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I) The gravitational field in a fluid sphere of uniform invariant density according, to the theory of relativity.

(2) Note on de Sitter' Universe.

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THE GRAVITATIONAL FIELD IN A FLUID SPHERE OF UNIFORM INVARIANT DENSITY, ACCORDING TO THE THEORY OF RELATIVITY.

Contents

I.- INTRODUCTION

The gravitational field within a fluid sphere of uniform density has been the object of many investigations, specially by Schwarzschild, Nardström and De Donder and was considered as a solved problem until Eddington

The Mathematical theory of Relativity-Cambridge 1923-p.121 sq. and p. 168 sq.

made some fundamental objections against the solution of these authors.

The density which was supposed to be uniform in these works was the component T_4^4 of the energy-tensor of the matter; Eddington contends that the true representation of the density is not T_4^4 but the associated invariant T. If this conception is exact, the solution of Schwarzschild is but an approximation and a solution is required for which the invariant density T is uniform throughout the sphere.

Let us consider with Eddington a fluid formed of a great number of moving particles. The fluid will be incompressible if a given closed surface contains the same number of particles whatever may be the pressure. The velocities of the particles and the intensity of the electro-magnetic field which acts between them will be generally modified by a change of pressure. Now T_4^4 refers to the apparent masses of the particles for an observer at rest, and is therefore increased for the same particles when their velocities are increased. In the same way the electromagnetic field has a component T_4^4 of which we must take account in the total T_4^4 included in a given boundary; it varies with the variations of intensity of the field.

On the contrary, T refers to the invariant masses of the particles i.e. to their masses for an observer with respect to which each particle is at rest. It depends neither on the motion of the particles, nor on the electromagnetic field which acts between them, as the electromagnetic field gives no contribution to T (at least when Maxwell's equations are fulfilled). For these two reasons, the invariant density T must be prefered to Schwarzschild's density T_4^4 as a true representation of the density.

Although Eddington insists chiefly on the objection we have spoken of, he indicates another objection which does not seem to have real foundations: The condition of fluidity is not expressed in natural measure ~~s~~ and so would also require modification.+ The condition of fluidity may be written

$$T_a^b = -g_a^b \, p \qquad (a,b,=1,2,3) \qquad (1)$$

where the pressure p is an invariant. It is true that this relation is not tensorial for any change of the coordinates, but only for a change of the three spatial coordinates x_1, x_2, x_3. If this relation is true for a system axes in which the time-axis is the proper time of the matter i;e. for axis with respect to which the matter is at rest, it will be true for any axis which fulfills this condition and therefore will be true in natural measures whatever they may be.

Schwarschild's solution really refers to a perfect fluid but the density of the fluid (defined as Eddington has shown it must be defined) is not uniform.

The two solutions (T or T_4^4 constant) do not differ very much for ordinary values of the pressure p.

+ That statement is corrected in the second edition of "The Mathematical Theory of Relativity".

We have

$$T = T^I_I + T^2_2 + T^3_3 + T^4_4 = T^4_4 - 3p \tag{2}$$

and the natural units used in this formula are such that when the densities are expressed in gramms per c.c., the pressure p represents its value in C;G.S. units divided by the square of the velocity of light. Therefore, the pressure is small in any practical case.

But for considerations involving the existence of a maximum radius for a given density, the central pressure becomes infinite in Schwarschild's solution; then the invariant density tends to minus infinity so that such a solution " ceases to correspond to a problem of any physical importance " .

" It is unfortunate that the solution breaks down for large spheres, because the existence of a limit to the size of the sphere is one of the most interesting objects of the research. "

In fact, we shall find that there is a maximum radius when the invariant density is uniform; this maximum is smaller (about two thirds) than the value obtained in Schwarzchild's hypothesis. But a fundamental difference arises: Scharzchild's maximum occurred when the central pressure tended to infinity, now the maximum sphere has a finite central pressure (equal to about half of the product of the density by the square of the velocity of light). The difficulty is quite more striking than in Schwarzchild's solution. We must confess that we do not see in what way this paradoxical result might be eluded or explained.

Before leaving these considerations on the convenient representation of the physical entities in relativity notations it will be useful to make the following remark: Incompressibility means that, when the pressure is changed, the matter included in a given boundary does not pass through this boundary so that the invariant density T which characterizes the matter does not change. In non-relativistic considerations, (and it happens to be the same in Schwarzchild's solution) it follows that the mass included in the boundary, defined as the product of the density by the volume, does not change. But according to the theory of relativity, the geometry is not euclidean and the curvature of space may be a function of the pressure, so that the included mass may vary although the density and the boundary remains unchanged. In relativity the matter is primarily defined by the energy-tensor and the mass is but a mathematical expression which has no immediate physical significance. When computing the mass of an incompressible fluid, we must indicate ~~at~~ what pressure it is referred to and it will be convenient to compute the mass reduced to zero pressure.

II - EQUATIONS OF THE FIELD

I) STATIC FIELD WITH SPHERICAL SYMMETRY

We first recall the general properties of a static field with spherical symmetry, following Eddington's notations.

The element of interval may be written

$$ds^2 = -e^{\lambda} dr^2 - e^{\mu}(d\theta^2 + \sin^2\theta \, d\phi^2) + e^{\nu} dt^2$$

where λ, μ and ν are functions of r only.

By a change of the coordinate r, this expression may be reduced to

$$ds^2 = -e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) + e^{\nu} dt^2 \qquad (3)$$

A choice of coordinates cannot restrain the generality of the solution of tensorial equations as the nature of the tensors enables us to compute, from a solution in particular coordinates, the solution for a general change of the coordinates.

The coordinate r has now a definite physical meaning : It is not the distance from the center; but it is the radius of an euclidean sphere which has the same area as the sphere on which lie the points of coordinate r.

The general equation of a gravitational field is

$$G_a^b - \frac{1}{2} g_a^b G = -8\pi T_a^b - L g_a^b \qquad (4)$$

in which G_a^b and G are the Riemannian tensors, T_a^b the energy tensor of the matter, and L (ordinarly written λ) the cosmological constant.

The first member

$$S_a^b \equiv G_a^b - \frac{1}{2} g_a^b G$$

is a function of λ, ν and r. The non-vanishing components of this tensor are, for the actual symmetry:

Eddington, l.c. p. 169

$$S_1^1 = e^{-\lambda}(\nu'/r + 1/r^2) - 1/r^2 \tag{6}$$

$$S_2^2 = S_3^3 = e^{-\lambda}(\nu''/2 - \lambda'\nu'/4 + \nu'^2/4 + \nu'/2r - \lambda'/2r) \tag{7}$$

$$S_4^4 = e^{-\lambda}(-\lambda'/r + 1/r^2) - 1/r^2 \tag{8}$$

The third one (8) is a linear equation in $e^{-\lambda}$ whose solution is

$$e^{-\lambda} = 1 + \frac{1}{r}\int S_4^4\, r^2\, dr \tag{9}$$

The first one (6) gives ν'

$$\nu' = r\, e^{\lambda}\left(S_1^1 + \frac{1}{r^2}\right) - \frac{1}{r} \tag{10}$$

The three equations are connected by an identity, the divergence equation, $S_{ab}^{b} = 0$, which reduces to

$$\frac{dS_1^1}{dr} + \frac{2}{r}(S_1^1 - S_2^2) + \frac{\nu'}{2}(S_1^1 - S_4^4) = 0 \tag{11}$$

This may be computed by replacing in the general expression of the divergence, the Christoffel's symbols by their particular form,

Eddington, l.c. p. 84

or by direct verification from equations (6), (7) and (8).

The divergence equation expresses the relation which must be fulfilled by the density and the stresses in order that the matter remains in equilibrium.

By elimination of λ and ν, the condition of equilibrium becomes

$$\frac{dS_1^1}{dr} + \frac{2}{r}(S_1^1 - S_2^2) + \frac{r\,(S_1^1 - S_4^4)\left(S_1^1 - \frac{1}{r^3}\int S_4^4\, r^2 dr\right)}{2\left(1 + \frac{1}{r}\int S_4^4\, r^2\, dr\right)} = 0 \tag{12}$$

The constant must be taken the same in the two indefinite integrands.

These equations give the solution of the problem, when two of the components (for instance S_1^1 and S_4^4) are given functions of r.

These functions are not necessarely continuous; they may have isolated points of discontinuity. Then λ and ν remain continuous but the derivative of λ is discontinuous with S_4^4 that of ν with S_I^I . We will have occasion to make use of this property that ν is continuous when the stresses S_I^I are continuous. It is the case at the boundary of the sphere where the density falls suddenly to zero but where the pressure is zero inside as well ~~than~~ as outside of the sphere. The meaning of the continuity of ν' is that a free point at rest at the boundary of the sphere undergoes the same acceleration if we consider it as being inside or outside of the sphere. The discontinuity of λ' at a point of discontinuity of the matter may be interpreted by saying that the area of the surface of a sphere is a function of the distance to the center which has a discontinuous derivative at any point where the density of the matter changes suddenly.

The components of the material tensor are not generally given functions of r but they must fulfil some relations, for instance to be a perfect fluid/: $S_I^I = S_2^2$ and to be incompressible or to be a perfect gas of uniform temperature Etc.

2) FLUID IN EQUILIBRIUM.

In the case of a fluid of pressure p, Schwarzschild's density ρ and invariant density d, the general equation (II) becomes

$$S_I^I = S_2^2 = S_3^3 = 8\pi\, p - L \qquad (I3)$$

$$S_4^4 = -8\pi\, \rho - L \qquad (I4)$$

$$S = -8\pi\, d - 4L \qquad (I5)$$

From these equations; we see that it is always sufficient to solve the equations when the cosmological constant vanishes The effect of introducing a cosmological constant L is clearly to increase the pressure by $l=L/8\pi$ and to decrease Schwarzschild's density and the invariant density respectively by l and 4l. In other words, to obtain the equation with a cosmological constant we have but to replace p, ρ and d respectively by p-l, ρ+l and d+4l.

For L=0, equations (9), (II) and (I2) reduce to

$$e^{-\lambda} = 1 - \frac{8\pi}{r}\int \rho\, r^2\, dr \qquad (I6)$$

$$\frac{d\nu}{dr} = -\frac{2}{p+\rho}\,\frac{dp}{dr} \qquad (I7)$$

$$\frac{dp}{dr} + \frac{4\pi r\,(p+\rho)\left(p + \frac{1}{r^3}\int \rho\, r^2\, dr\right)}{1 - \frac{8\pi}{r}\int \rho\, r^2\, dr} = 0 \qquad (I8)$$

When the nature of the fluid is defined by a relation between pressure and density, these quantities may be computed by solving the equation (I8) and then λ and ν are computed by quadratures from (I6) and (I7).

M. Brillouin (C.R.-I74-I922-p.I585) found similar equations by starting from the Schwarzschild solution applied to successive shells of different density. Then supposing that the number of shells increases indefinitely and passing to the limit, he deduces the equations for a continuous variation of density from the solution when the density is a discontinuous step-function. He introduces some auxiliary functions which do not simplify the question very much.
8888

The indefinite integrand contains an arbitrary constant which may be determined by the value of $e^{-\lambda}$ for a given value of r. When the center r=0 is inside the matter, it is determined by the requirement that $e^{-\lambda}$ remains finite at the center; as (16) may be written

$$e^{-\lambda} = 1 - \frac{2m}{r} - \frac{8\pi}{r}\int_0^r \rho\, r^2\, dr$$

where 2m is the constant in question. In the case of a fluid without a nucleus this constant must vanish and the limits of the integrands in (16) and (17) are 0 and r.

3) SCHARZSCHILD'S SOLUTION

When Schwarzschild's density ρ is a constant, the equations (16)(17) and (18) are immediately integrated:

$$e^{-\lambda} = 1 - \frac{8\pi}{3}\rho\, r^2$$

$$e^{\nu} = \frac{C_1}{(p+\rho)^2}$$

$$\frac{dp}{(p+\rho)(p+\rho/3)} = \frac{-4\pi r\,dr}{1 - 8\pi\rho\, r^2/3}$$

or

$$\frac{p+\rho/3}{p+\rho} = C_2\sqrt{1 - 8\pi\rho\, r^2/3}$$

where C_1 and C_2 are two integration constants.

C_1 is immaterial, as it may be absorbed by a change of the unit of time. C_2 may be expressed in terms of the radius of the sphere i.e. the value a of r at which the pressure vanishes. We have

$$3\, C_2\sqrt{1 - 8\pi\rho\, a^2/3} = 1$$

or, when a cosmological constant $L = 8\pi\, l$ is introduced (which has the effect of decreasing p and increasing ρ of the same amount l in the equations);

$$3\rho\, C_2\sqrt{1 - 8\pi(\rho+l)\, a^2/3} = \rho - 2l$$

The pressure must remain finite, and therefore

$$C_2\sqrt{1 - 8\pi(\rho+l)\, r^2/3}$$

must remain smaller than 1 . This condition will be fulfilled

(IO)

everywhere if it is fulfilled at the center $r=0$ i.e. if C_2 is smaller than I or if

$$3\rho\sqrt{I - 8\pi(\rho+l)\,a^2/3} > \rho - 2l$$

We have therefore

$$8\pi(\rho+l)\,a^2 < (8\rho^2 + 4\rho l - 4l^2)/3\rho^2 = 4(2\rho-I)(\rho+l)/3\rho^2$$

or

$$8\pi\rho\,a^2 < 4(2\rho-I)/3\rho \qquad (24)$$

For Einstein's cylindrical universe, which is a special case of Schwarzschild's solution $l = \rho/2$ and the second member of (24) is equal to 2 as it must be. For l smaller than $\rho/2$, C_2 is negative and there is no maximum ; this case does not refer to the problem of the sphere but to that of a condensation of matter at the horizon or absolute of the center. It might be described as the problem of the homogeneous "wall" when the matter fills up the space comprised between the two surfaces equidistant to a plane. This problem is of spherical symmetry, but we do not intend to deal with it in this paper.

4) UNIFORM INVARIANT DENSITY.

For a uniform invariant density and a vanishing cosmological constant, we must replace ρ by its value $d + 3p$ in which d is now a constant.

Equations (I6), (I7) and (I8) become

$$e^{-\lambda} = I - \frac{8\pi}{r}\int_0^r (d + 3p)\,r^2\,dr \qquad (25)$$

$$\frac{d\nu}{dr} = -\frac{2}{4p + d}\,\frac{dp}{dr} \qquad (26)$$

$$\frac{dp}{dr} + \frac{4\pi r\,(4p+d)\left(p + \frac{I}{r^3}\int_0^r (d+3p)\,r^2\,dr\right)}{I - \frac{8\pi}{r}\int_0^r (d+3p)\,r^2\,dr} = 0 \qquad (27)$$

The second one can be integrated

$$e^{\nu} = C^t\,(4p + d)^{-I/2} \qquad (28)$$

and introducing the new variable

$$q = \frac{3}{r^3}\int_0^r p\ r^2\ dr \tag{29}$$

(25) becomes

$$e^{-\lambda} = I - 8\pi\ \left(\frac{d}{3} + q\right) r^2 \tag{30}$$

and (27)

$$\frac{dp}{dr} + 4\pi r\ \frac{(p+q+d/3)(4p+d)}{I - 8\pi\ (q+d/3)\, r^2} = 0 \tag{3I}$$

The definition of q may be written

$$\frac{dq}{dr} + 3\ \frac{q-p}{r} = 0 \tag{32}$$

with p=q for r=0.

The problem of the field of a sphere of uniform invariant density is so reduced to find only a solution of these two linear equations between the two functions p and q of r.

They may be standardised by the substitution

$$d = I2\ u,\ p = ux,\ q = uy\ ,\ 8\pi\ r^2 = t/u \tag{33}$$

The parameter u disappears from the equations which become

$$\frac{dx}{dt} + \frac{(x+y+4)(x+3)}{I - (y+4)t} = 0 \tag{34}$$

$$\frac{dy}{dt} + \frac{3}{2}\ \frac{y-x}{t} = 0 \tag{35}$$

When a change of the parameter u is adopted, density and pressure are multiplied by the same amount u and the distances (r) are divided by the square root of u.

The standardised equations give a solution (for u=I) in which the density is represented by twelve; x is the pressure and t is the double of the area of the sphere on which the points of coordinate t lie.

y is a kind of mean pressure in the interval (0,t) defined by the equation corresponding to (29)

$$y = \frac{3}{2} t^{-\frac{3}{2}} \int_0^t x \, t^{\frac{1}{2}} \, dt \qquad (36)$$

III - DISCUSSION OF THE EQUATIONS

I) SPECIAL SOLUTIONS

There are two solutions of the equations (34) and (35) for which x and y are constant throughout the field : $x = y = -2$ and $x = y = -3$;

A negative pressure has no physical sense; but when a cosmological constant is introduced these solutions have a very simple meaning. In that case the equations of standardisation (33) must be replaced by

$$d + 4l = 12\,u, \; p - l = ux, \; q - l = uy, \; 8\pi\, r^2 = t/u \qquad (33')$$

The solution $x = -2$ may be considered as representing a vanishing pressure, with a cosmological constant $l = 2\,u$ or $L = 16\,\pi\,u$. The corresponding density will be $d = 4u$, therefore $8\pi d = 2L$ which shows that this solution is Einstein's cylindrical Universe.

The solution $x = -3$, for $l = 3\,u$, gives similarly $p = 0$ and $d = 0$ and is therefore de Sitter's Universe.

2) EXISTENCE OF INTEGRANDS

From the general theorem on the existence of solutions in a system of differential equations in the normal form, it is clear that a solution of equations (34) and (35), and only one , is generally defined by arbitrary values of x and y at a given point t. Exceptions can occur only when $t=0$ or when $I-(y+4)t = 0$, as the ordinary existence test fails in these cases. We have therefore to discuss the equations for these two particular points.

A - CENTER OF THE SPHERE ($t=0$) (see Note at the end)

The theory of integrands fundamentally rests on the following point: Let us consider two approximate solutions x_I, y_I and x_2, y_2. We can deduce from every one of them new approximate solutions X_I, Y_I and X_2, Y_2 replacing x and y by the functions of t, x_I, y_I or x_2, y_2 in the expression of dx/dt and dy/dt and integrating. It is required that, when x_I and y_I tends uniformly to x_2 and y_2 in an interval, the new functions X_I and Y_I tend uniformly to X_2 and Y_2 in the same interval.

Now it is clear, from (36), that $Y_2 - Y_I$ will be smaller (in absolute value) than $x_2 - x_I$ (at least if x_I and x_2 have no extremum in this interval) so that the requirement will be fulfilled for the equation in dy/dt. It will be fulfilled also for the equation in dx/dt as the general test is applicable to this equation.

Therefore $t=0$ is not a critical point of the differential equations, a solution and only one is defined by the value of x equal to that of y at the initial value, it may be developed in power series of t and is a continuous function of the initial value.

B - HORIZON OF THE CENTER $e^{-\lambda} = I - (y+4)t = 0$

a) <u>Every solution of initial value greater than -3</u> (the only one of actual interest) <u>reaches the critical point at the horizon of the center</u>.

$I - (y+4)t = 0$ represents an equilateral hyperbola of asymptotes $t=0$ and $y= -4$. The y curve starting with a finite value of y at t=0 will certainly cross the hyperbola if y remains greater than -3. Now, from the equation (34) in dx/dt

$$\frac{d}{dt} \log (x+3) = - \frac{x+y+4}{I - (y+4)t} \qquad (37)$$

x can only become smaller than -3 if the denominator vanishes i.e. if y crosses the hyperbola. On the other hand, from the integrand form of the second equation (36), y is always greater than the maximum of x in the interval (0,t). Therefore y will certainly cross the hyperbola before crossing the line -3.

b) <u>When y tends to the hyperbola for the first time, x+3 does not vanish</u>.

From (37) it is clear that when x+3 vanishes y must tend to the hyperbola. Therefore x+3 would vanish for the first time and d log(x+3) would be negative. As $I - (y+4)t$ is positive $x+y+4$ must be positive also. When y+4 tends to I/t and x to -3, x+y+4 tends to -3+I/t and t must be smaller than I/3.

On the other hand, as y approaches the hyperbola for the first time, the derivative on the y curve must be greater than the derivative along the hyperbola; the first one is computed from (35) and turns out to be $-3(I-t)/2t^2$; the second one is obtained directly from the equation of the hyperbola and is $-I/t^2$. The condition is therefore

$$-3(I-t)/2 > -I$$

and t must be greater than I/3.

As t cannot be together greater and smaller than I/3, it follows that x+3 does not vanish and is positive at the critical point. Exception can but occur when $t = 1/3$; in that case x+y+4 vanishes.

c) <u>x cannot tend to infinity at the critical point</u>.

Let us write Y and T for the finite values of y and t at the critical point and introduce new variables η and τ by the substitution

$$y = Y + \eta , \qquad t = T - \tau^2$$

Equations (34) and (35) become

$$\frac{dx}{d\tau} = \frac{2\,\tau(x+3)(x+Y+4+\eta)}{-\eta T + (Y+4)\tau^2}$$

and

$$\frac{d\eta}{d\tau} = \frac{3\tau}{T-\tau^2}\,(Y+\eta-x)$$

If x would tend to infinity, they would reduce in the ~~neighbourhood~~ neighbourhood of the critical point to

$$\frac{dx}{d\tau} = \frac{2\tau x^2}{-\eta T + (Y+4)\tau^2}$$

and

$$\frac{d\eta}{d\tau} = -\frac{3\,x\tau}{T}$$

From the last equation, it is clear that $(Y+4)\tau^2$ is negligable with regards to ηT when x tends to infinity. By dividing the two equations, we obtain

$$\frac{dx}{x} = \frac{2}{3}\,\frac{d\eta}{\eta}$$

or

$$x = C^t\,\eta^{\frac{2}{3}}$$

from which it follows that x would tend to zero and not to infinity when η tends to zero. Then we can introduce a new variable ξ by the substitution

$$x = X + \xi$$

where X is the finite value of x at the critical point.

The equations become

$$\frac{d\xi}{d\tau} = \frac{2\,\tau(X+3+\xi)(X+Y+4+\xi+\eta)}{-\eta T + (Y+4)\tau^2}$$

$$\frac{d\eta}{d\tau} = \frac{3\tau}{T-\tau^2}\,(Y-X+\eta-\xi)$$

d) x+y+4 vanishes at the crtical point.

If x+y+4 would not vanish at the critical point, it would tend to a finite value A, and X would tend to A-(Y+4) Then $d\eta/d\tau$ would tend to

$$3\,\tau\,\frac{2Y+4-A}{T}$$

and η to

$$\frac{3}{2}\,\frac{2Y+4-A}{T}\,\tau^2$$

Then

$$\frac{d\xi}{d\tau} = \frac{2\,\tau\,(A-Y-I)A}{\left[-\frac{3}{2}\,(2Y+4-A)+Y+4\right]\tau^2} = \frac{2\,(A-Y-I)A}{(-2Y-2+3A/2)\,\tau}$$

and ξ and therefore x would be infinite of the order $\log\tau$, which is impossible from 3).

Therefore X+Y+4 vanishes and we have at the critical point XT+I=0 . The x curves end on another hyperbola, symmetrical of the hyperbola whereon the y curves finish, with regard to the line x or y = -2.

e) x is a power series of $\tau = \sqrt{T - t}$

In the neighbourhood of the critical point,

$$\eta = \frac{3\,(Y+2)}{T}\,\tau^2 \quad \text{and} \quad \frac{d\xi}{d\tau} = \frac{\xi}{\tau}$$

therefore ξ ~~ξ~~ tends to $a_I\tau$, where a_I is an arbitrary constant.

We can write

$$\begin{aligned} x &= X + a_I\tau + a_2\tau^2 + a_3\,\tau^3 + \dots \\ y &= Y + \frac{3}{2}\,\frac{Y-X}{T}\,\tau^2 + b_3\,\tau^3 + \dots \end{aligned} \qquad (38)$$

This explains the nature of the singularity of the critical point and shows that there is an infinity of solutions,(for every value of a_I) with the same initial values X and Y at the critical point T.

Ih other words, a central condensation in a Universe is not determined, for a given cosmological constant, by the value of the pressure at the horizon of the center.

C - INFINITE CENTRAL PRESSURE

Before leaving this discussion of the equations, we must deal with the case where the solution is defined by the condition that the pressure is infinite at the center, t=0.

If we suppose that the center is a pole, we easily see from the equations that this pole must be of order one; and the solution is of the form

$$\begin{aligned} x &= \frac{I}{7t} + x_0 + x_I t + x_2 t^2 + \dots \\ y &= \frac{3}{7t} + y_0 + y_I t + y_2 t^2 + \dots \end{aligned} \tag{39}$$

where the coefficients may be actually determined.

IV - NUMERICAL COMPUTATIONS

I) PURPOSE OF THESE COMPUTATIONS

The x curves represent the pressure if there is no cosmological constant; when a cosmological constant is introduced, they represent the pressure reduced by $l=L/8\pi$; as we have seen in the above discussion, they join a point of the line $t=0$ to a point of the hyperbola $I+xt=0$ of abcissa greater than $I/3$. It follows from this fact that there is a locus of maxima of x (and also of minima) starting from a point of the arc of hyperbola and assymptotic to the line $t=0$. It might be that this locus of maxima would be the x curve with infinite central pressure, as it is the case in Schwarzschild's solution, or that the x curves have an envelope which is this locus of maxima. Actual computations show that this second possibility really occurs (although the minimum curve is very probably the curve of infinite central pressure).

This envelope has the following physical interpretation: The boundary of the sphere is the points where the pressure vanishes. Then x or, reintroducing the standardisation coefficient u, ux is equal to -1, while the invariant density and the radius are given by $d+4l = I2\ u$ and $8\pi\ r^2 u = t$. The radius $r=a$ on the envelope for a negative value of x is therefore the radius of the maximum sphere for a cosmological constant $L = -8\pi\ x$.

A knowledge of the envelope for negative values of x enables us to compute a relation between a, d and l which corresponds to the relation (24) we have found in the Schwarzschild problem.

The actual computations have been undertaken for this purpose. As a by-product some information has been obtained on the variations of the pressure for different central pressure, on the envelope of the y curves and on the minimum curve of x.

2) Method and results of computation

In order to build up a table of the envelope of the x curves, two x curves have been computed, for initial values 0 and 5. The variations on these two curves for an infinitesimal change of the initial value have been calculated as well as on the special solution $x = -2$. The points of the x curves where these variations vahish are points of the envelope.

Curves representing, for a given value of t, x as a function of its initial value x_0 are drawn from the three points which are known (for $x_0 = -2$, 0 and 5) and the corresponding tangents; ~~and~~ the maximum of these curves gives a point of the envelope with the corresponding central pressure.

The x curve for infinite central pressure has been computed in order to be sure that the envelope is really a locus of <u>absolute</u> maxima and that the curves of big central pressures do not pass above it. The computation seems to indicate that this curve is rather a locus of minimum.

Computations of the x and y curves have been done as ~~s~~ follows: We start with a Taylor developement in power series of t with the initial value of $x=y$. Then the curves are produced by graphical integration. The first and second derivatives of both variables are computed for values of t equal to 0.05, 0.I0, 0.I5 etc.; then using Euler's formula which gives the increment of a function in an interval when the two first derivatives are known at both ends of the interval, the graphical solution is checked and corrected by differential correction and the variations of x and y are computed for an infinitesimal variation dx=dy at the origin.

The results of the computations are given in the following tables and illustrated in the diagram. The curves for initial values -2, 0, 5 and ∞ are computed directly and they are represented in reinforced lines; the other curves are obtained by interpolation

Table I

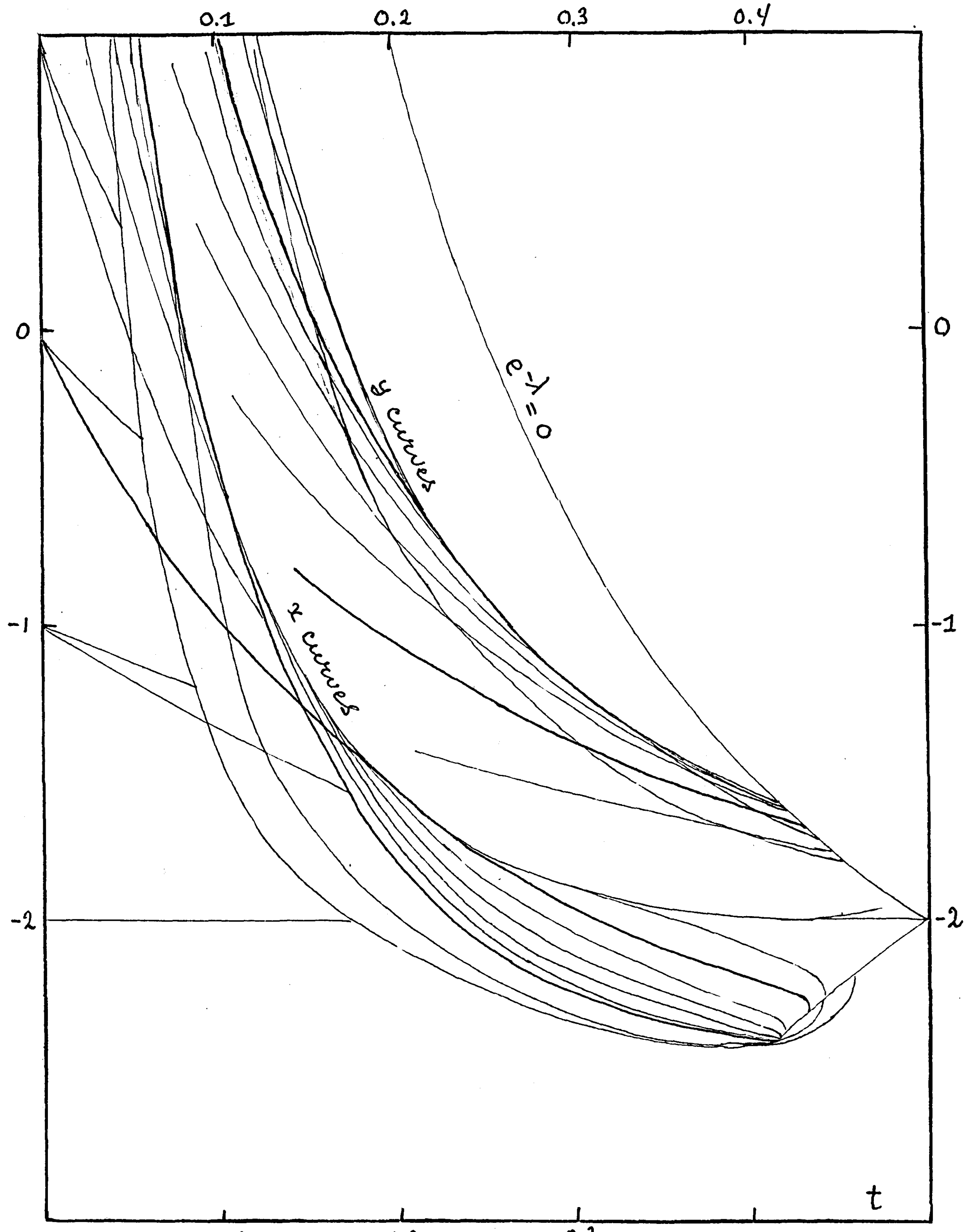
x curves

t										
0.00	-2	-I	0	I	2	3	4	5	IO	∞
0.05	-2	-I.20	-0.54	0.00	0.44	0.80	I.08	I.24	I.4	0.0I
0.I0	-2	-I.36	-0.96	-0.72	-0.58	-0.50	-0.46	-0.46	-0.80	-I.42
0.I5	-2	-I.50	-I.29	-I.20	-I.I6	-I.I7	-I.23	-I.32	-I.68	-I.87
0.20	-2	-I.64	-I.56	-I.57	-I.62	-I.68	-I.74	-I.79	-I.98	-2.09
0.25	-2	-I.80	-I.78	-I.85	-I.92	-I.98	-2.03	-2.08	-2.I8	-2.2I
0.30	-2	-I.90	-I.96	-2.04	-2.I2	-2.I8	-2.22	-2.24	-2.28	-2.30
0.35	-2	-I.99	-2.I0	-2.I9	-2.24	-2.28	-2.30	-2.3I	-2.32	-2.32

y curves

t										
0.00	-2	-I	0	I	2	3	4	5	IO	∞
0.05	-2	-I.I2	-0.33	0.32	0.9I	I.38	2.04	2.48		5.72
0.I0	-2	-I.25	-0.6I	-0.I5	0.24	0.56	0.82	I.03	I.58	I.44
0.I5	-2	-I.34	-0.85	-0.52	-0.28	-0.I0	0.03	0.I3	0.3I	0.I8
0.20	-2	-I.40	-I.05	-0.82	-0.68	-0.58	-0.52	-0.47	-0.4I	-0.69
0.25	--2	-I.50	-I.23	-I.08	-I.00	-0.94	-0.90	-0.90	-0.90	-I.I0
0.30	-2	-I.56	-I.38	-I.28	-I.23	-I.20	-I.I9	-I.I9	-I.20	-I.38
0.35	-2	-I.65	-I.52	-I.47	-I.44	-I.42	-I.42	-I.42	-I.44	-I.60
0.40	-2	-I.73	-I.64	-I.60	-I.58	-I.57	-I.57	-I.58	-I.63	-I.7I

fig. 1



The results for the envelopes are

Table II
x envelope (maximum)

t	x	x_0	y
0.05	I.4	11.0	
0.I0	-0.45	4.5	0.94
0.I5	-I.I7	2.3	-0.22
0.20	-I.56	0;5	-0.92
0.25	-I.77	-0.4	-I.32
0.30	-I.90	-I.0	-I.67
0.35	-I.96	-I.4	-I.80
0.40	-2.00	-2.0	-2.00

table III
y envelope (maximum)

t	y	y_0	x
0;I0	I.66	I3.8	-I.I0
0.I5	0.32	II.3	-I.74
0.20	-0.40	9.3	-I.96
0.25	-0.88	7.4	-2.I4
0.30	-I.20	5.8	-2.26
0.35	-I.57	3.5	-2.30

table IV
minimum curves ($P_0 = \infty$)

t	x	y
0;05	0.0I	5.72
0.I0	-I.42	I.44
0.I5	-I.87	0.I8
0.20	-2.09	-0.69
0.25	-2.2I	-I.I0
0.30	-2.30	-I.38
0.35	-2.32	-I.60
0.40		-I.7I

The ends of the curves (for $t > 0.35$) have not been computed but are drawn as an illustration of the nature of the critical point as it results from the above discussion.

V - INTERPRETATION OF THE RESULTS

The purpose of our computations was to find the relation between the invariant density d, the maximum radius a and the cosmological constant $L = 8\pi l$

When x and t are taken on the envelope, according to table II, we have from (33') (p.12)

$$ux = -1, \qquad 8\pi\, a^2 u = t, \qquad d + 4l = I2u$$

or, by elimination of the standardisation coëfficient u,

$$8\pi\, a^2 d = 4t\,(x+3) \tag{40}$$

and

$$\frac{l}{d} = -\frac{x}{4(x+3)} \tag{4I}$$

This enables us to compute $8\pi a^2 d$ for any value of $l/d = L/\,8\pi d$. The result is given in table V columns I and 2.

Table V
Maximum sphere

$\frac{L}{8\pi d}$	$8\pi a^2 d$	$\frac{2p_0}{d}$	$\frac{m}{\frac{4}{3}\pi a^3\ d}$	$\frac{V_0}{\frac{4}{3}\ \pi a^3}$	$\frac{a}{a_S}$	R^2L	$\frac{a_\infty}{a}$
0.00	I.00	I.00	I.40	I.04	0.6I	3.00	0.772
0.05	I.03	0.97	I.40	I.07	0.63	3.I3	0.763
0.I0	I.07	0.98	I.39	I.09	0.65	3.23	0.759
0.I5	I.I0	0.86	I.38	I.I2	0.67	3.33	0.75I
0.20	I.I3	0.80	I.36	I.I6	0.69	3.40	0.746
0.25	I.I6	0.73	I.33	I.20	0.7I	3.46	0.740
0.30	I.I9	0.56	I.30	I.24	0.73	3.50	0.73I
0.35	I.23	0.58	I.25	I.28	0.75	3.55	0.726
0.40	I.29	0.48	I.I9	I.33	0.78	3.59	0.72I
0.45	I.39	0.35	I.II	I.39	0.83	3.65	0.709
0.50	I.60	0.00	I.00	I.49	0.90	3.73	0.657

Column 3 gives the central pressure p_o. It is computed from the values of x_o (table II) by the formula

$$\frac{2\,p_o}{d} = \frac{x_o - x}{2(x+3)} \tag{42}$$

This is written in natural units. In arbitrary units the heading of the first three column must be read

$$\frac{L}{\varkappa d}, \quad , \quad \varkappa a^2 d \;, \quad \frac{2\,p_o}{c^2\,d}$$

where $\varkappa$ is Einstein constant equal to I.87 10^{-27} in C.G.S. units.

Column 4 gives the ~~central pressure~~ apparent mass of the sphere as it must be deduced from the gravitational field outside of the sphere. It is the coefficient m in the expression

$$e^{-\lambda} = I - \frac{2m}{r} - \frac{L}{3}\,r^2 \tag{43}$$

m is computed by the formula

$$\frac{m}{\frac{4}{3}\pi a^3\,d} = \frac{3(y+4)\,t}{8\pi a^2 d} - \frac{1}{d} \tag{44}$$

$\frac{4}{3}\pi a^3$ is not the volume of the sphere, as the space is not euclidean; column 5 gives the ratio of the real volume to the euclidean volume. The real volume is reduced to zero pressure according to the remark we have done in the introduction (P. 4)

It is computed as follows:

$$V_o = \int_o^r 4\pi\ r^2\ e^{\frac{\lambda}{2}}\ dr$$

and when the pressure is supposed to vanish

$$e^{-\lambda} = I - \frac{8\pi}{3}\,(d+1)\ r^2$$

The integration gives

$$\frac{V_o}{\frac{4}{3}\,\pi\ a^3} = \frac{\frac{9\pi\sqrt{3}}{4}\left(\frac{\chi}{\pi/2} - \frac{\sin 2\chi}{\pi}\right)}{(8\pi a^2 d)^{3/2}\ (I+1/d)^{3/2}} \tag{45}$$

with

$$\sin^2 \chi = \frac{4\ (I+\ell/d)t}{I + 4l/d} \qquad (46)$$

Column 6 gives the ratio of the maximum radius a for an uniform invariant density to the maximum radius a_S for an uniform Schwarzschild's density. The latter is computed from (24) where ρ is replaced by d.

If we suppose that there is no matter outside of the sphere, the radius R of the space will satisfy the equation $e^{-\lambda}=0$ or, according to (43)

$$I - \frac{2m}{R} - \frac{L}{3}\ R^2 = 0$$

Column (7) gives the value of $R^2 L$. It is a root of the cubic

$$(LR^2)^{3/2} - 3(LR^2)^{I/2} - \frac{m}{\frac{4}{3}\pi a^2 d}\left(\frac{1}{d}\right)^{I/2}(8\pi a^2 d)^{3/2} = 0 \qquad (47)$$

$LR^2 = 3$ is the value for an empty space (de Sitter's Universe) It is remarkable that the introduction of a material sphere increases the radius of Universe (at least in the case of a maximum sphere). This is rather astonishing as in the homogeneous space full of matter (Einstein's Universe) the radius is smaller; we have indeed in this case: $LR^2 = I$.

The relations

$$8\pi\ a^2 d = I, \quad p_0 = \frac{I}{2}\ dc^2$$

obtained for L=0, are obtained by numerical computation and we have no reason to believe that they are theoretically exact. The same is true for

$$m = \frac{4}{3}\pi a^3 d$$

obtained for $L/8\pi d = 0.50$.

When the central pressure increases, the radius of the sphere begins to increase, passes through a maximum a and then decreases to tend endly to the value a_∞ it takes for an infinite pressure. Column 8 gives the ratio of these two radii. It is computed from measures of t on the diagram (fig.I) for the value of x corresponding to $L/8\pi d$.

In Schwarschild's solution no sphere exists for which $L/8\pi d$ is greater than I/2, i.e. no sphere exists of a density smaller than the density of an Einstein's Universe of the same cosmological constant.

For an uniform invariant density, $L/8\pi d$ may be greater than I/2. In that case, if the x curves have no minimum (which is the case when $x_0 < 3$) the sphere may fill up the whole space. The maximum is then given by the equation of the critical point

$$I + xt = 0$$

or

$$a^2 L = I$$

This holds until the curves have a minimum. This happens at about x=-2.4 corresponding to $L/8\pi d$ = I.

Our computations are not accurate enough to decide if the minimum of the curves (for which x_0 is greater than about 3) occurs for values of x smaller than -2.4.

We can sum up our results as follows:

<u>When the density is greater than that of an Einstein's Universe , the radius of a sphere of uniform invariant density reaches a maximum for a finite value of the central pressure; this maximum is smaller</u> (6 to 9/IO) <u>than the value found by Schwarschild. If the pressure is supposed to increase further on,</u>

the radius diminishes to 7 or 8/10 of its maximum value until the pressure tends to infinity. The radius of the empty space which lies outside the maximum sphere is greater than that of an empty de Sitter's universe of the same cosmological constant.

Contrary to what happens in Schwarzschild's solution spheres may exist with a density smaller than that of an Einstein's Universe of the same cosmological constant, but not smaller than about half of this density. They may fill up the whole space; in that case the radius of the space is the same as that of an Einstein's Universe of the same cosmological constant.

For higher central pressure than about $p=2dc^2$, the density may be yet increased and then a maximum radius occurs again with a vanishing gradient of pressure at the boundary and free space outside of the sphere.

If matter is in the neighbourhood of a sphere with vanishing gradient of pressure at the boundary, it will not be attracted by the sphere as from (26) e^{ν} (the double of the Newtonian potential) is constant and we have seen (p.7) that de^{ν}/dr is continuous at the boundary.

A maximum radius with a non-vanishing gradient of pressure at the boundary really means that matter cannot exist outside of the sphere as, if it would exist, it could be brought in the neighbourhood of the boundary and then would be attracted and would increase the maximum sphere, which is impossible.

The solution for an invariant density (for which the central pressure is finite when the radius is maximum) excludes such speculations as were suggested by Schwarzschild's solution (for which the central pressure is infinite) on the occurence of a "catastrophe" ,if matter would be added to the maximum sphere.

Th. de Donder - Catastrophe dans le champ de Schwarzschild: Premiers compléments de la Gravifique einsteinienne -complément 3; Annales de l' Observatoire royal de Belgique, 3^{e} série tome I. or Gauthier Villars 1922.

It is a pleasure for me to express my thanks for the kind assistance I received from Professor Paul Heymans and Dr. Vallarta of the Massachusetts Institute of Technology in the course of this work.

I am also very much indebted to Professor Eddington who directed my attention on the problem of the sphere with uniform invariant density and gave me valuable informations as to the manner of dealing with the numerical solution of differential equations.

APPENDIX

DETAILS OF THE COMPUTATIONS.

I) TAYLOR'S DEVELOPEMENTS

By derivation of the equations

$$x'(I-yt-4t)+(x+y+4)(x+3) = 0 \qquad (48)$$

and

$$2y't+3(y-x)=0 \qquad (49)$$

we obtain

$$x^{(n+1)}(1-yt-4t)+\left[\left(2-\frac{3n}{2}\right)x+\left(1+\frac{n}{2}\right)y+7-4n\right]x^{(n)}+(x+3)y^{(n)} \qquad (50)$$
$$+\sum_{i=1}^{n-1}\frac{n!}{i!(n-i)!}\frac{x^{(i)}}{2(n-i+1)}\left[(2n-5i+2)x^{(n-i)}+(2n-i+2)y^{(n-i)}\right]=0$$

and

$$2t\,y^{(n+1)}+(2n+3)\,y^{(n)}-3x^{(n)}=0 \qquad (51)$$

For t=0 they reduce to

$$x^{(n+1)}+\frac{1}{3}\left[(-2n^2+3n+12)x+(-8n^2+2n+30)\right]y^{(n)}$$
$$+\frac{1}{3}\sum_{i=1}^{n-1}\frac{n!}{i!(n-i)!}(2n-5i+6)\,x^{(i)}y^{(n-i)} \qquad (52)$$

and

$$y^{(n)}=\frac{3}{2n+3}\,x^{(n)} \qquad (53)$$

Let us suppose that, when n tends to infinity, the ratio $\frac{n x^{(n)}}{x^{(n+1)}}$ tends to a limit T. Then the series will converge for t smaller than T. From (52) and (53), we have

$$I-(y+4)T-y'T^2-\frac{y''}{2!}T^3-\frac{y'''}{3!}T^4-\ldots=0$$

or

$$1-(Y+4)T=0$$

where Y is the value of y for t=T. This shows that the Taylor developement converges until y reaches the critical point at the horizon of the center.

This proof depends on the hypothesis that the ratio

$n\, x^{(n)} / x^{(n+I)}$ tends to a definite limit, so that it concerns only the most common way in which the series may cease to converge. Further investigation of this point does not seem to be needed, as ~~practically~~ in practice we only compute a few terms of the developement and in any case it will be necessary to test the result by other means.

Table VI gives the terms of the developement for $x_0=5$ and $t=0.05$, As a check the values of the derivatives are computed by derivation of the series and from the differential equations

table VI

$x_0=5$ $t=0.05$

x	y	x't	y't
+5;00000	5.00000		
-5.60000	-3.36000	-5.60000	-3.36000
2.49200	I.06800	4.98400	2.I3600
-0.83I2	-0.27707	-2.49360	-0.83I20
0.2I997	0.05999	0.87989	0.23999
-0.04673	-0.0I078	-0.23364	-0.05392
0.00779	0.00I56	0.04673	0.00935
-0.00094	-0.000I6	-0.00655	- 0.00II6
0.00006	0.0000I	0.00043	0.00007
00.000006		0;00004	0.000007
		-0.0000I6	-0.000002
		-0.00000I	-0.000000I
x= I.24096	y= 2.48I55	-2.42268	-I.86088
		-x'=48;4536	-y'=37.2I76

Verification

y+4	= 6.48I55
(y+4)t	= 0.324077
I-(y+4)t	= 0.675923
x+3	= 4.24096
x+y+4	= I.24059

$$x' = -\frac{(x+3)(x+y+4)}{I-(y+4)t} = -48.4536$$

$$y' = -\frac{3}{2}\,\frac{y-x}{t} = -37.2I76$$

Table VII gives similarly the terms of the developement for $x_0=5$ and $t=0.I0$.

Equality of y' in the two ways of computing is but a numerical check which has nothing to do with the convergence of the developement. The convergence may be appreciated from the correspondance of the values of x' from the developement and from the differential equations

Table VII

$x_0=5$ $t=0.I0$

	x		y	x't	y't
	5.0000		5.0000		
	-II.2000		-6.7200	-II.2000	-6.7200
	9.9680		4.2720	I9.9360	8.5440
	-6.6495		-2.2I65	-I9.9485	-6.6495
	3.5I96		0.9599	I4.0784	3.8395
	-I.4953		-0.345I	-7.4765	-I.7254
	0.4985		0.0997	2.99I0	0.6980
	-0.II97		-0.02II	-0.8379	-0.I478
	0.0I42		0.0022	0.II36	0.0I79
	0.0028		0.0004	0.0248	0.0036
	-0.00I5		-0.0002	-0.0I63	-0.002I
	-0.000I		-0.00002	-0.00I5	-0.0002
	-0.0000I		-0.000002	-0.0002	-0.00002
x=	-0.4632	y=	I.03I3	-2.337I	-2.2420
				x' = -23.27I	y'=-22.420

Verification

y+4	= 5.03I3		
(y+4)t	= 0.503I3	x'	= -23.323
I-(y+4)t	= 0.49687		
x+3	= 2.5368	y'	= -22.42
x+y+4	= 4.568I		

Table IX

$x_0 = \infty$

	t=0.05	0.I0	0.I5	0.20	0.25	0.30
	2.857I	I.4286	0.9524	0.7I44	0.57I4	0.4762
	-2.857I	-2.857I	-2.857I	-2.857I	-2.857I	-2.857I
	0.0084	0.0I68	0.0252	0.0336	0.0420	0.0504
	0.0006	0.0022	0.0050	0.0089	0.0I39	0.020I
	0.00004	0.0003	0.00II	0.0026	0.005I	0.0088
		0.00005	0.0003	0.0008	0.0020	0.0032
			0.00006	0.0003	0.0008	0.0020
				0.000I	0.0004	0.00I0
				0.00003	0.000I5	0.0005
					0.00007	0.0003
x=	0.0090	-I.4092	-I.873I	-2.0964	-2.22I2	-2.2946
y=	5.7I96	I.4397	0.0I77	-0.6888	-I.I09I	-I.385I
-x't	0.2847	I.4060	0.9I24	0.6500	0.4702	0.3I45
	0.2847	I.406I	0.9I24	0.6505	0.4703	0.324I

Table VIII gives the terms of the developement of x for $x_0=0$ and for t= 0.05, 0.I0,...,0.35. The values of x't are computed from the developement and from the differential equation.

Table VIII
$x_0=0$

	t=0.05	0.I0	0.I5	0.20	0.25	0.30	0.35
	-0.6000	-I.2000	-I.8000	-2.4000	-3.0000	-3.6000	-4.2000
	0.0720	0.2880	0.6480	I.I520	I.8000	2.5920	3.5280
	-0.0063	-0.0507	-0.I7I3	-0.4060	-0.7928	-I.3700	-2.I756
	0.0004	0.0070	0.0355	0.II23	0.2743	0.5687	I.0534
	-0.00002	-0.0008	-0.0059	-0.0247	-0.0755	-0.I879	-0.4063
		0.00007	0.0007	0.0043	0.0I62	0.0486	0.I225
			-0.00007	-0.0005	-0.0025	-0.0089	-0.0264
				0.00003	0.0002	0.0008	0.0029
					0.00003	0.000DI	0.0005
						-0.00005	-0.0002
							0.0000
x=	-0.5339	-0.9564	-I.2929	-I.5626	-I.780I	-I.9566	-2II0I2
y=	-0.33II	-0.6II7	-0.8509	-I.0559	-I.2326	-I.3858	-I.5I93
-x't=	0.4734	0.75I6	0.9009	0.9660	0.9768	0.9546	0.9I32
	0.4734	0.75I6	0.9008	0.9659	0.9774	0.9540	0.9060

For an infinite central pressure, the developement is of the form

$$x = X + I/7t \qquad\qquad y = Y + 3/7t$$

where X and Y are analytical functions which fulfill the equations

$$X' \left(\frac{4}{7} - Yt - 4t \right) + (X+3)(X+Y+4) + \frac{5X+2Y+20}{7t} = 0$$

and

$$Y' + \frac{3}{2}\,\frac{Y-X}{t} = 0$$

At t = 0 we must have X = Y = -20/7, and the derivatives are computed by

$$\frac{3}{7}\,\frac{6n+77}{2n+5}X^{(n+1)} + [(-2n^2+3n+12)x+(-8n^2+2n+6]\,Y^{(n)}/3 + \sum_{i=1}^{n-1}\binom{n}{i}(2n-5i+6)X^{(i)}Y^{(n-i)}/3 \qquad (54)$$

and

$$Y^{(n)} = \frac{3}{2n+3}\,X^{(n)} \qquad (55)$$

the developement is given in table IX (p.30)

2) TAYLOR DEVELOPEMENT FOR AN INFINITESIMAL VARIATION OF THE INITIAL VALUE.

The cöë fficients are given by differentiation of the formulae(52) and (53). The results are given in the following tables.

Table X

$x_0=5$

t=0.05	0.10	0.09
I.000	I.000	I.000
-I.500	-3.000	-2.700
I.03I	4.I26	3.342
-0.467	-3.736	-2.724
0.I56	2.50I	I.640
-0.040	-I.287	-0.760
0.008	0.503	0.268
-0.000I	-0.I39	-0.066
	0.0I9	00008
	-0.002	0.0007
0.I88	-0.0II	0.009

Table XI

$x_0=0$

	t=0.05	0.I0	0.I5	0.20	0.25	0.30	0.35	0.40
	I.000	I.000	I.000	I.000	I.000	I.000	I.000	I.000
	-0.500	-I.000	-I.500	-2.000	-2.500	-3.000	-3.500	-4.000
	0.099	0.396	0.89I	I.584	2.475	3.584	4.85I	6.446
	-0.0I2	-0.097	-0.329	-0.780	-I.523	-2.630	-4.I80	-6.239
	0.00I	0.007	00.088	0.278	0.679	I.409	2.6I0	4.453
		-0.002	-0.0I8	-0.074	-0.225	-0.563	-I.2I7	-2.374
		0.0002	0.003	0.0I6	0.060	0.I78	0.450	I.002
			-0.0003	-0.002	-0.0II	-0.039	-0.II5	-0.294
				0.0002	0.00I	0.005	0.0I8	0.052
					0.0003	0.00I	0.006	0.0I9
						0.000I	0.0006	0.002
dx=	0.588	0.3I4	0.I35	0.022	-0.044	-0.076	+0.0068	-0.043
dy=	0.739	0.54I	0.393	0.280	0.I96	0.0I3	0.009	0.006

Table XII
$x_0=-2$

	t=0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	-0.100	-0.200	-0.300	-0.400	-0.500	-0.600	-0.700	-0.800
	-0.001	-0.004	-0.009	-0.016	-0.025	-0.036	-0.049	-0.064
		-0.0003	-0.001	-0.003	-0.005	-0.009	-0.015	-0.022
			-0.0002	-0.0006	-0.002	-0.003	-0.006	-0.010
				-0.0002	-0.0005	-0.0013	-0.003	-0.005
				~~-0.0002~~	-0.0002	-0.0006	-0.0015	-0.0032
						-0.0003	-0.0008	-0.0020
						-0.0001	-0.0004	-0.0013
							-0.0003	-0.0009
							-0.0001	-0.0004
								-0.0003
dx=	0.989	0.796	0.690	0.580	0.467	0.349	0.224	0.090
dy=	0.939	0.878	0.816	0.752	0.687	0.620	0.551	0.009

3) DIFFERENTIAL CORRECTIONS

For an interval t_0, t Euler's formula is †

$$x-x_0 = \frac{x'+x'_0}{2}(t-t_0) - \frac{x''-x''_0}{12}(t-t_0)^2 + \frac{x^V}{720}(t-t_0)^5 + h \qquad (56)$$

h is a residuum which would be zero if the values of would be exact.

We have similar equations in y with a residuum k.

The value of x^V may be estimated by forming a table of the second derivatives , and writing

$$\Delta^3 x'' = x^V (\Delta t)^5.$$

If we apply differential corrections dx, dy to x and y the corresponding variations of the derivatives will be of the form (for an interval $t-t_0$:1/20)

$$\begin{aligned} dx'/40 &= -a\,dx - b\,dy & dy'/40 &= p\,dx - p\,dy \\ dx''/4800 &= l\,dx + m\,dy & dy''/4800 &= -q\,dx + r\,dy \end{aligned} \qquad (57)$$

This correction applied to Euler's formula must absorb the residua h and k and we must have

† C de la Vallée Poussin - Cours d'Analyse Infinitésimale 2e edit. tome II p 376

Table XIII

$x_0=5$

t	0.05	0.I0	0.I5	0.20	0.25	0.30	0.35
x	I.24	-0.46	-I.32	-I.79	-2.07	-2.24	-2.35
y	2.48	I.03	0.I3	-0.47	-0.90	-I.2I	-I.42
$-x$	-I.2400	0.4600	I.3200	I.7900	2.0700	2.2400	2.3500
x_0	5.0000	I.2400	-0.46000	-I.3200	-I.7900	-2.0700	-2.2400
$x'/40$	-I.2I05	-0.5839	-0.3I00	-0.I790	-0.I064	-0.064I	-0.0386
$x_0'/40$	-2.8000	-I.2I05	-0.5839	-0.3I00	-0.I790	-0.I064	-0.064I
$-x''/4800$	-0.I58I	-0.0660	-0.0306	-0.0I56	-0.0085	-0.0052	-0.0044
$-x_0''/4800$	0.4I52	0.I58I	0.0660	0.0306	0.0I56	0.0085	0.0052
+R	-0.0047	-0.00I8	-0.0006	-0.0002	-0.000I	0.0000	0.0000
-h	0.00I9	-0.004I	0.0009	-0.0042	0.00I6	0.0028	0.008I
$-y$	-2.4800	-I.0300	-0.I300	0.4700	0.9000	I.2I00	I.4200
y_0	5.0000	2.4800	I.0300	0.I300	0.4700	0.9000	I.2I00
$y'/40$	-0.9300	-0.5588	-0.3625	-0.2477	-0.I755	-0.I287	-0.0995
$y_0'/40$	-I.6800	-0.9300	-0.5588	-0.3625	-0.2477	-0.I755	-0.I287
$-y''/4800$	-0.0850	-0.0434	-0.0246	-0.0I46	-0.0093	-0.0063	-0.0045
$-y_0''/4800$	0.I780	0.0850	0.0434	0.0246	0.0I46	0.0093	0.0063
+R	-0.00I0	-0.0006	-0.0002	0.000I	0.0000	0.0000	0.0000
-k	0.0020	0.0022	-0.0023	-0.0003	0.0I24	0.0088	-0.0I64
a	0.442	0.358	0.295	0.25I	0.2I8	0.20I	0.27
b	0.246	0.246	0.233	0.225	0.222	0.234	0.34
p	0.750	0.275	0.250	0.287	0.I50	0.I25	0.I0
l	0.02I	0.024	0.0I8	0.0I3	0.008	0.006	0.00
m	0.097	0.060	0.046	0.037	0.029	0.025	0.02
q	0.430	0.I23	0.059	0.035	0.023	0.0I7	0.0I
r	0.250	0.047	0.0I5	0.005	0.00I	-0.00I	0.00
I+a+l	I.463	I.382	I.3I3	I.264	I.226	I.207	I.27
b+m	0.343	0.306	0.279	0.262	0.25I	0.259	0.3I
-(p+q)	-I.I80	-0.498	-0.309	-0.222	-0.I73	-0.I4I	-0.II
I+p+r	2.000	I.422	I.265	I.I92	I.I5I	I.I24	I.II
$I-a_0+l_0$	0.579	0.666	0.72I	0.762	0.790	0.805	0.8I
$-(b_0-m_0)$	-0.I49	-0.I86	-0.I87	-0.I88	-0.I93	-0.209	-0.29
p_0-q_0	0.320	0.252	0.I9I	0.I52	0.I27	0.I08	0.09
$I-p_0+r_0$	0.500	0.689	0.765	0.8I8	0.85I	0.874	0.92

$$(I+a+l)\,dx + (b+m)\,dy = (I-a_0+l_0)\,dx_0 - (b_0-m_0)\,dy_0 - h$$
$$-(p+q)\,dx + (I+p+r)\,dy = (p_0-q_0)\,dx_0 + (I-p_0+r_0)\,dy_0 - k \qquad (58)$$

The values of x and y are found by graphical integration

The coëfficients are computed from the formulae

$$a = \frac{1}{40}\,\frac{2x+y+7}{1-4t-4t} \qquad p = \frac{3}{80t}$$

$$l = \frac{1}{40}\,\frac{x+3-x't}{1-4t-4t} \qquad q = \frac{1}{240t}(5p+3a)$$

$$r = \frac{1}{240t}(5p-3l) \qquad (59)$$

$$l = \frac{a}{120}\,\frac{2x+3-y't}{1-yt-4t} - \frac{1}{4800}\,\frac{2x'+y'}{1-yt-4t} - \frac{pl}{3}$$

$$m = \frac{b}{120}\,\frac{2x+3-y't}{1-yt-4t} + \frac{1}{4800}\,\frac{x''t}{1-yt-4t} + \frac{pl}{3}$$

The rests

$$R = \frac{1}{30}\,\frac{x^{(v)}dt^5}{4!} = \frac{1}{60}\,\frac{\Delta^3 x''}{4800}$$

are computed from the table of $x''/4800$ and $y''/4800$

t	x"/4800				-R
0.00					0.0078
					0.0047
0.05	0.I58I				
		-0.092I			0.00I8
0.I0	0.0660		0.0567		
		-0.0354		-0.0367	0.0006
0.I5	0.0306		0.0200		
		-0.0I50		-0.0I2I	0.0002
0.20	0.0I56		0.0079		
		-0.007I		-0.004I	0.000I
0.25	0.0085		0.0038		
		--0.0033		-0.00I3	
0.30	0.0052		0.0025		
		-0.0008			
0.35	0.0044				

t	y"/4800				-R
0.00					0.0047
					0.00I0
0.05	0.0850				0.00I0
		-0.04I6			0.0006
0.I0	0.0434		0.0228		[illegible]
		-0.0I88		-0.0I40	0.0002
0.I5	0.0246		0.0088		
		-0.0I00		-0.004I	0.000I
0.20	0.0I46		0.0047		
		-0.0053		-0.0024	
0.25	0.0093		0.0023		
		-0.0030		-0.00II	
0.30	0.0063		0.00I2		
		-0.00I8			
0.35	0.0045				

Formula (58) applied with the values of h and k found in table XIII gives the differential corrections. When it is applied with h=k=0 and $dx_0=dy_0=I$ for t=0 it gives the variations of x and y for an infinitesimal variations of x and y at the origin. The results are given in table XIV

Table XIV
$x_0=5$

t	differential corrections dx	dy	variations dx	dy
0.00	0.0000	0.0000	I.0000	I.0000
0.05	0.0009	0.00I5	0.I88	0.42I
0.IO	-0.0032	0.00I3	-0.009	0.I87
0.I5	-0.00I3	-0.002I	-0.050	0.087
0.20	-0.0033	+0.0024	-0.049	0.038
0.25	-0.002I	0.0084	-0.039	0.0I5
0.30	-0.0032	0.0I33	-0.028	0.003
0.35	0.0032	-0.0043	-0.0I8	-0.002

4) INTERPOLATIONS

The results of the above computations are gathered in the following tables

Table XV $x_0=5$

t	x	y	dx/dx_0	dy/dy_0
0.00	5.000	0.000	I.000	I.000
0.05	I.24I	2.482	0.I88	0.42I
0.IO	-0.463	I.03I	-0.009	0.I87
0.I5	-I.32I	0.I28	-0.050	0.087
0.20	-I.793	-0.472	-0.049	0.038
0.25	-2.072	-0.892	-0.039	0.0I5
0.30	-2.243	-I.I97	-0.028	0.003
0.35	-2.347	-I.424	-0.0I8	-0.002

Table XVI $x_0=0$

t	x	y	dx/dx_0	dy/dy_0
0.00	0.000	0.000	I.000	I.000
0.05	-0.534	-0.33I	0.588	0.739
0.IO	-0.956	-0.6I2	0.3I4	0.542
0.I5	-I.293	-0.85I	0.I35	0.393
0.20	-I.563	-I.056	0.022	0.280
0.25	-I.780	-I.233	-0.044	0.I96
0.30	-I.967	-I.286	-0.056	0.0I3
0.35	-2.I0I	-I.5I9	-00.076	0.009

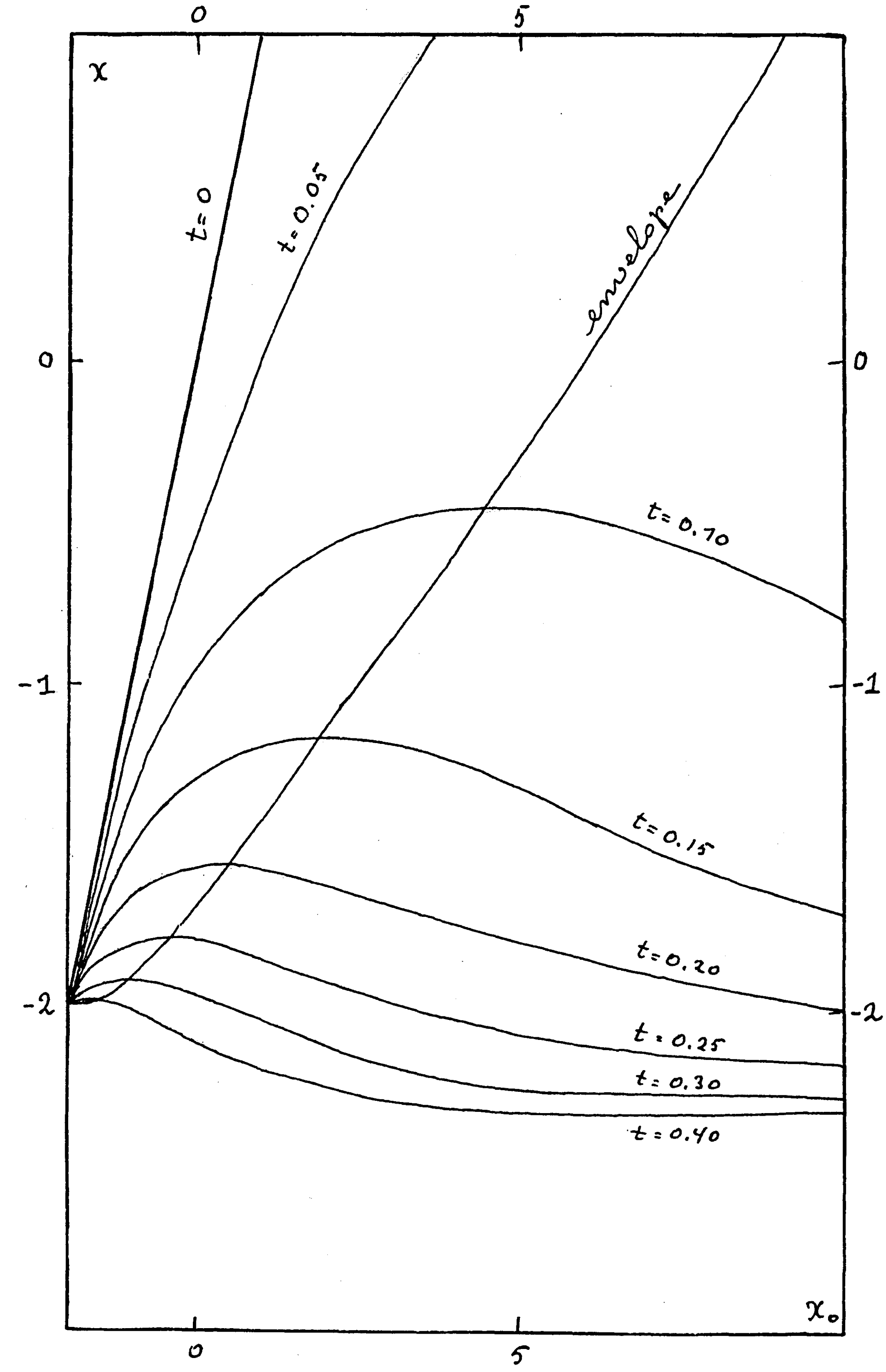
Table XVII $x_0 = -2$

t	dx/dx_0	dy/dy_0
0.00	I.000	I.000
0.05	0.898	0.937
0.I0	0.796	0.878
0.I5	0.690	0.8I6
0.20	0.580	0.752
0.25	0.467	0.687
0.30	0.349	0.620
0.35	0.224	0.55I

These data enable us to draw curves taking as abcissa the initial value x_0 and as ordinate X for every value of t. this curves are defined by three point of abcissae $x_0=-2$, 0 and 5, and the tangent at these points. Further more the assymptots are known for $x_0=\infty$. The locus of maxima of these curves ~~lines~~ corresponds to the points of the envelope.

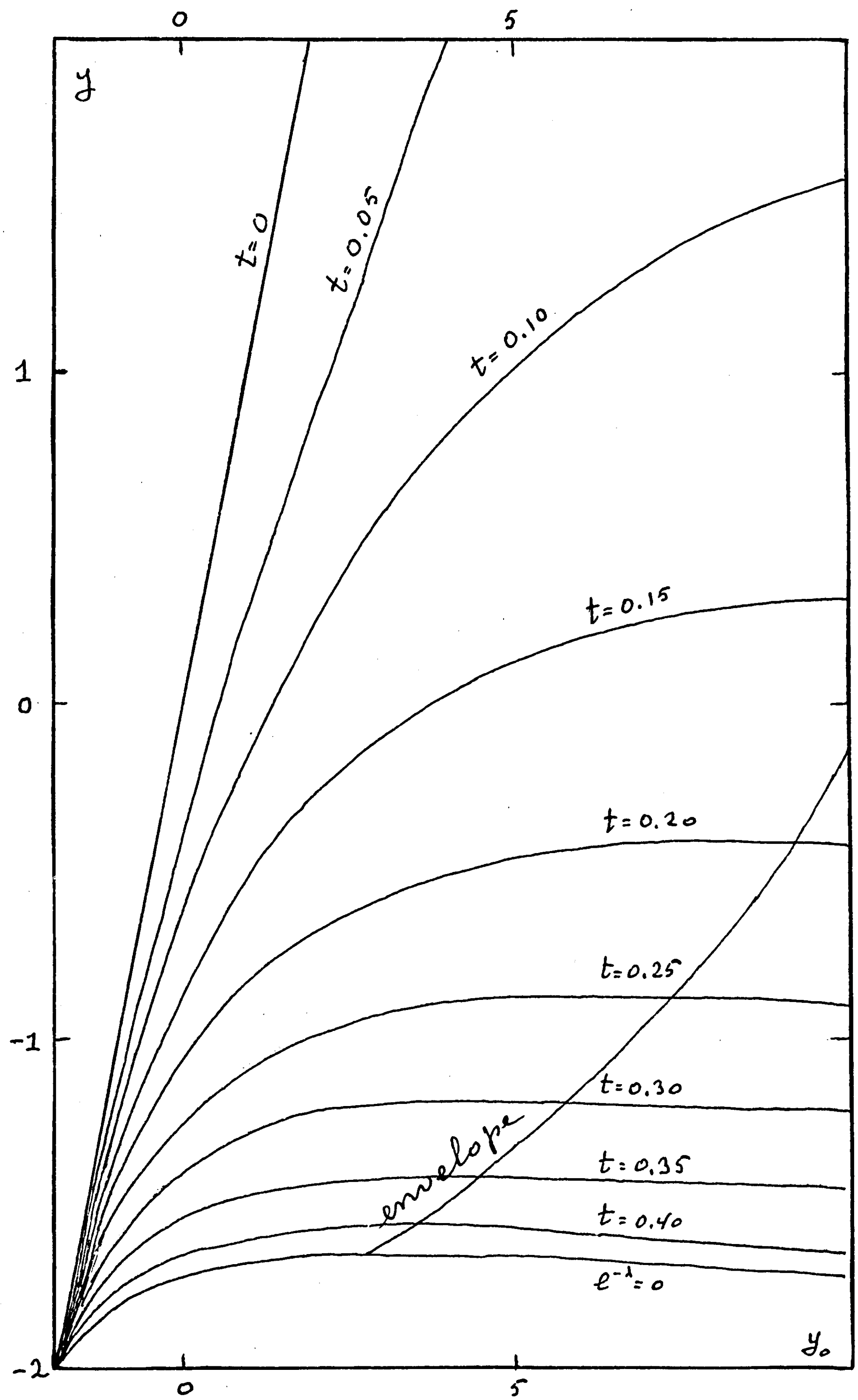
Fig. 2 and 3 give these curves for x and y respectively It is from these diagrams that the data of tables I, II, III and IV have been taken.

Fig. 2



0
5
x
t=0
t=0.05
envelope
0
0
t=0.10
-1
-1
t=0.15
t=0.20
-2
-2
t=0.25
t=0.30
t=0.40
x_0
0
5

Fig. 3



0
5
y
t=0
t=0.05
t=0.10
1
1
t=0.15
0
0
t=0.20
t=0.25
-1
-1
t=0.30
envelope
t=0.35
t=0.40
$e^{-\lambda}=0$
y_0
-2
0
5

The Gravitational Field Of a fluid Sphere of Uniform invariant Density, according to the Theory of Relativity.

Summary

According to Schwarzschild's classical solution for the field of an homogeneous sphere, the density which is supposed to be constant is represented by the component T_4^4 of the material tensor. Eddington has shown that physical requirements would be better fulfilled if the constant density would be represented by the invariant T and not by the component T_4^4. This identification seems to be the best macroscopic representation available, although it may be modified by a detailed knowledge of the internal structure of matter.

The purpose of this work is to solve the equations of the gravitational field of a fluid homogeneous sphere according to Eddington's hypothesis of a constant invariant density T.

These equations may be written as follows

$$\frac{dx}{dt} + \frac{(x+3)(x+y+4)}{1-(y+4)t} = 0 \qquad y = \frac{3}{2} t^{-\frac{3}{2}} \int_0^t x\, t^{\frac{1}{2}} dt$$

where x is twelf times the ratio of the variable pressure to the constant density, y is an auxiliary variable (a kind of mean pressure) and t is one sixth of the product of the density by the area of the sphere at the level considered. Pressure and density are evaluated in natural units and the cosmological constant is supposed to vanish. When the cosmological constant λ does not vanish, the same equations may be used, but the density found for a vanishing cosmological

constant must be reduced by $\lambda/2\varkappa$ and the pressure increased by $\lambda/8\varkappa$.

The special solutions x=y=-2 and x=y=-3 represent respectively Einstein's cylindrical Universe and de Sitter's empty world.

When the pressure is determined, the gravitational potentials may be readily computed by

$$ds^2 = -\frac{dr^2}{1-(y+4)t} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + C^b \frac{d\tau^2}{\sqrt{x+3}}$$

(as explained above, t is proportional to r^2)

The differential equations have two singular points, one for t=0, i.e. at the center of the sphere, the other for $1-(y+4)t=0$, i.e. at the horizon of the center.

The singularity at the center is but an apparent one. It is the appearence of a singularity which occurs in nearly every problem of gravitation with spherical symmetry. The equations are of the general type

$$\frac{dx}{dt} = F(x,y,t) \qquad y = \frac{1}{\phi(t)}\int_0^t x\,\frac{d\phi(t)}{dt}\,dt \quad \text{with} \quad \phi(0)=0$$

When, in the neighbourhood of t=0, F and $d\phi/dt$ are continuous and F satisfies Lipschitz' condition, these equations have a solution and only one for any initial value of x. This is shown by extending Picard's process of integration by computing the successive approximations by the formulae

$$y_n(t) = \frac{1}{\phi(t)}\int_0^t x_n(t)\,\frac{d\phi(t)}{dt}\,dt, \qquad x_{n+1}(t) = x_0 + \int_0^t F[x_n(t), y_n(t), t]\,dt$$

Taylor developement may thus be used for any finite initial value of x, i.e. for any central pressure. For an infinite central pressure x has a simple pole at the origin and a power series can be written.

Considering only initial values greater than -3, (the only one of physical interest), it is shown, from a discussion of the equations, that the pressure x reaches the critical point at the horizon of the center for a value of x lying on the hyperbola $xt+I=0$ between the points $t=I/3$ and $t=I$. The critical point X,T is a real singularity. In its neighbourhood, x may be developed in power series of $\sqrt{T-t}$.

For any value of t, there is a maximum value of x. Two cases might occur: I) the locus of the maxima may be one of the x curves, e.g. that of infinite central pressure as in Schwarschild's solution, or 2) it may be an envelope of the x-curves. As regards to physical interpretation the first case would mean that, when the central pressure grows up to infinity, the radius of the sphere tends to a definite limit; in the second case, this radius would have a maximum for a finite central pressure and then become smaller for increasing central pressure.

Numerical computations have been carried on and prove that this second case really does occur.

Integrals have been computed for $x_0=(-3,-2),0,5,\infty$ and then, for successive values of t (0.05, 0.I0, etc.) curves of x as function of x_0 have been plotted from the computed values at the five points and the tangents at these points. The value of x on the envelope is given as the locus of maxima of the curves t=Ct. The numerical integrations have been carried on, starting with the Taylor's developement and then by trials checked up and differentially corrected by using Euler-Maclaurin formula.

As to physical interpretation, the results may be contrasted with that obtained in Schwarzschild's hypothesis.

For an uniform Schwarzschild's density, the radius of the sphere increases with the central pressure and tends to a maximum when the central pressure tends to infinity. Even in this limiting case, the sphere does not fill up the space, there remains free space outside of the sphere. Furthermore there is no solution when the density is smaller than that of an Einstein's cylindrical universe of the same cosmological constant.

For an uniform invariant density,

I) When the density is greater than that of an Einstein's universe of the same cosmological constant, the radius of the sphere increases with the central pressure, passes through a maximum for a finite value of the central pressure and then diminishes until this pressure tends to infinity.

2) The density may be smaller than the density of an Einstein's universe of the same cosmological constant, but it cannot be smaller than about one half of this density.

Then the material sphere may fill up the whole space which has the same radius as an Einstein's universe of the same cosmological constant.

3) When the density approaches its minimum, the pressure curves have a minimum which corresponds to the boundary of a maximum sphere with free space outside of the sphere. The gradient of pressure vanishes at the boundary and the gravitation force is a repulsion outside of the sphere.

In the first case numerical informations may be gathered in the following table

$\frac{\lambda}{8\pi d}$	$8\pi a^2 d$	$\frac{2p_0}{c^2 d}$	$\frac{m}{\frac{4}{3}\pi a^3 d}$	$\frac{V_0}{\frac{4}{3}\pi a^3}$	$\frac{a}{a_S}$	$R^2\lambda$	$\frac{a_\infty}{a}$
0.00	I.00	I.00	I.40	I.04	0.6I	3.00	0.77
0.05	I.03	0.99	I.40	I.07	0.63	3.I3	0.77
0.I0	I.07	0.98	I.39	I.09	0.65	3.23	0.76
0.I5	I.I0	0.86	I.38	I.I2	0.67	3.33	0.75
0.20	I.I3	0.80	I.36	I.I6	0.69	3.40	0.75
0.25	I.I6	0.73	I.33	I.20	0.7I	3.46	0.74
0.30	I.I9	0.66	I.30	I.24	0.73	3.50	0.73
0.35	I.23	0.58	I.25	I.28	0.75	3.55	0.73
0.40	I.29?	0.48	I.I9	I.33	0.78	3.59	0.72
0.45	I.39	0.35	I.II	I.39	0.83	3.65	0.7I
0.50	I.60	0.00	I.00	I.49	0.90	3.73	0.66

where d is the density, λ the cosmological constant, p_0 the central pressure, a the maximum radius, a_S the maximum radius in Schwarzschild's hypothesis, a_∞ the radius for an infinite central pressure, m the apparent mass of the sphere as it must be deduced from the gravitational field outside of the sphere, V_0 the true volume which the matter would occupy if the pressure of the incompressible material would be reduced to zero (the pressure modifies the curvature of space), R the radius of the space outside of the material sphere. This is greater than that of an empty de Sitter's space of the same cosmological constant. The simple values obtained at the first line of this table are obtained by numerical computations and there is no reason to believe that they are rigorously exact.

If matter would be introduced in the free space in the neighbourhood of the maximum sphere, it would fall on the sphere and tend to increase the radius of the incompressible sphere. Nevertheless no solution would be possible with a greater radius. No explanation has been found of this paradoxical result which has already been raised against Schwarzschild's solution. But the difficulty is now more striking as the central pressure of the maximum sphere is finite. Infinite pressure suggests that the equations cease to keep their physical meaning and, as it has been said, that some kind of a "catastrophe" would occur. This way of eluding the difficulty is excluded in the case of an uniform invariant density.

NOTE ON A SPECIAL KIND OF SINGULARITY IN DIFFERENTIAL EQUATIONS.

In questions of theoretical physics dealing with gravitation field of spherical symmetry, equations occur which fail to satisfy the ordinary test of existence of a solution for the center of symmetry. However, a Taylor's developement may be computed when the initial values of the variables are properly connected, and it is possible to start with a numerical computation of the solution. The object of this note is to give a formal justification of this procedure.

We consider equations which may be written

$$\frac{dx}{dt} = F(x,y,t), \qquad y = \frac{1}{\phi(t)} \int_0^t x \frac{d\phi(t)}{dt} dt$$

where F and ϕ are regular functions for t=0, but where

$$\phi(0) = 0,$$

AND WE HAVE TO SHOW THAT A CONTINUOUS SOLUTION AND ONLY ONE DOES EXIST, WHICH HAS A GIVEN INITIAL VALUE OF x, $x=x_0$ FOR t=0.

Equations

$$\frac{dx}{dt} = -\frac{(x+3)(x+y+4)}{1-(y+4)t}, \qquad y = \frac{3}{2} t^{-\frac{3}{2}} \int_0^t x t^{\frac{1}{2}} dt$$

which occur in my thesis are clearly of this type.

Similarly Emden's equation

$$\frac{d^2u}{dz^2} + \frac{2}{z}\frac{du}{dz} + u^n = 0$$

which is fundamental in the theory of radiative equilibrium of a star reduces to

$$\frac{dx}{dt} = -\frac{n}{6} x^{\frac{n-1}{n}} y, \qquad y = \frac{3}{2} t^{-\frac{3}{2}} \int_0^t x t^{\frac{1}{2}} dt$$

by the substitution

$$x = u^n, \quad y = -6\frac{du}{dt}, \quad t = z^2.$$

We may notice that Bessell' functions of the first kind have at the origin a singularity of the same character; equation

$$t\frac{d^2x}{dt^2} + (1+n)\frac{dx}{dt} - x = 0$$

may be written

$$\frac{dx}{dt} = \frac{1}{n+1} y, \qquad y = \frac{n+1}{t^{n+1}} \int_0^t x t^n dt.$$

We suppose that F(x,y,t) is a continuous function of x,y,t in a domain D

$$x_0 - a < x < x_0 + a$$
$$y_0 - a < y < y_0 + a$$
$$0 \le t < b$$

has a maximum absolute value M in this domain and satisfies Lipschitz' condition

$$\left| F(x,y,t) - F(x',y',t) \right| < A\left|x - x'\right| + B\left|y - y'\right| .$$

$\phi(t)$ is supposed to vanish for t=0, and to have a positive derivative in an interval

$$0 \le t < c$$

We shall prove that there is a solution which satisfies any initial value

$$x = y = x_0 \qquad (t = 0)$$

and is continuous in an interval $0 \le t < h$ where h is the smallest of the three numbers A/M, a, and B.

We proceed by successive approximations computed by the formulae

$$y_n(t) = \frac{1}{\phi(t)} \int_0^t x_n(t) \frac{d\phi(t)}{dt} dt$$

$$x_{n+1}(t) = x_0 + \int_0^t F\left[x_n(t), y_n(t), t\right] dt$$

~~Then~~, We start with x_0 equal a constant and then y_0 will be the same constant.

The way we compute $y_n(t)$ has the following property: If x,y,x',y' are functions of t such that

$$y = \frac{1}{\phi(t)} \int_0^t x \frac{d\phi(t)}{dt} dt$$

$$y' = \frac{1}{\phi(t)} \int_0^t x' \frac{d\phi(t)}{dt} dt$$

we have

$$y' - y = \frac{1}{\phi(t)} \int_0^t (x' - x) \frac{d\phi(t)}{dt} dt$$

and, applying the theorem of the mean, we see that in any interval (0,t), $|y'-y|$ is smaller than the maximum of $|x-x'|$ in the same interval.

This property enables us to deduce, from any inequality established for the x, a similar inequality for the y. It is then possible to extend every step of the demonstration of Picard to the actual case.

We have first to show that the approximations may be continued indefinitely.

In the interval (0,h), we have

$$|x_1 - x_0| < Mh < a$$

and therefore also

$$|y_1 - y_0| < a$$

Replacing in F, x and y by x_I and y_I, we get functions of t which are continuous in (0,h) and are smaller than M in absolute value.

Similarly, in (0,h)

$$|x_2 - x_0| < a \quad , \quad |y_2 - y_0| < a$$

and in general

$$|x_n - x_0| < a \quad , \quad |y_n - y_0| < a .$$

$F(x_n, y_n, t)$ being continuous in (0,h) and of absolute value smaller than M.

Thus the process can be continuated keeping in the required domain.

The next step is to show that x_n and y_n tend respectively to definite limits.

We have, in (0,h)

$$|x_1(t) - x_0| < Mt$$

and therefore

$$|y_1(t) - y_0| < Mt$$

Then

$$x_2(t) - x_1(t) = \int_0^t \{ F[x_1(t), y_1(t), t] - F[x_0, y_0, t] \} dt$$

and

$$|x_2(t) - x_1(t)| < \int_0^t \{ A\,|x_1(t) - x_0| + B\,|y_1(t) - y_0| \} dt$$

or

$$|x_2(t) - x_1(t)| < (A+B) M \frac{t^2}{2!}$$

and therefore

$$|y_2(t) - y_1(t)| < (A+B) M \frac{t^2}{2!}$$

Similarly

$$|x_n(t) - x_{n-1}(t)| < (A+B)^{n-1} M \frac{t^n}{n!}$$

and therefore

$$|y_n(t) - y_{n-1}(t)| < (A+B)^{n-1} M \frac{t^n}{n!}$$

Every term of the series

$$x_0 + (x_1 - x_0) + (x_2 - x_1) + \dots + (x_n - x_{n-1}) + \dots$$

$$y_0 + (y_1 - y_0) + (y_2 - y_1) + \dots + (y_n - y_{n-1}) + \dots$$

is a continuous function of t in (0,h) and these series converge uniformly in that interval.

It remains to see that the sums X(t), Y(t) of these series are solutions of the equations.

When n tends to infinity, we get

$$X(t) = \frac{1}{\phi(t)} \int_0^t X(t) \frac{d\phi(t)}{dt} dt$$

$$X(t) = x_0 + \int_0^t F[X(t), Y(t), t] \, dt$$

as $X(t) - x_{n-1}(t)$ and $Y(t) - y_{n-1}(t)$ tend uniformly to zero and the integrals

$$\frac{1}{\phi(t)} \int_0^t \{X(t) - x_{n-1}(t)\} \frac{d\phi(t)}{dt} dt$$

$$\int_0^t \{F[X(t), Y(t), t] - F[x_{n-1}(t), y_{n-1}(t), t]\} \, dt$$

tend to zero when n tends to infinity.

Therefore X(t) and Y(t) are solution of the equations in the interval (0,h) and they are continuous in this interval.

THE SOLUTION IS UNIQUE.

Let us suppose that two sets of solutions x,y and X,Y would satisfy the equations, with the same initial values

$$x = y = X = Y \quad \text{for } t = 0$$

We have to show that these two sets of solutions are identiqal and it is sufficient to do so for any finite interval (0,T).

It is possible to find an interval (0,k), such that x,y,X,Y keep in the domain D, when t is in this interval. Then we have, in (0,k)

$$|X(t) - x(t)| \leq \cancel{\int_0^t \{F[X(t), Y(t), t] - F[x(t), y(t), t]\} dt} \int_0^t \{A|X - x| + B|Y - y|\} \, dt$$

On the other hand

$$|Y(t) - y(t)| < \max \eta \ |X(t) - x(t)| \text{ in } (0, t)$$

If ξ is the maximum of $|X-x|$ in (0,t), we have

$$\xi < (A+B)\xi t$$

This equation cannot be satisfied for $t < I/(A+B)$ except if $\xi = 0$, i.e. if x and X are identical. Then y and Y will be identical also. Therefore the two solutions are identical in an interval (0,T) where T is the smallest of the two numbers k and I/(A+B).