$\Delta_1(\lambda,\rho\,\sigma) = k^2 n_2 \qquad (220\,j)$$

$$M_{11}(\lambda) = k^2 b_1 \qquad (220\,k)$$

$$M_{11}(\rho) = k^2 r_1 \qquad (220\,l)$$

$$M_{11}(\sigma) = k^2 b_5 \qquad (220\,m)$$

$$M_{11}^{(1)}(\lambda,\rho) = k^2 b_2 \qquad (220\,n)$$

$$M_{11}^{(1)}(\sigma,\rho) = k^2 b_4 \qquad (220\,p)$$

$$M_{11}^{(1)}(\lambda,\sigma) = k^2 s_1 \qquad (220\,q).$$

The rule of formation of these equivalence equations can be seen from the system of equations (195) for the two-mesh network. Any array of real positive numbers satisfying the system of equations (220) become the parameters of a network having (219) for an impedance function, provided the non-diagonal elements in the determinants of the coefficients are less or equal to the diagonal elements - that is provided the mutual parameters are less than the total parameters of the network.

Note that the system of equations (220) is symmetrical in λ, ρ and σ as in the two-mesh case.

Let us see what the impedance of the four-mesh network is, and the corresponding equivalence equations. The determinant of the four-mesh network containing inductance, resistance and capacity elements is

$$D(p) = \begin{vmatrix} \lambda_{11} p + \rho_{11} + \frac{\sigma_{11}}{p} & \lambda_{12} p + \rho_{12} + \frac{\sigma_{12}}{p} & \lambda_{13} p + \rho_{13} + \frac{\sigma_{13}}{p} & \lambda_{14} p + \rho_{14} + \frac{\sigma_{14}}{p} \\ \lambda_{12} p + \rho_{12} + \frac{\sigma_{12}}{p} & \lambda_{22} p + \rho_{22} + \frac{\sigma_{22}}{p} & \lambda_{23} p + \rho_{23} + \frac{\sigma_{23}}{p} & \lambda_{24} p + \rho_{24} + \frac{\sigma_{24}}{p} \\ \lambda_{13} p + \rho_{13} + \frac{\sigma_{13}}{p} & \lambda_{23} p + \rho_{23} + \frac{\sigma_{23}}{p} & \lambda_{33} p + \rho_{33} + \frac{\sigma_{33}}{p} & \lambda_{34} p + \rho_{34} + \frac{\sigma_{34}}{p} \\ \lambda_{14} p + \rho_{14} + \frac{\sigma_{14}}{p} & \lambda_{24} p + \rho_{24} + \frac{\sigma_{24}}{p} & \lambda_{34} p + \rho_{34} + \frac{\sigma_{34}}{p} & \lambda_{44} p + \rho_{44} + \frac{\sigma_{44}}{p} \end{vmatrix} \qquad (221)$$

This becomes, using our abbreviated determinant notation

$$D(p) = \frac{1}{p^4} \left| \lambda_{11} p^2 + \rho_{11} p + \sigma_{11} \quad \lambda_{12} p^2 + \rho_{12} p + \sigma_{12} \quad \lambda_{13} p^2 + \rho_{13} p + \sigma_{13} \quad \lambda_{14} p^2 + \rho_{14} p + \sigma_{14} \right|$$

Expanding this determinant in the usual manner we obtain

$$D(p) = \frac{1}{p^4}\Big[|\lambda_{11}\lambda_{12}\lambda_{13}\lambda_{14}|\, p^8 + \big\{ |\lambda_{11}\lambda_{12}\lambda_{13}\rho_{14}| + |\lambda_{11}\lambda_{12}\rho_{13}\lambda_{14}| + |\lambda_{11}\rho_{12}\lambda_{13}\lambda_{14}| + |\rho_{11}\lambda_{12}\lambda_{13}\lambda_{14}| \big\}\, p^7$$

$$+ \Big\{ \big(|\lambda_{11}\lambda_{12}\lambda_{13}\sigma_{14}| + |\lambda_{11}\lambda_{12}\sigma_{13}\lambda_{14}| + |\lambda_{11}\sigma_{12}\lambda_{13}\lambda_{14}| + |\sigma_{11}\lambda_{12}\lambda_{13}\lambda_{14}| \big)$$

$$+ \big(|\lambda_{11}\lambda_{12}\rho_{13}\rho_{14}| + |\lambda_{11}\rho_{12}\lambda_{13}\rho_{14}| + |\lambda_{11}\rho_{12}\rho_{13}\lambda_{14}| + |\rho_{11}\lambda_{12}\lambda_{13}\rho_{14}| + |\rho_{11}\lambda_{12}\rho_{13}\lambda_{14}| + |\rho_{11}\rho_{12}\lambda_{13}\lambda_{14}| \big) \Big\}\, p^6$$

$$+ \Big\{ \big(|\lambda_{11}\lambda_{12}\rho_{13}\sigma_{14}| + |\lambda_{11}\lambda_{12}\sigma_{13}\rho_{14}| + |\lambda_{11}\rho_{12}\lambda_{13}\sigma_{14}| + |\lambda_{11}\rho_{12}\sigma_{13}\lambda_{14}| + |\lambda_{11}\sigma_{12}\lambda_{13}\rho_{14}| + |\lambda_{11}\sigma_{12}\rho_{13}\lambda_{14}|$$

$$+ |\rho_{11}\lambda_{12}\lambda_{13}\sigma_{14}| + |\rho_{11}\lambda_{12}\sigma_{13}\lambda_{14}| + |\rho_{11}\sigma_{12}\lambda_{13}\lambda_{14}| + |\sigma_{11}\lambda_{12}\lambda_{13}\rho_{14}| + |\sigma_{11}\lambda_{12}\rho_{13}\lambda_{14}| + |\sigma_{11}\rho_{12}\lambda_{13}\lambda_{14}| \big)$$

$$+ \big(|\lambda_{11}\rho_{12}\rho_{13}\rho_{14}| + |\rho_{11}\lambda_{12}\rho_{13}\rho_{14}| + |\rho_{11}\rho_{12}\lambda_{13}\rho_{14}| + |\rho_{11}\rho_{12}\rho_{13}\lambda_{14}| \big) \Big\}\, p^5$$

$$+ \Big\{ \big(|\lambda_{11}\lambda_{12}\sigma_{13}\sigma_{14}| + |\lambda_{11}\sigma_{12}\lambda_{13}\sigma_{14}| + |\lambda_{11}\sigma_{12}\sigma_{13}\lambda_{14}| + |\sigma_{11}\lambda_{12}\lambda_{13}\sigma_{14}| + |\sigma_{11}\lambda_{12}\sigma_{13}\lambda_{14}| + |\sigma_{11}\sigma_{12}\lambda_{13}\lambda_{14}| \big)$$

$$+ \big(|\lambda_{11}\rho_{12}\rho_{13}\sigma_{14}| + |\lambda_{11}\rho_{12}\sigma_{13}\rho_{14}| + |\lambda_{11}\sigma_{12}\rho_{13}\rho_{14}| + |\rho_{11}\lambda_{12}\rho_{13}\sigma_{14}| + |\rho_{11}\lambda_{12}\sigma_{13}\rho_{14}|$$

$$+ |\rho_{11}\rho_{12}\lambda_{13}\sigma_{14}| + |\rho_{11}\rho_{12}\sigma_{13}\lambda_{14}| + |\rho_{11}\sigma_{12}\lambda_{13}\rho_{14}| + |\rho_{11}\sigma_{12}\rho_{13}\lambda_{14}|$$

$$+ |\sigma_{11}\lambda_{12}\rho_{13}\rho_{14}| + |\sigma_{11}\rho_{12}\lambda_{13}\rho_{14}| + |\sigma_{11}\rho_{12}\rho_{13}\lambda_{14}| \big) + \big(|\rho_{11}\rho_{12}\rho_{13}\rho_{14}| \big) \Big\}\, p^4$$

$$+ \Big\{ \big(|\lambda_{11}\rho_{12}\sigma_{13}\sigma_{14}| + |\lambda_{11}\sigma_{12}\rho_{13}\sigma_{14}| + |\lambda_{11}\sigma_{12}\sigma_{13}\rho_{14}| + |\rho_{11}\lambda_{12}\sigma_{13}\sigma_{14}| + |\rho_{11}\sigma_{12}\lambda_{13}\sigma_{14}|$$

$$+ |\rho_{11}\sigma_{12}\sigma_{13}\lambda_{14}| + |\sigma_{11}\lambda_{12}\rho_{13}\sigma_{14}| + |\sigma_{11}\lambda_{12}\sigma_{13}\rho_{14}| + |\sigma_{11}\rho_{12}\lambda_{13}\sigma_{14}| + |\sigma_{11}\rho_{12}\sigma_{13}\lambda_{14}|$$

$$+ |\sigma_{11}\sigma_{12}\lambda_{13}\rho_{14}| + |\sigma_{11}\sigma_{12}\rho_{13}\lambda_{14}| \big) + \big(|\rho_{11}\rho_{12}\rho_{13}\sigma_{14}| + |\rho_{11}\rho_{12}\sigma_{13}\rho_{14}| + |\rho_{11}\sigma_{12}\rho_{13}\rho_{14}|$$

$$+ |\sigma_{11}\rho_{12}\rho_{13}\rho_{14}| \big) \Big\}\, p^3$$

$$+ \Big\{ \big(|\lambda_{11}\sigma_{12}\sigma_{13}\sigma_{14}| + |\sigma_{11}\lambda_{12}\sigma_{13}\sigma_{14}| + |\sigma_{11}\sigma_{12}\lambda_{13}\sigma_{14}| + |\sigma_{11}\sigma_{12}\sigma_{13}\lambda_{14}| \big)$$

$$+ \big(|\rho_{11}\rho_{12}\sigma_{13}\sigma_{14}| + |\rho_{11}\sigma_{12}\rho_{13}\sigma_{14}| + |\rho_{11}\sigma_{12}\sigma_{13}\rho_{14}| + |\sigma_{11}\rho_{12}\rho_{13}\sigma_{14}|$$

$$+ |\sigma_{11}\rho_{12}\sigma_{13}\rho_{14}| + |\sigma_{11}\sigma_{12}\rho_{13}\rho_{14}| \big) \Big\}\, p^2$$

$$+\left\{|\rho_{11}\sigma_{12}\sigma_{13}\sigma_{14}|+|\sigma_{11}\rho_{12}\sigma_{13}\sigma_{14}|+|\sigma_{11}\sigma_{12}\rho_{13}\sigma_{14}|+|\sigma_{11}\sigma_{12}\sigma_{13}\rho_{14}|\right\}p+|\sigma_{11}\sigma_{12}\sigma_{13}\sigma_{14}|\Bigg] \tag{222}$$

This may be written, by means of our symbolic notation as follows

$$D(p)=\frac{1}{p^4}\Bigg[\Delta(\lambda)p^8+\Delta_1(\lambda,\rho)p^7+\left\{\Delta_1(\lambda,\sigma)+\Delta_2(\lambda,\rho)\right\}p^6+\left\{\Delta_2(\lambda,\rho,\sigma)+\Delta_1(\rho,\lambda)\right\}p^5$$

$$+\left\{\Delta_2(\lambda,\sigma)+\Delta_2(\rho,\lambda,\sigma)+\Delta(\rho)\right\}p^4+\left\{\Delta_2(\sigma,\rho,\lambda)+\Delta_1(\rho,\sigma)\right\}p^3$$

$$+\left\{\Delta_1(\sigma,\lambda)+\Delta_2(\rho,\sigma)\right\}p^2+\Delta_1(\sigma,\rho)p+\Delta(\sigma)\Bigg] \tag{223}$$

The minor of D(p) is given by (221) and is

$$M_{11}(p)=\begin{vmatrix} \lambda_{22}p+\rho_{22}+\frac{\sigma_{22}}{p} & \lambda_{23}p+\rho_{23}+\frac{\sigma_{23}}{p} & \lambda_{24}p+\rho_{24}+\frac{\sigma_{24}}{p} \\ \lambda_{23}p+\rho_{23}+\frac{\sigma_{23}}{p} & \lambda_{33}p+\rho_{33}+\frac{\sigma_{33}}{p} & \lambda_{34}p+\rho_{34}+\frac{\sigma_{34}}{p} \\ \lambda_{24}p+\rho_{24}+\frac{\sigma_{24}}{p} & \lambda_{34}p+\rho_{34}+\frac{\sigma_{34}}{p} & \lambda_{44}p+\rho_{44}+\frac{\sigma_{44}}{p} \end{vmatrix} \tag{224}$$

But this is exactly in the form of the determinant of the three mesh network with inductance, resistance and capacity elements, the expansion of which is given by (215). Hence (215) is the expansion of (224) except that we replace the Δ's in (215) by M's. Thus

$$M_{11}(p) = \frac{1}{p^3}\Big[M_{11}(\lambda)p^6 + M_{11}^{(1)}(\lambda,\rho)p^5 + \{M_{11}^{(1)}(\rho,\lambda) + M_{11}^{(1)}(\lambda,\sigma)\}p^4$$

$$+ \{M_{11}^{(1)}(\lambda,\rho,\sigma) + M_{11}(\rho)\}p^3 + \{M_{11}^{(1)}(\sigma,\lambda) + M_{11}^{(1)}(\rho,\sigma)\}p^2 + M_{11}^{(1)}(\sigma,\rho)p + M_{11}(\sigma)\Big] \quad (225)$$

Hence

$$Z(p) = \frac{D(p)}{M_{11}(p)}$$

$$= \frac{\begin{array}{l}\Delta(\lambda)p^8 + \Delta_1(\lambda,\rho)p^7 + \{\Delta_1(\lambda,\sigma) + \Delta_2(\lambda,\rho)\}p^6 + \{\Delta_2(\lambda,\rho,\sigma) + \Delta_1(\rho,\lambda)\}p^5 \\ + \{\Delta_2(\lambda,\sigma) + \Delta_2(\rho,\lambda,\sigma) + \Delta(\rho)\}p^4 + \{\Delta_2(\sigma,\rho,\lambda) + \Delta_1(\rho,\sigma)\}p^3 \\ + \{\Delta_1(\sigma,\lambda) + \Delta_2(\rho,\sigma)\}p^2 + \Delta_1(\sigma,\rho)p + \Delta(\sigma)\end{array}}{\begin{array}{l}p\big[M_{11}(\lambda)p^6 + M_{11}^{(1)}(\lambda,\rho)p^5 + \{M_{11}^{(1)}(\rho,\lambda) + M_{11}^{(1)}(\lambda,\sigma)\}p^4 \\ + \{M_{11}^{(1)}(\lambda,\rho,\sigma) + M_{11}(\rho)\}p^3 + \{M_{11}^{(1)}(\sigma,\lambda) + M_{11}^{(1)}(\rho,\sigma)\}p^2 + M_{11}^{(1)}(\sigma,\rho)p + M_{11}(\sigma)\big]\end{array}} \quad (226)$$

This is the impedance function of the most general four-mesh network containing inductance, resistance and capacity elements. As in the two and three-mesh case, the networks equivalent to a given network are obtained from the equivalence equations. Thus if

$$Z(p) = \frac{\begin{array}{l}a_0p^8 + a_1p^7 + (m_1+n_1)p^6 + (m_2+n_2)p^5 + (m_3+n_3+l_3)p^4 \\ + (m_4+n_4)p^3 + (m_5+n_5)p^2 + a_7p + a_8\end{array}}{\begin{array}{l}p\big[b_1p^6 + b_2p^5 + (r_1+s_1)p^4 + (r_2+s_2)p^3 + (r_3+s_3)p^2 \\ + b_6p + b_7\big]\end{array}} \quad (227)$$

represents the impedance function of a physical network, the networks equivalent to this network is obtained from the equivalence equations, which are

$$\Delta(\lambda) = k^2 a_0 \qquad (228a)$$

$$\Delta(\rho) = k^2 l_3 \qquad (228b)$$

$$\Delta(\sigma) = k^2 a_8 \qquad (228c)$$

$$\Delta_1(\lambda,\rho) = k^2 a_1 \qquad (228d)$$

$$\Delta_1(\rho,\sigma) = k^2 n_4 \qquad (228e)$$

$$\Delta_1(\sigma,\lambda) = k^2 m_5 \qquad (228f)$$

$$\Delta_1(\rho,\lambda) = k^2 n_2 \qquad (228g)$$

$$\Delta_1(\sigma,\rho) = k^2 a_7 \qquad (228h)$$

$$\Delta_1(\lambda,\sigma) = k^2 m_1 \qquad (228i)$$

$$\Delta_2(\lambda,\rho) = k^2 n_1 \qquad (228j)$$

$$\Delta_2(\rho,\sigma) = k^2 n_5 \qquad (228k)$$

$$\Delta_2(\sigma,\lambda) = k^2 m_3 \qquad (228l)$$

$$\Delta_2(\lambda,\rho,\sigma) = k^2 m_2 \qquad (228m)$$

$$\Delta_2(\rho,\lambda,\sigma) = k^2 n_3 \qquad (228n)$$

$$\Delta_2(\sigma,\rho,\lambda) = k^2 m_4 \qquad (228p)$$

$$M_{11}(\lambda) = k^2 b_1 \qquad (228q)$$

$$M_{11}(\rho) = k^2 s_1 \qquad (228r)$$

$$M_{11}(\sigma) = k^2 a_8 \qquad (228s)$$

$$M_{11}^{(1)}(\lambda,\rho) = k^2 b_2 \qquad (228t)$$

$$M_{11}^{(1)}(\sigma,\rho) = k^2 b_6 \qquad (228u)$$

$$M_{11}^{(1)}(\lambda,\sigma) = k^2 s_1 \qquad (228v)$$

$$M_{11}^{(1)}(\rho,\lambda) = k^2 r_1 \qquad (228w)$$

$$M_{11}^{(1)}(\rho,\sigma) = k^2 r_3 \qquad (228x)$$

$$M_{11}^{(1)}(\sigma,\lambda) = k^2 s_3 \qquad (228y)$$

$$M_{11}^{(1)}(\lambda,\rho,\sigma) = k^2 r_2 \qquad (228z)$$

The equivalence equations for networks of any number of meshes can be obtained by induction. These equations are symmetrical in λ, ρ and σ. It can be readily seen that although these equations express themselves rather simply in our symbolic notation, they are nevertheless quite complex when written in open form. While it is not difficult in the two-mesh case to obtain the networks having a given impedance function, it does become quite difficult to do so for networks with three or more meshes. The equivalence equations, which we have obtained for networks of any number of meshes, will give all the networks having a given impedance function. and the same m, n, etc as the given!

CHAPTER VII.

The Infinite Group of Equivalent Networks.

It will now be shown that networks form a group of which the impedance function is an absolute invariant, and that it is possible to proceed in a continuous manner from one network to its equivalent network by means of a linear affine transformation of the instantaneous mesh currents and charges of the net. Every network will be shown to determine one, two or three quadratic forms, depending upon whether the network has one, two or three kinds of elements. These quadratic forms are constructed from the three respective matrices which have as elements, the elements of the network. Hence there is a one to one correspondence between a network and a group of one, two or three matrices. Given a network, the corresponding matrices can at once be written down, and given the matrices, it is a simple matter to construct the corresponding network. Now a linear affine transformation of the quadratic forms will preserve the form of the quadratic forms. The transformed quadratic forms will have different matrices of its coefficients. These different matrices however will correspond to a network having different network elements but having the same impedance function.

Before proceeding to demonstrate this theory, it will be useful to briefly review some of quadratic form theory.[29]

The general quadratic form in n variables is

$$
\left.\begin{aligned}
\sum_1^n a_{ij} x_i x_j \equiv\; & a_{11} x_1^2 + a_{12} x_1 x_2 + \cdots + a_{1n} x_1 x_n \\
& + a_{21} x_2 x_1 + a_{22} x_2^2 + \cdots + a_{2n} x_2 x_n \\
& \cdots\cdots\cdots\cdots\cdots\cdots \\
& \cdots\cdots\cdots\cdots\cdots\cdots \\
& + a_{n1} x_n x_1 + a_{n2} x_n x_2 \cdots + a_{nn} x_n^2
\end{aligned}\right\} \qquad (229)
$$

where

$$a_{ij} = a_{ji}$$

If we subject the quadratic form (229) to the linear transformation

$$
\begin{aligned}
x_1 &= c_{11} x_1' + \cdots\cdots + c_{1n} x_n' \\
&\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots \\
x_n &= c_{n1} x_1' + \cdots\cdots + c_{nn} x_n'
\end{aligned}
$$

we get a new quadratic form

$$\sum_1^n a_{ij}' x_i' x_j' \qquad (229a)$$

29. For an excellent discussion of Quadratic forms and Invariant theory see M. Bôcher "Introduction to Higher Algebra" 1927; See also T.J.I'A. Bromwich, "Quadratic Forms and their Classification by Means of Invariant-Factors"; F. Faà di Bruno, "Einleitung in die Theorie der Binaren Formen", 1881; P. Gordan, "Invariantentheorie", 1885; E. Pascal, "Repertorium der Höheren Mathematik", 1900.

This may be readily seen by actually making the substitution and collecting terms.

The matrix containing the coefficients of the quadratic form (229), namely

$$A = \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{Vmatrix} \tag{230}$$

is called the matrix of the quadratic form (229). The determinant of (230) is called the discriminant of (229) and the rank of (230) the rank of (229). The quadratic form(229) is singular if its discriminant vanishes.

If in the quadratic form (229) with the matrix A, we subject the x's to a linear transformation with matrix

$$C = \begin{Vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & & & \vdots \\ \vdots & & & \vdots \\ c_{n1} & \cdots & \cdots & c_{nn} \end{Vmatrix} \tag{231}$$

we obtain a new quadratic form with the matrix C' A C, where C' is the conjugate of C. That is, the matrix of the new quadratic form would be obtained by multiplying together the three matrices.

$$\begin{Vmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & & & \vdots \\ \vdots & & & \vdots \\ c_{1n} & \cdots & \cdots & c_{nn} \end{Vmatrix} \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{Vmatrix} \begin{Vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & & & \\ & & & \\ c_{n1} & & & c_{nn} \end{Vmatrix} \tag{232}$$

This can be easily verified by making the substitution. From this it follows that the rank of a quadratic form is not changed by a non-singular linear transformation and that the discriminant of a quadratic form is a relative invariant of weight two.

The polar form of (229a) is the bilinear quadratic form

$$\sum_1^n a'_{ij}\, y'_i z'_j \tag{229b}$$

It can be shown that if we transform the y's and z's of the polar form $\sum_1^n a_{ij}\, y_i z_j$ of (229) by the same transformation

$$C = \begin{Vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & & & \vdots \\ \vdots & & & \vdots \\ c_{n1} & \cdots & \cdots & c_{nn} \end{Vmatrix} \tag{231}$$

we get a (new) bilinear form

$$\sum_1^n \bar{a}_{ij}\, y_i z_j \tag{233}$$

where

$$\bar{a}_{ij} = a'_{ij}$$

Now consider the pair of quadratic forms

$$\phi(x_1 \cdots x_n) \equiv \sum_1^n a_{ij}\, x_i x_j \tag{234a}$$

$$\psi(x_1 \cdots x_n) \equiv \sum_1^n b_{ij}\, x_i x_j \tag{234b}$$

and form from them the pencil of quadratic forms

$$\phi + \lambda\psi = \sum_1^n (a_{ij} + \lambda b_{ij})\, x_i x_j \tag{235}$$

The discriminant of this pencil

$$F(\lambda) = \begin{vmatrix} a_{11}+\lambda b_{11} & \cdots & a_{1n}+\lambda b_{1n} \\ \vdots & & \vdots \\ a_{n1}+\lambda b_{n1} & \cdots & a_{nn}+\lambda b_{nn} \end{vmatrix} \qquad (236)$$

is a polynomial which is in general of degree n and may be written

$$F(\lambda) = \Delta(a)\lambda^n + \Delta_1(a,b)\lambda^{n-1} + \Delta_2(a,b)\lambda^{n-2} + \cdots + \Delta_2(b,a)\lambda^2 + \Delta_1(b,a)\lambda + \Delta(b) \qquad (237)$$

where the Δ, Δ_1, Δ_2, etc. have the same meaning as given in chapter IV.

It can be shown that the coefficients $\Delta(a)$, $\Delta_1(a,b)$ etc. are integral rational invariants of weight two of the pair of quadratic forms ϕ, ψ . Furthermore, the roots of the λ equation,

$$F(\lambda) = 0$$

of the pair of quadratic forms are absolute irrational invariants of this pair of forms with regard to a non-singular linear transformation. The multiplicities of the roots of the λ equation are then arithmetical invariants of the pair of quadratic forms with regard to non-singular linear transformations. It can also be shown that every integral rational invariant of a quadratic form is a constant multiple of some power of the discriminant.

All of the above is well-known in matrix theory and can be found in Bôcher's excellent book mentioned above. Any other important facts or theorem in matrix theory will be pointed out as we need them.

Before proceeding with the general n-mesh problem involving all three kinds of network elements, let us consider first a special case which we have already treated by another method. Thus consider the system of networks having (113) page 81, for an impedance function. We saw that by multiplying (113) by k^2 and by using the invariant equations (109) page 78 we could arrive at the complete infinite set of networks having (113) for an impedance function. This, we recall, could be done rather nicely by the use of the mutual parameter plane, the points of which plane corresponded to networks and their images. That is, there was a one to one correspondence between an electrical network and a point in the mutual parameter plane. Positive values of k gave a certain network and negative values of k gave its image. We saw, for example, that a certain region AB, figure 24, in the mutual parameter plane contained networks having (113) as an impedance function, indicating that we can go from one network to its equivalent network by a continuous transformation. The region CD, figure 24, it is recalled contained the image networks of the region AB.

Let us consider now the same impedance function (113), namely

$$Z(p) = \frac{p^2 + 4p + 3}{p + 2} \qquad (113)$$

and let us consider a simple network, say the one shown in figure 27, page 87 which we saw had (113) for an impedance function. This network is shown once more in figure 66.

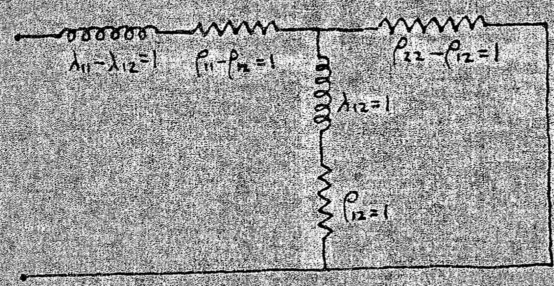


FIG. 66

Now let us see if it is possible to obtain the transformations which will carry over this network into its complete set of equivalent networks. We know that there must exist some such continuous transformation from our study of our mutual parameter plane.

First let us construct the quadratic forms of the network. It may be mentioned here that once given a network, it is a simple matter to construct its quadratic forms and the corresponding matrices containing the coefficients of the quadratic forms. Furthermore, from the mathematical point of view, we can talk about matrices instead of networks, since every group of one, two or three matrices or tensors, determines at once a network, the elements of the matrices being the parameters of the network.

Thus the quadratic forms of any two-mesh network containing simply inductance and resistance elements is, as we have seen,

$$T = \frac{1}{2}(\lambda_{11} i_1^2 + 2\lambda_{12} i_1 i_2 + \lambda_{22} i_2^2) \qquad (238a)$$

$$R = \frac{1}{2}(\rho_{11} i_1^2 + 2\rho_{12} i_1 i_2 + \rho_{22} i_2^2) \qquad (238b)$$

These are of course merely the total instantaneous magnetic energy in the coils, and 1/2 the total instantaneous power loss in the network. These quadratic forms (238) are of the form (229) with $n = 2$.

The matrices containing the coefficients of these forms are of course

$$\left\| \begin{matrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{matrix} \right\| , \qquad \left\| \begin{matrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{matrix} \right\| \qquad (239)$$

It is a simple matter, as is readily seen, to construct the network from these two matrices.

The network parameters for our particular network, figure 66, are $\lambda_{11} = 2$, $\lambda_{22} = 1$, $\lambda_{12} = 1$, $\rho_{11} = 2$, $\rho_{22} = 2$, $\rho_{12} = 1$
Hence the corresponding quadratic forms (238) are

$$T = \frac{1}{2}(2 i_1^2 + 2 i_1 i_2 + i_2^2) \qquad (240a)$$

$$R = \frac{1}{2}(2 i_1^2 + 2 i_1 i_2 + 2 i_2^2) \qquad (240b)$$

and the matrices of the coefficients of these forms are

$$\left\|\begin{matrix} 2 & 1 \\ 1 & 1 \end{matrix}\right\|, \qquad \left\|\begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}\right\| \qquad (241)$$

Now let us perform the following linear affine transformations of the instantaneous currents in the network

$$\begin{aligned} i_1 &= i_1' \\ i_2 &= a_{21} i_1' + a_{22} i_2' \end{aligned} \qquad (242)$$

where the a's are any real numbers, positive or negative. Substituting these values for i_1 and i_2 in the quadratic forms (240a) and (240b), we have

$$T = \frac{1}{2}\left[2 i_1'^2 + 2(i_1')(a_{21} i_1' + a_{22} i_2') + (a_{21} i_1' + a_{22} i_2')^2\right]$$

$$R = \frac{1}{2}\left[2 i_1'^2 + 2(i_1')(a_{21} i_1' + a_{22} i_2') + 2(a_{21} i_1' + a_{22} i_2')^2\right]$$

Collecting terms, we have

$$T = \frac{1}{2}\left[(2 + 2a_{21} + a_{21}^2) i_1'^2 + (2a_{22} + 2a_{21}a_{22}) i_1' i_2' + (a_{22}^2) i_2'^2\right] \qquad (243a)$$

$$R = \frac{1}{2}\left[(2 + 2a_{21} + 2a_{21}^2) i_1'^2 + (2a_{22} + 4a_{21}a_{22}) i_1' i_2' + (2a_{22}^2) i_2'^2\right] \qquad (243b)$$

Thus the transformation (242) gives the new quadratic forms (243). The two matrices containing the coefficients of these new forms are then

$$\left\| \begin{matrix} 2+2a_{21}+a_{21}^2 & a_{22}+a_{21}a_{22} \\ a_{22}+a_{21}a_{22} & a_{22}^2 \end{matrix} \right\| , \quad \left\| \begin{matrix} 2+2a_{21}+2a_{21}^2 & a_{22}+2a_{21}a_{22} \\ a_{22}+2a_{21}a_{22} & 2a_{22}^2 \end{matrix} \right\| \qquad (244)$$

These two matrices determine now an infinite set of networks equivalent to the network shown in figure 66. The different networks are obtained by assigning different real values to a_{21} and a_{22}. Now, for example, from our previous work, we know that the network shown in figure 26, page 86 is equivalent to the network shown in figure 66, and so has the same impedance function (113). Let us draw the network of figure 26 again. It is shown now in figure 67.

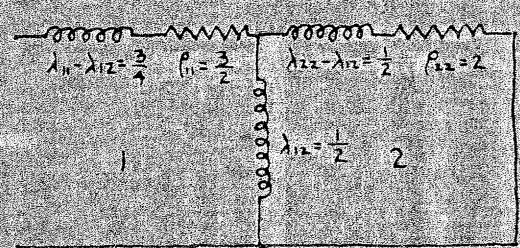


FIG.67

The parameters of this network are of course

$$\lambda_{11} = \frac{5}{4}, \ \lambda_{22} = 1, \ \lambda_{12} = \frac{1}{2}; \ \rho_{11} = \frac{3}{2}, \ \rho_{22} = 2, \ \rho_{12} = 0 \qquad (245)$$

Now let us see if the matrices (244) will give this equivalent network. Of course any real values assigned to a_{21}

and a_{22} will give possible networks equivalent to the network of figure 66, although some of these networks may contain negative elements.

To obtain the network of figure 26 from the matrices (244), we note, comparing the elements of the matrices with the parameters of the network of fig. 26, we must have in (244)

$$a_{22}^2 = 1$$

and

$$a_{22} + a_{21}\, a_{22} = \frac{1}{2}$$

Thus

$$a_{22} = \pm 1 \tag{246}$$

When $\qquad a_{22} = +1$

Then, $\qquad a_{21} = \frac{1}{2} - 1 = -\frac{1}{2}$

Now let us see what the elements of the matrices (244) are, substituting the values for a_{21} and a_{22} of $+1$ and $-\frac{1}{2}$ respectively. The matrices (294) become under these substitutions

$$\begin{Vmatrix} 2+2(-\frac{1}{2})+(-\frac{1}{2})^2 & 1+(-\frac{1}{2})1 \\ 1+(-\frac{1}{2})1 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} 2+2(-\frac{1}{2})+2(-\frac{1}{2})^2 & 1+2(-\frac{1}{2})1 \\ 1+2(-\frac{1}{2})1 & 2(1) \end{Vmatrix}$$

which are simplified to

$$\begin{Vmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{Vmatrix}, \quad \begin{Vmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{Vmatrix} \tag{247}$$

But note the very important and surprising result that these two matrices contain as elements exactly the parameters (245) of the network of figure 26. Thus this network can at once be constructed from the matrices (247), which are of course the matrices containing the coefficients of the quadratic forms of this network.

In the same way we can obtain, if we so desire, by assigning proper values to a_{21} and a_{22}, all the networks which we have already obtained in Chapter III, having (113) as an impedance function, and in fact all the networks given by the mutual parameter plane.

Note this further surprising and beautiful result, that if we take the other sign, for a_{22} in (246), we get exactly the image network of the network in figure 26, that is we get this network with the branches in mesh 2 interchanged.

Thus if $\qquad a_{22} = -1$

Then $\qquad a_{21} = -1 - \frac{1}{2} = -\frac{3}{2}$

Our matrices now become

$$\begin{Vmatrix} 2+2(-\frac{3}{2})+(-\frac{3}{2})^2 & -1+(-\frac{3}{2})(-1) \\ -1+(-\frac{3}{2})(-1) & 1 \end{Vmatrix}, \quad \begin{Vmatrix} 2+2(-\frac{3}{2})+2(-\frac{3}{2})^2 & -1+2(-\frac{3}{2})(-1) \\ -1+2(-\frac{3}{2})(-1) & 2 \end{Vmatrix}$$

which simplify to

$$\begin{Vmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{Vmatrix}, \quad \begin{Vmatrix} \frac{7}{2} & 2 \\ 2 & 2 \end{Vmatrix} \qquad (248)$$

But these are exactly the matrices of the network whose parameters are $\lambda_{11} = \frac{5}{4}$, $\lambda_{22} = 1$, $\lambda_{12} = \frac{1}{2}$; $\rho_{11} = \frac{7}{2}$, $\rho_{22} = 2$, $\rho_{12} = 2$ which are the parameters of the image of the network shown in figure 67. This network has already been found by means of the mutual parameter plane and corresponds to the point D, figure 24, which is the image of point A. The network itself is shown in figure 30, page 90, and the parameters of the network are seen to be identical with the elements of the matrices (248).

Note however, that it is unnecessary to go through the work of substituting (242) in (240) to obtain the matrices (244) of the quadratic forms (243). We merely make use of the theorem on matrices, given on the bottom of page 229, namely that if we subject the x's in a quadratic form with matrix A to a linear transformation with matrix C, we obtain a new quadratic form with the matrix

$$C' \; A \; C$$

where C' is the conjugate of C. In open form, the matrix of the new quadratic form is obtained by multiplying together the three matrices

$$\begin{Vmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & & & \vdots \\ \vdots & & & \vdots \\ c_{1n} & \cdots & \cdots & c_{nn} \end{Vmatrix} \times \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{Vmatrix} \times \begin{Vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & & & \vdots \\ \vdots & & & \vdots \\ c_{n1} & \cdots & \cdots & c_{nn} \end{Vmatrix} \qquad (232)$$

In our problem, the linear transformation is (242), the matrix of which is

$$\begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix}$$

which corresponds to the C matrix. Hence using this matrix and the matrices (241), we obtain for the matrices of the transformed quadratic forms

$$\begin{Vmatrix} 1 & a_{21} \\ 0 & a_{22} \end{Vmatrix} \times \begin{Vmatrix} 2 & 1 \\ 1 & 1 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix} \qquad (248a)$$

$$\begin{Vmatrix} 1 & a_{21} \\ 0 & a_{22} \end{Vmatrix} \times \begin{Vmatrix} 2 & 1 \\ 1 & 2 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix} \qquad (248b)$$

Performing the multiplication of the matrices we have

$$\begin{Vmatrix} 1 & a_{21} \\ 0 & a_{22} \end{Vmatrix} \times \begin{Vmatrix} 2 & 1 \\ 1 & 1 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix}$$

$$= \begin{Vmatrix} 2+a_{21} & 1+a_{21} \\ a_{22} & a_{22} \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix}$$

$$= \begin{Vmatrix} 2+a_{21}+a_{21}+a_{21}^2 & a_{22}+a_{21}a_{22} \\ a_{22}+a_{21}a_{22} & a_{22}^2 \end{Vmatrix}$$

$$= \begin{Vmatrix} 2+2a_{21}+a_{21}^2 & a_{22}+a_{21}a_{22} \\ a_{22}+a_{21}a_{22} & a_{22}^2 \end{Vmatrix} \qquad (249a)$$

Note that this is exactly the left hand matrix of (244), page 236 which was obtained from the transformed quadratic forms (243). In the same way, performing the matrix multiplication in (248b) we have

$$\begin{Vmatrix} 1 & a_{21} \\ 0 & a_{22} \end{Vmatrix} \times \begin{Vmatrix} 2 & 1 \\ 1 & 2 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix}$$

$$= \begin{Vmatrix} 2+a_{21} & 1+2a_{21} \\ a_{22} & 2a_{22} \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix}$$

$$= \begin{Vmatrix} 2+2a_{21}+2a_{21}^2 & a_{22}+2a_{21}a_{22} \\ a_{22}+2a_{21}a_{22} & 2a_{22}^2 \end{Vmatrix} \qquad (249b)$$

which is exactly the right hand matrix of (244).

Thus, no matter how complicated a network may be, and no matter how many meshes it may contain, a transformation (232) will give the complete set of equivalent networks. If we are interested only in obtaining networks with positive elements, then the only restriction on the matrix C of the transformation is that its elements be reals and of such values that the matrix of the new quadratic form, obtained by multiplying the matrices in (232), have all the elements in its main diagonal positive and greater in absolute value than the corresponding non-diagonal elements representing the mutual elements.

Before proceeding to the general case, it will be useful to see how we can arrive at the networks containing the least number of elements having (113) for an impedance function. These networks, shown in fig.33-36, we obtained by means of the mutual parameter plane, though they could also be obtained, as we have shown, by means of partial and continued fraction expansion - the method used by Foster and Cauer.

Rewriting the matrices representing the infinite number of networks having (113) for an impedance function, we have

$$\left\|\begin{matrix} 2+2a_{21}+a_{21}^2 & a_{22}+a_{21}a_{22} \\ a_{22}+a_{21}a_{22} & a_{22}^2 \end{matrix}\right\|, \quad \left\|\begin{matrix} 2+2a_{21}+2a_{21}^2 & a_{22}+2a_{21}a_{22} \\ a_{22}+2a_{21}a_{22} & 2a_{22}^2 \end{matrix}\right\| \qquad (244)$$

From these matrices it is a simple matter to obtain the networks containing the least number of elements. The matrices, furthermore, tell us at a glance, by the number of arbitrary constants, namely a_{21} and a_{22}, that we can eliminate at most two elements, thus leaving four elements as the number for the minimal forms. This is very useful, since we can tell at a glance, from the matrices the least number of elements that a network of the group may have. If $\lambda_{11}, \lambda_{22}, \lambda_{12}, \rho_{11}, \rho_{22}, \rho_{12}$ represent the parameters of the infinite set of networks having (113) for an impedance function, the corresponding matrices would of course be

$$\left\|\begin{matrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{matrix}\right\|, \quad \left\|\begin{matrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{matrix}\right\|$$

The eight minimal forms are then obtained by making the parameters take on values so that the following equations are satisfied.

$$\left.\begin{array}{lll}
(1) & \lambda_{12} = 0, & \rho_{12} = \rho_{11} \\
(2) & \lambda_{12} = 0, & \rho_{12} = \rho_{22} \\
(3) & \rho_{12} = 0 & \lambda_{12} = \lambda_{11} \\
(4) & \rho_{12} = 0 & \lambda_{12} = \lambda_{22} \\
(5) & \lambda_{12} = \lambda_{11} & \rho_{12} = \rho_{11} \\
(6) & \lambda_{12} = \lambda_{11} & \rho_{12} = \rho_{22} \\
(7) & \lambda_{12} = \lambda_{22} & \rho_{12} = \rho_{11} \\
(8) & \lambda_{12} = \lambda_{22} & \rho_{12} = \rho_{22}
\end{array}\right\} \qquad (250)$$

Note that these conditions on the parameters are identical with those given on page 96a under figures 33, 34, 35 and 36.

Applying these conditions to (244) we see that a_{21} and a_{22} must satisfy the following equations for the minimal networks.

$$(1) \quad \begin{aligned} a_{22} + a_{21} a_{22} &= 0 \\ a_{22} + 2a_{21} a_{22} &= 2 + 2a_{21} + 2a_{21}^2 \end{aligned}$$

$$(2) \quad \begin{aligned} a_{22} + a_{21} a_{22} &= 0 \\ a_{22} + 2a_{21} a_{22} &= 2a_{22}^2 \end{aligned}$$

$$(3) \quad \begin{aligned} a_{22} + 2a_{21} a_{22} &= 0 \\ a_{22} + a_{21} a_{22} &= 2 + a_{21} + a_{21}^2 \end{aligned}$$

(4) $$a_{22} + 2a_{21}a_{22} = 0$$
$$a_{22} + a_{21}a_{22} = a_{22}^2$$

(5) $$a_{22} + a_{21}a_{22} = 2 + 2a_{21} + a_{21}^2$$
$$a_{22} + 2a_{21}a_{22} = 2 + 2a_{21} + 2a_{21}^2$$

(6) $$a_{22} + a_{21}a_{22} = 2 + 2a_{21} + a_{21}^2$$
$$a_{22} + 2a_{21}a_{22} = 2a_{22}^2$$

(7) $$a_{22} + a_{21}a_{22} = a_{22}^2$$
$$a_{22} + 2a_{21}a_{22} = 2 + a_{21} + 2a_{21}^2$$

(8) $$a_{22} + a_{21}a_{22} = a_{22}^2$$
$$a_{22} + 2a_{21}a_{22} = 2a_{22}^2$$

Each equation from (1) to (8) will give a minimal form.

Let us actually obtain some of the minimal networks. Finding the values of a_{21} and a_{22} in (1) page 243 , we have

$$a_{22} + a_{21}a_{22} = 0 \qquad (251)$$
$$a_{22} + 2a_{21}a_{22} = 2 + 2a_{21} + 2a_{21}^2 \qquad (251a)$$

Hence

$$a_{22}(1 + a_{21}) = 0$$

And
$$a_{21} = -1$$
$$\text{or} \quad a_{22} = 0$$

Substituting the value $a_{21} = -1$ in $(251a)$ we have

$$a_{22} - 2a_{22} = 2 - 2 + 2$$
$$\therefore a_{22} = -2$$

Hence for the first minimal form, with $\lambda_{12} = 0$ and $\rho_{12} = \rho_{11}$, we must have

$$a_{21} = -1$$
$$a_{22} = -2$$

Substituting these values of a_{21} and a_{22} back in the matrices (244), these become

$$\begin{Vmatrix} 2-2+1 & -2+(-1)(-2) \\ -2+(-1)(-2) & (-2)^2 \end{Vmatrix}, \quad \begin{Vmatrix} 2-2+2(-1)^2 & -2+2(-1)(-2) \\ -2+2(-1)(-2) & 2(-2)^2 \end{Vmatrix}$$

$$= \begin{Vmatrix} 1 & 0 \\ 0 & 4 \end{Vmatrix}, \quad \begin{Vmatrix} 2 & 2 \\ 2 & 8 \end{Vmatrix} \qquad (253)$$

The corresponding network is readily constructed from these matrices, and is shown in figure 68.

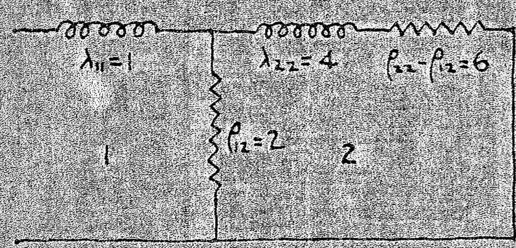


FIG. 68

Note that this is exactly figure 33a, page 99, which was obtained by means of the mutual parameter plane and the equivalence equations.

This illustrates how simple it is to obtain the minimal networks from the matrices (244), as compared to arriving at them by partial or continued fraction expansion, or by means of the equivalence equations and the mutual parameter plane.

It is to be recalled that one other value that a_{22} could have was zero (equation 252). Substituting this value of a_{22} in (251a), we have

$$0 = 2 + 2a_{21} + 2a_{21}^2$$

$$\therefore \quad a_{21}^2 + a_{21} + 1 = 0$$

$$\therefore \quad a_{21} = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

And

$$a_{21} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

Here a_{21} becomes complex, which gives a network with complex elements. At present we can give no physical meaning to a complex element be it inductance, resistance or capacity, although it may have important mathematical significance, or even perhaps future physical significance. Thus for example, while for some time negative resistance had no physical significance in electrical theory, it now definitely has it in radio communication. Thus, for example, a device which has negative resistance can behave as an amplifier, a generator of continuous oscillations.[30]

Let us now obtain the minimal form given by the equations (2) page 243, which are

$$a_{22} + a_{21} a_{22} = 0 \qquad (254a)$$

$$a_{22} + 2a_{21} a_{22} = 2a_{22}^2 \qquad (254b)$$

From (254a) we have

$$a_{22}(1 + a_{21}) = 0$$

And

$$a_{21} = -1$$

Or

$$a_{22} = 0$$

30. See H.J. Van der Bijl, The Thermeonic Vacuum Tube and its Applications, pp. 48, 108, 379, 1920; L.S.Palmer, Wireless Principles and Practice, p. 357, 1928; and V. Bush, Operational Circuit Analysis, p. 267, 1929.

Substituting $a_{21}=-1$ in (254b) we have

$$a_{22} - 2a_{22} = 2a_{22}^2$$

And

$$a_{22} = -\tfrac{1}{2}$$

Thus the values of a_{21} and a_{22} satisfying (2) page 243, are

$$a_{21} = -1$$

$$a_{22} = -\tfrac{1}{2}$$

Substituting these values in the matrices (244), we have for the matrices of the network

$$\begin{Vmatrix} 2+(-2)+1 & -\frac{1}{2}+(-1)(-\frac{1}{2}) \\ -\frac{1}{2}+(-1)(-\frac{1}{2}) & (-\frac{1}{2})^2 \end{Vmatrix}, \quad \begin{Vmatrix} 2+(-2)+2 & -\frac{1}{2}+2(-1)(-\frac{1}{2}) \\ -\frac{1}{2}+2(-1)(-\frac{1}{2}) & 2(-\frac{1}{2})^2 \end{Vmatrix}$$

$$= \begin{Vmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{Vmatrix}, \quad \begin{Vmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{Vmatrix} \qquad (255)$$

The parameters of the network are then

$$\lambda_{11} = 1, \quad \lambda_{22} = \frac{1}{4}, \quad \lambda_{12} = 0 \; ; \; \rho_{11} = 2, \quad \rho_{22} = \frac{1}{2}, \quad \rho_{12} = \frac{1}{2}$$

and the network is shown in figure 69.

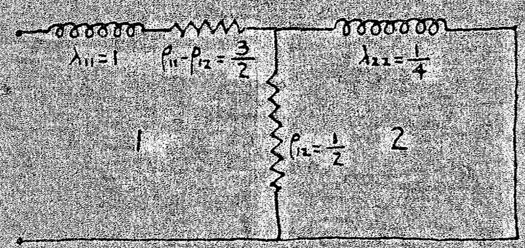


FIG. 69

Note that this network is exactly the network shown in figure 35b, page 111, which was obtained from the equivalence equations and the mutual parameter plane.

It is a simple matter to proceed in this manner and obtain the other minimal forms and their respective images. It is only necessary to satisfy, for each minimal form, one of the equations 1 to 8, page 243, 244.

Let us now obtain the matrices containing the elements of all the networks having all three kinds of network elements, namely inductance, resistance and capacity elements, all of which networks have the same impedance function. We shall see that exactly the same method is used except that we will now have three matrices representing a network instead of two.

To fix ideas, let us again consider an impedance function and the group of networks associated with it, which we have already treated by means of the mutual parameter plane and the equivalence equations. Thus consider the network shown in figure 70, which appears also in figure 60, page 200

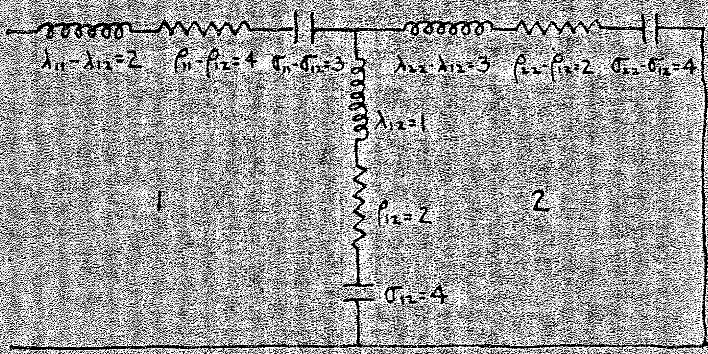


FIG. 70

The parameters of this network are of course

$$\lambda_{11}=3,\ \lambda_{22}=4,\ \lambda_{12}=1\ ;\ \rho_{11}=6,\ \rho_{22}=4,\ \rho_{12}=2\ ;\ \sigma_{11}=7,\ \sigma_{22}=8,\ \sigma_{12}=4$$

and the impedance function is given by (207'), page 200.
Note, however, that in our transformation method, we do not need to know what the impedance function is - it vanishes, so to speak, from the picture.

Let us proceed then to set up the three matrices corresponding to our network. These are

$$\begin{Vmatrix} 3 & 1 \\ 1 & 4 \end{Vmatrix},\quad \begin{Vmatrix} 6 & 2 \\ 2 & 4 \end{Vmatrix},\quad \begin{Vmatrix} 7 & 4 \\ 4 & 8 \end{Vmatrix} \qquad (256)$$

Performing the substitution C' A C that is (232) on each of the three matrices of (256) we obtain the matrices or tensors corresponding to the complete infinite group of networks having (207') for an impedance function. Our C matrix for this two-mesh case is the same as before, namely

$$\begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix} \tag{257}$$

Proceeding with the transformation on the inductance matrix, we have

$$\begin{Vmatrix} 1 & a_{21} \\ 0 & a_{22} \end{Vmatrix} \times \begin{Vmatrix} 3 & 1 \\ 1 & 4 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix}$$

$$= \begin{Vmatrix} 3+a_{21} & 1+4a_{21} \\ a_{22} & 4a_{22} \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix}$$

$$= \begin{Vmatrix} 3+2a_{21}+4a_{21}^2 & a_{22}+4a_{21}a_{22} \\ a_{22}+4a_{21}a_{22} & 4a_{22}^2 \end{Vmatrix} \tag{258}$$

This represents the inductance tensor of our infinite group of networks.

The transformation of the resistance matrix gives

$$\begin{Vmatrix} 1 & a_{21} \\ 0 & a_{22} \end{Vmatrix} \times \begin{Vmatrix} 6 & 2 \\ 2 & 4 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix}$$

$$= \begin{Vmatrix} 6 + a_{21} & 2+4a_{21} \\ 2a_{22} & 4a_{22} \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix}$$

$$= \begin{Vmatrix} 6 + 4a_{21} + 4a_{21}^2 & 2a_{22} + 4a_{21}a_{22} \\ 2a_{22} + 4a_{21}a_{22} & 4a_{22}^2 \end{Vmatrix} \qquad (259)$$

This represents the resistance tensor of our infinite group of networks.

Finally, the transformation of the elastance matrix gives

$$\begin{Vmatrix} 1 & a_{21} \\ 0 & a_{22} \end{Vmatrix} \times \begin{Vmatrix} 7 & 4 \\ 4 & 8 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix}$$

$$= \begin{Vmatrix} 7+4a_{21} & 4+8a_{21} \\ 4a_{22} & 8a_{22} \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{Vmatrix}$$

$$= \begin{Vmatrix} 7+8a_{21}+8a_{21}^2 & 4a_{22}+8a_{21}a_{22} \\ 4a_{22}+8a_{21}a_{22} & 8a_{22}^2 \end{Vmatrix} \qquad (260)$$

This represents the elastance tensor of our infinite group of networks.

Thus the three matrices representing the infinite group of networks having (207), page 200 for an impedance function are, in the order of inductance, resistance and elastance matrices

$$\begin{Vmatrix} 3+2a_{21}+4a_{21}^2 & a_{22}+4a_{21}a_{22} \\ a_{22}+4a_{21}a_{22} & 4a_{22}^2 \end{Vmatrix}, \quad \begin{Vmatrix} 6+4a_{21}+4a_{21}^2 & 2a_{22}+4a_{21}a_{22} \\ 2a_{22}+4a_{21}a_{22} & 4a_{22}^2 \end{Vmatrix}, \quad \begin{Vmatrix} 7+8a_{21}+8a_{21}^2 & 4a_{22}+8a_{21}a_{22} \\ 4a_{22}+8a_{21}a_{22} & 8a_{22}^2 \end{Vmatrix} \qquad (261)$$

Before proceeding further, note that the number of arbitrary constants, namely a_{21} and a_{22}, tells us, as in the previous case, the number of network elements which we may eliminate from the network. Thus, the minimal forms, that is the networks with the least number of elements, will contain in general seven elements.

As before, let us consider a network which we know, from our previous work, to be equivalent to the network shown in figure 70, and let us see if our matrices (261) will give this equivalent network. Thus, the network shown in figure 62, page 206, is equivalent, as we have seen, to the network of figure 70. This equivalent network is shown again in figure 71.

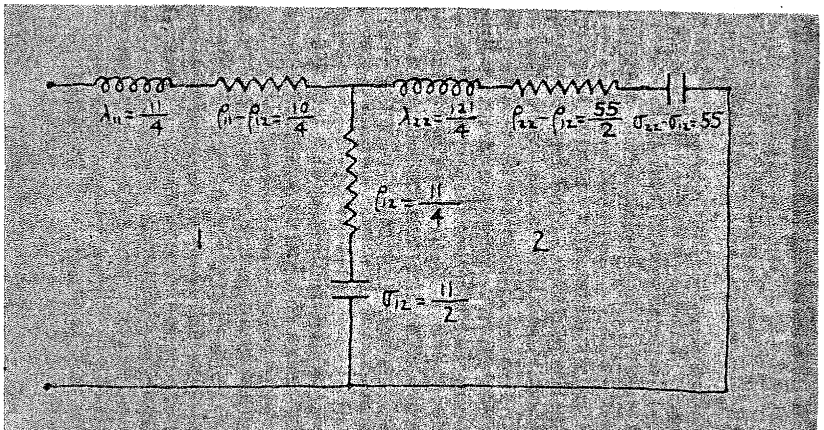


FIG. 71

The parameters of this network are

$$\lambda_{11} = \frac{11}{4}\,,\ \lambda_{22} = \frac{121}{4}\,,\ \lambda_{12} = 0\ ;\ \rho_{11} = \frac{21}{4}\,,\ \rho_{22} = \frac{121}{4}\,,\ \rho_{12} = \frac{11}{4}\ ;\ \sigma_{11} = \frac{11}{2}\,,\ \sigma_{22} = \frac{121}{2}\,,\ \sigma_{12} = \frac{11}{2}$$

and the corresponding matrices are

$$\left\|\begin{matrix} \frac{11}{4} & 0 \\ 0 & \frac{121}{4} \end{matrix}\right\|, \qquad \left\|\begin{matrix} \frac{21}{4} & \frac{11}{4} \\ \frac{11}{4} & \frac{121}{4} \end{matrix}\right\|, \qquad \left\|\begin{matrix} \frac{11}{2} & \frac{11}{2} \\ \frac{11}{2} & \frac{121}{2} \end{matrix}\right\| \qquad (262)$$

Comparing these matrices with those in (261), we see that we must have, for example

$$4a_{22}^{2} = \frac{121}{4}$$

And

$$a_{22} = \pm\ \frac{11}{4}$$

Further

$$a_{22} + 4\,a_{21}\,a_{22} = 0$$

And

$$a_{21} = -\frac{1}{4}$$

Thus, using the values

$$a_{21} = -\frac{1}{4}$$

$$a_{22} = -\frac{11}{4}$$

The matrices of (261) become respectively

$$\begin{Vmatrix} 3 + 2a_{21} + 4a_{21}^2 & a_{22} + 4a_{21}a_{22} \\ a_{22} + 4a_{21}a_{22} & 4a_{22}^2 \end{Vmatrix} = \begin{Vmatrix} \frac{11}{4} & 0 \\ 0 & \frac{121}{4} \end{Vmatrix}$$

$$\begin{Vmatrix} 6 + 4a_{21} + 4a_{21}^2 & 2a_{22} + 4a_{21}a_{22} \\ 2a_{22} + 4a_{21}a_{22} & 4a_{22}^2 \end{Vmatrix} = \begin{Vmatrix} \frac{21}{4} & \frac{11}{4} \\ \frac{11}{4} & \frac{121}{4} \end{Vmatrix}$$

$$\begin{Vmatrix} 7 + 8a_{21} + 8a_{21}^2 & 4a_{22} + 8a_{21}a_{22} \\ 4a_{22} + 8a_{21}a_{22} & 8a_{22}^2 \end{Vmatrix} = \begin{Vmatrix} \frac{11}{2} & \frac{11}{2} \\ \frac{11}{2} & \frac{121}{2} \end{Vmatrix}$$

The three matrices are thus

$$\begin{Vmatrix} \frac{11}{4} & 0 \\ 0 & \frac{121}{4} \end{Vmatrix}, \quad \begin{Vmatrix} \frac{21}{4} & \frac{11}{4} \\ \frac{11}{4} & \frac{121}{4} \end{Vmatrix}, \quad \begin{Vmatrix} \frac{11}{2} & \frac{11}{2} \\ \frac{11}{2} & \frac{121}{2} \end{Vmatrix} \qquad (263)$$

These are seen to be identical with those of (262) which represent the matrices of the network of figure 71. Thus we see that the procedure for networks containing inductance, resistance and capacity elements is exactly the same as for networks containing only two kinds of elements. By assigning then, different values to a_{21} and a_{22}, we can run through the complete infinite group of networks having (207'), page 200, for an impedance function. Furthermore, we can proceed in a manner similar to that of obtaining the minimal forms of networks with two kinds of network elements, to obtain the minimal forms of networks containing all three kinds of network elements.

For the general case then of networks of any number of meshes containing all three kinds of network elements, namely inductance, resistance and elastance elements, we have the following three matrices which represent or definitely fix the network.

$$\begin{Vmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{12} & & & \vdots \\ \vdots & & & \vdots \\ \lambda_{1n} & \cdots & \cdots & \lambda_{nn} \end{Vmatrix}, \quad \begin{Vmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & & & \vdots \\ \vdots & & & \vdots \\ \rho_{1n} & \cdots & \cdots & \rho_{nn} \end{Vmatrix}, \quad \begin{Vmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & & & \vdots \\ \vdots & & & \vdots \\ \sigma_{1n} & \cdots & \cdots & \sigma_{nn} \end{Vmatrix} \qquad (264)$$

Making a linear affine transformation of the instantaneous mesh currents or charges in the network, we have

$$\left.\begin{aligned}
i_1 &= i_1' \\
i_2 &= a_{21} i_1' + a_{22} i_2' + \text{-------} + a_{2n} i_n' \\
i_3 &= a_{31} i_1' + a_{32} i_2' + \text{-------} + a_{3n} i_n' \\
&\text{-------------------} \\
&\text{-------------------} \\
i_n &= a_{n1} i_1' + a_{n2} i_2' + \text{-------} + a_{nn} i_n'
\end{aligned}\right\} \qquad (265a)$$

for the currents, and

$$\left.\begin{aligned}
q_1 &= q_1' \\
q_2 &= a_{21} q_1' + a_{22} q_2' + \text{-------} + a_{2n} q_n' \\
q_3 &= a_{31} q_1' + a_{32} q_2' + \text{-------} + a_{3n} q_n' \\
&\text{-------------------} \\
&\text{-------------------} \\
q_n &= a_{n1} q_1' + a_{n2} q_2' + \text{-----} + a_{nn} q_n'
\end{aligned}\right\} \qquad (265b)$$

for the charges.

The three fundamental forms of the electric network of n meshes, whose coefficients are determined from the three matrices (264), are respectively

$$T = \frac{1}{2}\sum_1^n \lambda_{jk}\, i_j i_k \qquad (266a)$$

$$F = \frac{1}{2}\sum_1^n \rho_{jk}\, i_j i_k \qquad (266b)$$

$$V = \frac{1}{2}\sum_1^n \sigma_{jk}\, q_j q_k \qquad (266c)$$

The substitution of the transformations (265) in (266), results in three new quadratic forms, namely

$$T' = \frac{1}{2}\sum_1^n \lambda'_{jk}\, i'_j i'_k \qquad (267a)$$

$$F' = \frac{1}{2}\sum_1^n \rho'_{jk}\, i'_j i'_k \qquad (267b)$$

$$V' = \frac{1}{2}\sum_1^n \sigma'_{jk}\, q'_j q'_k \qquad (267c)$$

The coefficients of these new quadratic forms λ_{jk}', ρ_{jk}' and σ_{jk}' will of course be functions of the elements of the matrices (264) of the original quadratic forms (266) and the a coefficients of the transformations (265). This has already been noted in our previous two-mesh examples. From physical considerations, the quadratic forms 266 and 267 are respectively equal, that is, quadratic forms are invariant to a linear affine transformation of the instantaneous currents.

The transformation matrix, which contains the coefficients of the transformations (265) may be written

$$C = \begin{Vmatrix} 1 & 0 & - & - & - & 0 \\ a_{21} & a_{22} & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{n1} & a_{n2} & - & - & - & a_{nn} \end{Vmatrix} \qquad (268)$$

The matrices containing the coefficients of the new quadratic forms (267) are of course

$$\begin{Vmatrix} \lambda_{11}' & - & - & - & \lambda_{1n}' \\ | & & & & | \\ | & & & & | \\ | & & & & | \\ \lambda_{1n}' & - & - & - & \lambda_{nn}' \end{Vmatrix}, \quad \begin{Vmatrix} \rho_{11}' & - & - & - & \rho_{1n}' \\ | & & & & | \\ | & & & & | \\ | & & & & | \\ \rho_{1n}' & - & - & - & \rho_{nn}' \end{Vmatrix}, \quad \begin{Vmatrix} \sigma_{11}' & - & - & - & \sigma_{1n}' \\ | & & & & | \\ | & & & & | \\ | & & & & | \\ \sigma_{1n}' & - & - & - & \sigma_{nn}' \end{Vmatrix} \qquad (269)$$

The matrices contain the complete infinite group of networks having for an impedance function the impedance of the network of (264). The impedance function is thus an <u>absolute</u> invariant to a linear affine transformation of the instantaneous currents or charges of the networks. The matrices (269) include within them the matrices (264), which are obtained from (269) by the <u>identity</u>

transformation, namely

$$\left.\begin{array}{l} i_1 = i_1' \\ i_2 = \quad i_2' \\ i_3 = \qquad i_3' \\ \text{-----------------} \\ \text{-----------------} \\ i_n = \qquad\qquad i_n' \end{array}\right\} \qquad (270)$$

The C matrix corresponding to this transformation is the identity matrix

$$\begin{Vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{Vmatrix} \qquad (271)$$

The matrices (269), which contain the complete infinite group of networks equivalent to the networks represented by the matrices (264) may be called tensors. The matrices (264) correspond to the matrices (256), page 249 in our two-mesh example, the transformation matrix C (268) corresponds to (257) and the tensors (269) to the tensors (261).

As in the two-mesh example, we can avoid the actual substitution of the transformations (265) in the quadratic forms

by making use of the transformation theorem given on page 239
Thus the tensors (269) are obtained at once from the matrices (264), and the transformation matrix C (268) by the following matrix multiplications

$$\begin{Vmatrix} 1 & a_{21} & & a_{n1} \\ 0 & a_{22} & & a_{n2} \\ & & & \\ 0 & a_{2n} & & a_{nn} \end{Vmatrix} \times \begin{Vmatrix} \lambda_{11} & - - - & \lambda_{1n} \\ | & & | \\ | & & | \\ \lambda_{1n} & - - - & \lambda_{nn} \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 & - - - - & 0 \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{Vmatrix} \qquad (272a)$$

$$\begin{Vmatrix} 1 & a_{21} & & a_{n1} \\ 0 & a_{22} & & a_{n2} \\ & & & \\ 0 & a_{2n} & & a_{nn} \end{Vmatrix} \times \begin{Vmatrix} \rho_{11} & - - - & \rho_{1n} \\ | & & | \\ | & & | \\ \rho_{1n} & - - - & \rho_{nn} \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 & & 0 \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{Vmatrix} \qquad (272b)$$

$$\begin{Vmatrix} 1 & a_{21} & & a_{n1} \\ 0 & a_{22} & & a_{n2} \\ & & & \\ 0 & a_{2n} & & a_{nn} \end{Vmatrix} \times \begin{Vmatrix} \sigma_{11} & - - - & \sigma_{1n} \\ | & & | \\ | & & | \\ \sigma_{1n} & - - - & \sigma_{nn} \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 & & 0 \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{Vmatrix} \qquad (272c)$$

The result of the matrix multiplications will be the three tensors (269) where the elements λ', ρ' and σ' are expressed in terms of the elements of the given network λ, ρ and σ, and the elements a of the transformation matrix C (268). The matrix multiplications (272) correspond in our two-mesh example to the matrix multiplications (258), (259) and (260).

This method of arriving at the complete infinite group of networks equivalent to a given network is thus very powerful, and with one sweep gives all the networks of any number of meshes with all three kinds of network elements, equivalent to a given network. A glance at our equivalence equations, merely for the three-mesh network, with only two kinds of network elements (chapter V) will indicate the power of this method. It will be recalled there, that after setting up the equivalence equations in open form (169), page 166 , which is in itself laborious, we went to considerable labor to obtain the solution (177) for just one total parameter λ_{11} . The evaluation of the two fourth order determinants of (177) is quite a task, and this has to be done for all the total parameters. For networks with more than three meshes, the method of the equivalence equations is prohibitive. Also, a perusal of Foster's and Cauer's paper on the two-mesh network with all three network elements present, will show at once the simplification introduced by the above transformation method. Furthermore, the methods of both Foster and Cauer were different for networks with two kinds of elements, where the partial and continued fraction methods was used, than for networks with all three kinds of elements present, where the equivalence equations were used. Our transformation method thus unifies the treatment of all networks, as well as simplifies considerably the method arriving at the equivalent networks. Finally, it solves the general n-mesh problem, with which both Foster and Cauer have been much concerned with for some time.[31]

The above linear affine transformations have been made for preserving the invariance of the driving-point impedance function. These transformations need not be so limited. Thus transformations may be made whereby the invariance of the transfer-impedance function may be preserved. This gives rise to a new notion of equivalence, namely equivalence with respect to a definite mesh. Thus, for example, two n-mesh networks may be equivalent with respect to the k^{th} mesh. This means that the instantaneous currents in the k^{th} mesh of both equivalent networks are identical, or to put it in another way, the currents in the k^{th} mesh of both equivalent networks are equal at all frequencies. The complete infinite group of n-mesh networks equivalent with respect to the k^{th} mesh are obtained exactly in the same manner as above, except that now the transformations are

$$\left.\begin{array}{l} i_1 = a_{11} i_1' + \cdots\cdots\cdots + a_{1n} i_n' \\ i_2 = a_{21} i_1' + \cdots\cdots\cdots + a_{2n} i_n' \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ i_k = i_k' \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ i_n = a_{n1} i_1' + \cdots\cdots\cdots + a_{nn} i_n' \end{array}\right\} \qquad (273a)$$

31. See R.M.Foster "Theorems Regarding the Driving-Point Impedance of Two-Mesh Circuits", Bell System Technical Journal, vol. 3, 1924, p.657 and W. Cauer, "Die Verwirklichung von Wechselstromwiderstanden usw.", Archiv fur Elektrotechnik, Heft IV, Band XVII, 1926, p. 385.

for the currents, and

$$\left.\begin{aligned}
q_1 &= a_{11}q_1' + \cdots\cdots + a_{1n}q_n' \\
q_2 &= a_{21}q_1' + \cdots\cdots + a_{2n}q_n' \\
&\cdots\cdots\cdots\cdots\cdots \\
q_k &= 1\,q_k' \\
&\cdots\cdots\cdots\cdots\cdots \\
q_n &= a_{n1}q_1' + \cdots\cdots + a_{nn}q_n'
\end{aligned}\right\} \qquad (273b)$$

As for the equivalence of the driving-point impedance, the tensors representing the complete infinite group of networks in which ~~now~~ the transfer-impedance function in the k^{th} mesh is now an invariant, are obtained by the following matrix multiplications n

C' $\qquad\qquad A \qquad\qquad C$

$$\left\|\begin{matrix} a_{11} & \cdots & a_{k-1,1} & 1 & a_{k+1,1} & \cdots & a_{n1} \\ a_{12} & \cdots & a_{k-1,2} & 0 & a_{k+1,2} & \cdots & a_{n2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & \cdots & a_{k-1,n} & 0 & a_{k+1,n} & \cdots & a_{nn} \end{matrix}\right\| \times \left\|\begin{matrix} \lambda_{11} & & \lambda_{1n} \\ & & \\ \lambda_{1n} & & \lambda_{nn} \end{matrix}\right\| \times \left\|\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{k-1,1} & a_{k-1,2} & \cdots & a_{k-1,n} \\ 1 & 0 & \cdots & 0 \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}\right\| \qquad (274a)$$

$$\left\|\begin{matrix} & & \\ & '' & \\ & & \end{matrix}\right\| \times \left\|\begin{matrix} \rho_{11} & & \rho_{1n} \\ & & \\ \rho_{1n} & & \rho_{nn} \end{matrix}\right\| \times \left\|\begin{matrix} & & \\ & '' & \\ & & \end{matrix}\right\| \qquad (274b)$$

$$\left\|\begin{matrix} & & \\ & '' & \\ & & \end{matrix}\right\| \times \left\|\begin{matrix} \sigma_{11} & & \sigma_{1n} \\ & & \\ \sigma_{1n} & & \sigma_{nn} \end{matrix}\right\| \times \left\|\begin{matrix} & & \\ & '' & \\ & & \end{matrix}\right\| \qquad (274c)$$

The result of the matrix multiplications will be three tensors like (269), where the elements λ', ρ' and σ' will, as before, be expressed in terms of the elements of the given network λ, ρ and σ and the elements a of the transformation matrix. Thus we obtain the tensors representing the complete infinite group of networks which are equivalent with respect to the k^{th} mesh

It was mentioned that the number of arbitrary constants in the transformation matrix exactly determined the number of elements which may be eliminated from the network without disturbing the invariance of the impedance function. Thus the least number of elements necessary in any network to realize a definite driving-point impedance function, or a definite transfer-impedance function, is readily determined. This is very important, since we can now tell at once whether any communication network of any number of meshes has superfluous elements.

Instead of imposing conditions on the a coefficients of the transformation to give minimal networks, it may be possible to obtain equivalence with respect to more than one mesh in a network, that is, to make the instantaneous currents in both the k-mesh and r-mesh say, identical for the complete infinite group of networks. This is done by resorting to a more general linear affine transformation than the one given in (265). In this more general transformation we shall have two identical instantaneous mesh currents or charges, that is $i_k = i_k'$ and $i_r = i_r'$. Finally, we may obtain equivalence with respect to say j meshes, by using a still more general linear affine transformation.

In all of the above theory, we have limited ourselves to networks of a finite number of meshes, that is, to networks with n degrees of freedom. There is no reason physically why all of the above theory can't apply exactly to networks of an infinite number of meshes, that is, an infinite number of degrees of freedom. Here we have very interesting problems arising, bearing intimately on mathematical theory, acoustics, electromagnetic wave theory, elastic waves, - in short, all branches of physics involving oscillations. This is true for the finite problem, for all the theory explained above can be applied to any physical vibrational problem involving a finite number of degrees of freedom, not just electric circuit theory; although the latter province seems best able to offer fertile soil for further investigation, and provide a physical picture of the phenomena taking place.

In the problem involving networks with an infinite number of degrees of freedom we have to deal with matrices and tensors containing an infinite number of elements as well as with quadratic forms which are power series. The matrices containing the coefficients of the three fundamental quadratic forms will contain an infinite number of elements, as well as the transformation matrix and the resulting tensors. But for a physical network of an infinite number of degrees of freedom, we <u>know physically</u> that the total instantaneous magnetic energy in the coils, the total instantaneous electrostatic energy in the condensers and the total instantaneous power lost in the resistances are finite quantities. Hence the three fundamental quadratic forms, which are now power series, are properly convergent. Likewise, the linear transformations, which are linear forms of an infinite number of terms, have meaning, as ~~has~~ have the infinite transformation matrices which contain the coefficients of the transformations. Finally the resulting tensors, which contain an infinite number of elements have physical meaning. They represent the complete infinite group of networks of an infinite number of degrees of freedom which are equivalent in one or all the ways defined above.

The network with an infinite number of degrees of freedom is merely a continuous system such as the smooth transmission or communication line. Thus, not only may there be an infinite number of different terminal networks which may perform the same function in a communication system, but also an infinite number of communication lines, which may perform the same function.

It will be recalled that the above investigation has been limited to two terminal networks. By theprinciple of superposition, it is possible to extend the above theory to networks of any number of terminals. This extension is of considerable importance, since by means of it, any section of a communication network can be removed and replaced by an equivalent section.

C O N C L U S I O N

The purpose of this dissertation has been essentially two-fold. First, to clarify and simplify the existing knowledge of the impedance function and the various methods by which it could be made to yield its associated networks, and, in particular, its minimal networks. Second, to extend the present results of the two-mesh network to the general case of the n-mesh network. To do so, it was necessary first to review thoroughly the work of Foster and Cauer and point out exactly what they had done. This, it should be mentioned, I found difficult to do at the beginning until I had independently reached some of the results of Foster and Cauer. Thus, for example, I had already arrived at the equivalence equations for the n-mesh case in my symbolic notation, before I realized that Foster's equations 46-53[32] and Cauer's equations 63-71[33] were exactly the equivalence equations for the two-mesh case.

Routh's excellent treatise "Advanced Rigid Dynamics" was very useful in clarifying the situation. Of course, the problem of current and charge in an electrical network is identical with the problem of velocities and displacement in a dynamical networks, and Lagrange's equations in dynamics hold exactly for

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32. R. M. Foster, "Theorem Regarding the Driving-Point Impedance of Two-Mesh Circuits", Bell System Technical Journal Vol. 3, 1924, p. 684.

33. W. Cauer, "Die Verwirklichung von Wechselwiderstanden usw", Archiv fur Elektrotechnik, Heft 4, Band XVII, 1926, p. 373.

both. Thus, for example, the Kirchhoff equations of the network (which correspond to Newton's laws of motion in dynamics) were obtained by substituting the three fundamental forms of energy of the electric circuit (which correspond to the kinetic, potential and dissipation functions in dynamics) in Lagrange's equations. It was observed that the matrices containing the coefficients of these three quadratic forms were of considerable importance. From these matrices, we could at once set up the network. In fact, we might forget the picture of the actual electrical network, and work entirely with the matrices. Mathematically, an electrical network is a group of one, two or three matrices, depending on whether the network has one, two or three kinds of network elements. Furthermore, the impedance function could at once be constructed from these matrices, by means of a convenient symbolic notation. This, we saw, saved considerable labor in more complicated networks, in arriving at the impedance function. Also it simplified considerably the problem of obtaining the two-mesh equivalent network by means of the equivalence equations. These equivalence equations, which expressed the relations between the elements of two networks in order that they be equivalent, were obtained for the n-mesh network, although their solutions were very difficult for network of more than two meshes.

By means of the equivalence equations, and the use of the mutual parameter plane, it was a simple matter to arrive at the complete infinite set of two-mesh networks all having the same impedance function, and the corresponding minimal networks.

There was a one to one correspondence between a point in the plane and a network which was a member of the group of equivalent networks. To exhaust the entire plane, it was found necessary to consider also networks with negative elements. A complete exploration of the mutual parameter plane for the two-mesh network revealed regions which were images of each other. That is, networks in one region were identical with the networks in the image region, except that the branches in mesh 2 were interchanged. Points of discontinuity occured about which reflection took place. The points representing the minimal forms for the two-mesh network with inductance and resistance elements were indicated in the plane. There were really eight minimal forms, four of them being images of the other four. No reasons could be found for the peculiar position of these points, although a complete exploration of the mutual parameter plane would no doubt reveal the manner in which networks transformed continuously to cover the complete infinite group of networks of the plane.

This continuous transformation of one network into its equivalent network suggested the idea of trying to find the transformations which would transform a network into its equivalent network. Naturally, it was desirable that these transformations be linear. To do this a thorough study of quadratic forms was made, and it was found that they were invariant to a linear affine transformation of the instantaneous mesh currents or charges of the network. This meant that the total instantaneous magnetic and electrostatic energies and the power loss were invariant to a linear affine transformation of the instantaneous mesh currents or charges. Similarly, the impedance function is invariant to this transformation. Now it was noted that the matrices which contained the coefficients of the three fundamental quadratic forms represented a definite network. Thus, beginning with a definite network, the matrices containing the coefficients of the quadratic forms could at once be constructed, and hence the quadratic forms themselves. By a linear affine transformation of the variables of the quadratic forms (which are, of course, the instantaneous mesh currents or charges), new quadratic forms were obtained. By constructing the matrices of these new quadratic forms, it was surprisingly found that these new matrices exactly represented an equivalent network. Thus, by assigning different values to the coefficients of the transformation, the complete infinite group of networks equivalent to a given network could be obtained. However, instead of actually performing the transformation, which is rather tedious, a matrix multiplication gave the tensor at once, which contained the complete infinite group of matrices representing networks all of which had the same impedance function. This matrix multiplication was merely C'AC, where A represented the original matrix, C the transformation matrix and C' its conjugate. This rather simple matrix multiplication at once solves the general n-mesh problem.

The notion of equivalence was then extended to include equivalence of networks with respect to any mesh, that is, equivalence with respect to transfer impedance. Also the possibility of the extension of equivalence to include equivalence with respect to a certain number of meshes was indicated, and it was shown by the principle of superposition that the above theory could be made to apply to networks with any number of terminals. Finally, by extending the results to networks with an infinite number of meshes, that is, an infinite number of degrees of freedom, we are confronted with very interesting problems of accoustics, electromagnetic theory - problems expressed in terms of partial differential equations instead of total. Thus we are led to continuous systems such as the smooth transmission or communication line; and here, too, there exists an infinite group of lines all having the same impedance function. It is hardly necessary to stress the important practical significance of this. In these problems of networks with an infinite number of degrees of freedom we are led to quadratic forms which are now power series and matrices which have an infinite number of elements. The problems which arise here are interesting as they are varied.

At this point it will be useful to point out the effects of the foregoing results on future electrical theory, and to suggest problems for further investigation.

First it should be mentioned that present undergraduate work, and even graduate work in electrical engineering begin essentially with Ohm's and Kirchhoff's laws; and the energy and power relations (which are the quadratic forms) are given a secondary importance. The reverse should be the case. The total instantaneous kinetic energy, potential energy and power loss are of fundamental importance, and the manner in which these give, by means of Lagrange's equations, the Kirchhoff equations

~~the Kirchhoff equations~~ of the network should be emphasized. Furthermore the impression is given that there is a one to one correspondence between a network and its impedance, that is, that to a given impedance there corresponds one network which has that impedance. The fact that to a given impedance there may correspond an infinite number of networks should be pointed out even in undergraduate work. This conception may be very useful in simplifying many problems. Thus, for example, the solution for the instantaneous currents of a network of the group at once results in the solutions for the instantaneous currents of all of the infinite number of networks in the group, since these currents are obtained from the former by a simple linear transformation. Furthermore, the solution for the instantaneous currents in one network in the group may be much simpler than for another; and there may be one network in the group for which the computations are least complicated. Hence, if it is necessary to obtain currents and voltages in one network, it may be simpler first to transform the network to an equivalent one, for which the computations are much simpler. This is already recognized, for example, when we transform from Y to Δ and vice versa.

It is to be noted that in the matrix multiplication which gives the tensor containing the complete infinite group of equivalent networks, the impedance function vanishes from the picture. This suggests the possibility that the notion of the impedance function, which is a special creation of the electrical engineer, may disappear in the future. What we have to deal with are networks, currents and energies, and the impedance function, while it may be helpful for visualization, is not necessary to obtain the final important results.

As was pointed out before, the problem of currents and charges in an electrical network is identical with the problem of velocities and displacements in a dynamical system. This is in general recognized, and yet there is much in classical dynamic theory that still remains to be translated in appropriate language for electric circuit theory. In fact, lacking a knowledge of classical dynamics, the electrical engineer has often gone to considerable trouble in working out for himself thing he could have found, for example, in Routh's Dynamics. It was also mentioned that much of the inspiration and proof of Foster's two papers came from the similar dynamical problem of vibrations about a position of equilibrium.

Questions such as what in electric circuit theory corresponds to the principle of normal coordinates in dynamic theory, still remain unanswered. Is it possible to eliminate in the fundamental quadratic forms of the electric circuit, the cross product terms, thereby giving expressions which are sums of squares of the currents or charges? If it is, can a physical network be built realizing this?

Much remains to be done to explore completely the mutual parameter plane, and to explain the reasons for the peculiar positions of the points representing the minimal forms. Also, it appears that mathematics does not discriminate against negative network elements, which seems to indicate that they may be realized physically, though not, of course, by coils, resistors and condensers.

Problems of networks with an infinite mumber of degrees of freedom, equivalence with respect to transfer-impedance, equivalence with respect to more than one mesh, networks with more than two terminals, have only been touched upon. Likewise, the application of continued fraction theory to the electrical network has only just begun, and future work in this field is certain to reveal much both to the mathematician and the electrical engineer.

Finally, it should be added, that in the study of the electrical network and its response to an impressed electromotive force, one continually runs into many seemingly unrelated branches of mathematics, such as (1) Continued fractions (2) Cauchy residue theory (3) Asymptotic series (4) Fractional and Irrational Derivatives and Integrals (5) Group theory (6) Fourier Series and Transforms (7) Integral equations and what not. It seems almost as if there were something there, inarticulately trying to make itself understood - but perhaps it must await a modern Euler.

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A B S T R A C T.

On the Invariant Impedance Function and its Associated Group of Networks.

The dissertation is essentially a thorough investigation of the impedance function, its invariance to a change of network parameters, and the methods by which it could be made to yield its associated networks, and, in particular, its minimal networks.

It is well known that to adefinite network there corresponds one and only one impedance function, which for a finite network is a ratio of two polynomials in a ~~real~~ complex variable with real coefficients. It is not so well known, however, that to a given impedance function there corresponds an infinite number of networks, every one of which has for an impedance the given impedance function. That is, electrical networks form a group in which the impedance function is an invariant.

It is shown how the coefficients of the impedance function can be obtained directly from the matrices of the coefficients of the three fundamental quadratic forms of the electric circuit, namely the total instantaneous magnetic and electrostatic energies and the total instantaneous power loss of the network. These three quadratic forms, which may be called the inductance, elastance and resistance quadratic forms, are positive and definite and when substituted in Lagrange's equations yield the Kirchhoff equations of the network.

The coefficients of the impedance function are expressed in certain determinants and principal minors of the matrices of the coefficients of the quadratic forms. The elements of these determinants and minors are the mutual and total parameters of the network, or, what is the same thing, the coefficients of the quadratic forms. A symbolic determinantal notation is introduced, which seems to be a natural way to express the impedance function and the equivalence equations. The coefficients of the impedance function are shown to be invariants of weight two and hence the impedance function, which is the ratio of two relative invariants of the same weight, is itself an absolute invariant. By a change of network parameters but preserving the invariance of the impedance function, the most desirable network may be obtained. Of course, any one of the infinite number of networks of the group having the same impedance function may be substituted for each other in a communication system without affecting the system. Thus, it is not enough to design a network to perform a certain function and be satisfied when the network is finally built and performs its function satisfactorily. As long as there exists an infinite number of other networks which will perform identically the same function, the design is not a good one until the best and most economical network is selected.

The conditions are given for the invariance of the form of the impedance function in terms of the resultant of the numerator and denominator of the impedance function. The vanishing of the resultant is shown to correspond to short-circuiting a network. By removal of as many elements of the network as can be removed without violating the conditions for the invariance of the form of the impedance function, networks with the least number of elements result. These have been obtained by Foster by a partial fraction expansion of the impedance and admittance functions, and by Cauer by continued fraction expansions of the same. These minimal forms, as the networks with the least number of elements may be called, are shown for the two-mesh case with two kinds of elements to be really eight in number, four of which are, so to speak, images of each other, the branches in the second mesh being interchanged. The straight line equation in the mutual parameters, involving the resultant is plotted and a family of straight lines is obtained, every point of which, within certain regions, is a possible pair of mutual parameters of the network, the other parameters being obtained from the equivalence equations. The mutual parameter plane may be divided into regions which contain points representing six, five and four element networks. Thus, for example, the eight points corresponding to the eight minimal forms are indicated in the plane. Also, two regions in the plane may be images of each other, the interior of which regions represent six-element networks, and the boundary, five-elememt networks. The complete exploration of the mutual parameter plane will show regions whose points represent, respectively, six, five and four-element networks. Other regions will contain points representing networks which have both positive and negative elements, and finally still other regions will contain points representing networks having all negative elements. This suggests the possibility of making use of negative elements which it would be necessary to realize in other ways than through coils, resistors and condensers. The mutual parameter plane is also constructed for the general two-mesh network and similar results obtained. Here a vector notation is also introduced which seems capable of generalization to networks with any number of meshes. Also, the condtion that a function having the form of an impedance function be in fact the impedance of a physical network is given in terms of the resultant, and the equivalence equations for the general case of n meshes obtained.

Finally, and most important, a transformation method is developed, which, ~~with one sweep,~~ gives the complete infinite group of networks equivalent to a given network. This solves the n-mesh problem in a most elegant fashion, simplifying and unifying the procedure for all networks of any number of meshes. This is done by making a linear affine transformation of the variables of the fundamental quadratic forms of the network (which variables are of course the instantaneous mesh currents or charges). Thus, beginning with a definite network the matrices containing the coefficients of the quadratic forms, and hence the quadratic forms themselves, are readily constructed from the elements of the network. By a linear affine transformation of the variables of the quadratic forms, new quadratic forms are obtained. By constructing the matrices of these new quadratic forms, it is found that these new matrices represent an equivalent network. Then, by assigning different values to the coefficients of the transformation, the complete infinite group of networks equivalent to a given network are obtained. However, instead of actually performing the substitution, which is rather tedious, a simple matrix multiplication is used which gives at once tensor which

~~which~~ contains the complete infinite group of matrices representing networks all of which have the same impedance function. This matrix multiplication is merely C'AC, where A represents the original matrix, C the transformation matrix and C' its conjugate.

The notion of equivalence is then extended to include equivalence of networks with respect to any mesh, that is, equivalence with respect to transfer impedance. Also the possibility of the extension of equivalence to include equivalence with respect to a certain number of meshes is indicated. By the principle of superposition, the above theory can be extended to networks with any number of terminals. Finally, extensions to networks with an infinite number of meshes, that is, an infinite number of degrees of freedom, which lead to continuous systems are indicated, the importance of the further application of dynamic theory and continued fraction theory to electric circuit theory is stressed, and problems for further investigation are pointed out.