ABELIAN ALGEBRAS AND ADJOINT ORBITS

by

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ABSTRACT

We study the sets $X_d(\mathfrak{g})$ and $Q_d(\mathfrak{g})$ of d-dimensional abelian subalgebras of $\mathfrak{g}$ and d-dimensional tori of $\mathfrak{g}$ respectively, where $\mathfrak{g}$ is the Lie algebra of a semi-simple connected algebraic group G over an algebraicly closed field k of characteristic 0. $X_d(\mathfrak{g})$ is a closed subvariety of the Grassmannian $Gr_d(\mathfrak{g})$ of d-dimensional subspaces of $\mathfrak{g}$.

$Q_d(\mathfrak{g})$ is an irreducible, constructible subset of $X_d(\mathfrak{g})$ and its closure $\overline{Q_d(\mathfrak{g})}$ is easily an irreducible component of $X_d(\mathfrak{g})$ when $d \leq \ell$, where ℓ = rank of .
In general, $X_d(\mathfrak{g})$ has other irreducible components so that tori are not the general type of abelian subalgebra of $\mathfrak{g}$.

Using Kostant's description of the closed G-orbits on $X_d(\mathfrak{g})$ and generalizing a degeneration of his, we show that all these closed G-orbits lie in $\overline{Q_d(\mathfrak{g})}$ when

$d \leq \ell$. This means that the most specialized abelian subalgebras are limits of tori. In particular, then, all the irreducible components of $X_d(\mathfrak{g})$ meet $\overline{Q_d(\mathfrak{g})}$, so that $X_d(\mathfrak{g})$ is a connected variety when $d \leq \ell$.

A representation theoretic corollary is that $\mathcal{U}(\mathfrak{g}) \cdot \Lambda^d \mathfrak{t} = A_d(\mathfrak{g})$, where $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, $\mathfrak{t}$ is a maximal torus of $\mathfrak{g}$, and $A_d(\mathfrak{g})$ is the span in $\Lambda^d \mathfrak{g}$ of all the totally decomposable tensors corresponding to elements of $X_d(\mathfrak{g})$. This equality of representation spaces was first proved by King for $\mathfrak{g}$ a simple Lie algebra of exceptional type, and has various applications.

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§1 INTRODUCTION

We will study the set $X_d(\mathfrak{g})$ of d-dimensional abelian subalgebras of a semi-simple Lie algebra $\mathfrak{g}$. Here $\mathfrak{g}$ is the Lie algebra of a connected semi-simple algebraic group G over an algebraicly closed field of characteristic 0. The set $X_d(\mathfrak{g})$ forms a closed subvariety of the Grassmannian $Gr_d(\mathfrak{g})$ of d-dimensional subspaces of $\mathfrak{g}$. The adjoint action of G on induces actions of G on $Gr_d(\mathfrak{g})$ and $X_d(\mathfrak{g})$.

Viewing $X_d(\mathfrak{g})$ as an abstract projective variety with a G-action, we know, for instance, that the irreducible components of $X_d(\mathfrak{g})$ are G-invariant. Moreover, we can think of points in the boundary of a G-orbit $\mathcal{O}$ on $X_d(\mathfrak{g})$ as being limits or degenerations of elements of $\mathcal{O}$, so that the <u>closed</u> G-orbits on $X_d(\mathfrak{g})$ represent the most degenerate types of abelian subalgebras.

When d is less than or equal to the rank ℓ of $\mathfrak{g}$, the variety $X_d(\mathfrak{g})$ contains eminent elements, namely tori (subalgebras made up of commuting semi-simple elements of $\mathfrak{g}$). It is easy to see, using the conjugacy of maximal tori, that the set $Q_d(\mathfrak{g})$ of d-dimensional tori forms an irreducible, constructible subset of $X_d(\mathfrak{g})$ (always with respect to the Zariski topology). In fact, by considering the open dense subset of regular semisimple elements in $\mathfrak{g}$, we see (§2, Prop. 1.5) that $Q_d(\mathfrak{g})$ is dense in the irreducible component of $X_d(\mathfrak{g})$ in which it sits, i.e. that $\overline{Q_d(\mathfrak{g})}$ is an irreducible component of $X_d(\mathfrak{g})$, when $d \leq \ell$. In general $X_d(\mathfrak{g})$ has other irreducible components, and ones much larger in dimension than $\overline{Q_d(\mathfrak{g})}$ (§2, Prop. 2.1). So tori are not the general sort of abelian subalgebras (except when $d = 1$).

In the case $d = \ell$, then all the elements of $\overline{Q_\ell(\mathfrak{g})}$ are algebraic Lie subalgebras (§2, Corollary 2.8). This gives one way of showing that certain ℓ-dimensional subalgebras are not limits of tori.

In §3, we use Kostant's description (§3.2) of the closed G-orbits of $X_d(\mathfrak{g})$ to show that they all lie in $\overline{Q_d(\mathfrak{g})}$. This means that the most specialized abelian subalgebras can be gotten as limits of tori. One immediate corollary is that each irreducible component of $X_d(\mathfrak{g})$ meets $\overline{Q_d(\mathfrak{g})}$, so that $X_d(\mathfrak{g})$ is a <u>connected</u> variety when $d \leq \ell$.

Another corollary pertains to the projective embedding

$$X_d(\mathfrak{g}) \hookrightarrow Gr_d(\mathfrak{g}) \hookrightarrow \mathbb{P}(\Lambda^d\mathfrak{g}) ,$$

where the second map is the Plücker embedding of the Grassmannian. Let $A_d(\mathfrak{g})$ denote the linear span in $\Lambda^d\mathfrak{g}$ of the affine cone over the image of $X_d(\mathfrak{g})$ in $\mathbb{P}(\Lambda^d\mathfrak{g})$. Then, as $\mathbb{P}(A_d(\mathfrak{g}))$ is spanned by the closed orbits of $X_d(\mathfrak{g})$, we have the corollary that the linear spans of $Q_d(\mathfrak{g})$ and $X_d(\mathfrak{g})$ in $\mathbb{P}(\Lambda^d\mathfrak{g})$ are equal. Passing to affine cones, we have

$$\mathcal{U}(\mathfrak{g}) \cdot \Lambda^d\mathfrak{t} = A_d(\mathfrak{g}) , \; d \leq \ell .$$

This equality was proven by King [Ki] for the case of $\mathfrak{g}$ a simple Lie algebra of exceptional type. Applications of this result are discussed in §4.

§2.1 INTRODUCTION AND PRELIMINARY RESULTS

Let G be a connected, semi-simple algebraic group over an algebraicly closed field k of characteristic 0 . Let ℓ be the rank of G , and let $\mathfrak{g}$ be the Lie algebra of G .

<u>Definition 1.1</u> For each positive integer d , let $X_d(\mathfrak{g})$ denote the set of d-dimensional abelian subalgebra of $\mathfrak{g}$, and let $Q_d(\mathfrak{g})$ denote the set of d-dimensional tori (i.e., Lie subalgebras of $\mathfrak{g}$ made up of commuting semi-simple elements).

So $Q_d(\mathfrak{g})$ is non-empty only when $d \leq \ell$.

Recall the Grassmann variety $Gr_d(V)$ which parameterizes the d-dimensional linear subspaces (spaces thru the origin) of an affine linear space V . $Gr_d(V)$ is a smooth, projective variety, and $Gr_1(V)$ is just the projective space $\mathbb{P}(V)$. Our sets $X_d(\mathfrak{g})$ and $Q_d(\mathfrak{g})$ naturally sit inside $Gr_d(\mathfrak{g})$, and the next lemma implies that this embedding induces variety structures on them.

<u>Lemma 1.2</u> (a) $X_d(\mathfrak{g})$ is a closed subvariety of $Gr_d(\mathfrak{g})$.

(b) The collection $Q_d(\mathfrak{g})$ of tori is an irreducible, constructible subset of $X_d(\mathfrak{g})$.

<u>Proof</u>. (a) To see that $X_d(\mathfrak{g})$ forms a closed (we will always mean in the <u>Zariski</u> topology) subset of $Gr_d(\mathfrak{g})$, consider the bilinear bracket map $[-,-]$: $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Then $X_d(\mathfrak{g})$ is just the set of d-dimensional subspaces L of $\mathfrak{g}$ on which the bracket is zero, and this is easily a closed condition on the Grassmannian by the continuity of $[\ ,\ ]$.
(b) Fix a maximal torus $\mathfrak{t}$ of $\mathfrak{g}$. Then, by the conjugacy of maximal tori, we see $Q_d(\mathfrak{g})$ is the image of $G \times Gr_d(\mathfrak{t})$ under the natural morphism

$$G \times Gr_d(\mathfrak{t}) \to Gr_d(\mathfrak{g}) \quad \text{by} \quad (g,L) \mapsto gLg^{-1}, g \in G, L \in Gr_d(\mathfrak{t}) .$$

The stated properties of $Q_d(\mathfrak{g})$ now follow from the fact that $G \times Gr_d(\mathfrak{t})$ is an irreducible variety. □.

<u>Remarks 1.3</u> (1) It would be interesting to know if, in (a), the subscheme of $Gr_d(\mathfrak{g})$ determined by the vanishing of the bracket is <u>reduced</u>. For instance this question arises for the variety of unipotent elements of an algebraic group, and there it turns out that there the natural scheme <u>is</u> reduced (see [S1], for instance).

(2) The variety $Q_\ell(\mathfrak{g})$ of <u>maximal</u> tori is just the affine variety $G/N(T)$, where $N(T)$ is the normalizer of a maximal torus T of G .

(3) The varieties $Gr_d(\mathfrak{g})$, $X_d(\mathfrak{g})$, and $Q_d(\mathfrak{g})$ have a natural G-action deduced from the adjoint action of G on $\mathfrak{g}$.

Now the lemma implies that $Q_d(\mathfrak{g})$ lies in a single irreducible component of $X_d(\mathfrak{g})$. The next proposition says that $Q_d(\mathfrak{g})$ is actually <u>dense</u> in that component. For this, we require the notion of regular elements.

<u>Definition 1.4</u> An element $x \in \mathfrak{g}$ is <u>regular</u> if its orbit $G \cdot x$ under the adjoint action of G has maximal dimension; equivalently, if the stabilizer G^x has minimal dimension.

The regularity condition can also be phrased in terms of the adjoint action of $\mathfrak{g}$ on itself. Because, recall that for any $x \in \mathfrak{g}$, the centralizer $\mathfrak{g}^x = \{z \in \mathfrak{g} \mid [z,x] = 0\}$ is the Lie algebra of the identity component of G^x . (Here we are using the characteristic 0 hypothesis; in general it is just true that the Lie algebra of G^x is contained in $\mathfrak{g}^x$.) So $\dim G^x = \dim \mathfrak{g}^x$ and $\dim G \cdot x = \dim \mathfrak{g} \cdot x$. (Indeed, $x + \mathfrak{g} \cdot x$ is just the embedded tangent space to the orbit $G \cdot x$ in $\mathfrak{g}$ at the point x .)

One finds that [Kol,St] (1) x is regular $\Leftrightarrow \dim \mathfrak{g}^x = \ell$, (2) x regular $\Rightarrow \mathfrak{g}^x$ is an abelian subalgebra, and (3) the regular semi-simple elements $\mathfrak{g}^{reg}_{s.s.}$ form an open dense subset of $\mathfrak{g}$, whose complement has codimension 1.

In $\mathfrak{sl}_{n+1}$, for example, the regular semi-simple elements are precisely the diagonalizable matrices with <u>distinct</u> eigenvalues.

Call a torus $\mathfrak{T}$ of $\mathfrak{g}$ a <u>regular torus</u> if $\mathfrak{T}$ contains a regular element. Note a regular torus is contained in a unique maximal torus, namely the centralizer of that regular element.

<u>Proposition 1.5</u> The closure $\overline{Q_d(\mathfrak{g})}$ of the tori is an irreducible component of the variety $X_d(\mathfrak{g})$ of d-dimensional abelian subalgebras of $\mathfrak{g}$, and

$$\dim Q_d(\mathfrak{g}) = (\dim G) - \ell + d(\ell - d) , \quad 1 \leq d \leq \ell .$$

<u>Proof</u>. Since $\mathfrak{g}^{reg}_{s.s.}$ is open dense in $\mathfrak{g}$, the set

$$U = \{L \in Gr_d(\mathfrak{g}) \mid L \text{ meets } \mathfrak{g}^{reg}_{s.s.}\}$$

is open dense in $Gr_d(\mathfrak{g})$, and the set

$Q_d^{reg}(\mathfrak{g}) \overset{def.}{=} U \cap Q_d(\mathfrak{g}) = \{\text{regular } d\text{-dimensional tori}\}$

is open dense in the irreducible set $Q_d(\mathfrak{g})$.

Now for $X_d(\mathfrak{g})$, all we can say is that $U \cap X_d(\mathfrak{g})$ is open in $X_d(\mathfrak{g})$ and hence dense in each irreducible component which it meets. But $U \cap X_d(\mathfrak{g})$ is equal to the irreducible set $Q_d^{reg}(\mathfrak{g})$, because, if L is an abelian subalgebra containing a regular semi-simple element x , then $L \subset \mathfrak{g}^x$ = a maximal torus. So $\overline{Q_d^{reg}} = \overline{Q_d(\mathfrak{g})}$ is an irreducible component of $X_d(\mathfrak{g})$.

Now if T is a maximal torus of G with Lie algebra $\mathfrak{t}$, then the conjugation mapping in the proof of Lemma 1.2(b) obviously induces a dominant map of irreducible varieties

$$G/T \times Gr_d(\mathfrak{t}) \to \overline{Q_d(\mathfrak{g})} .$$

Since the normalizer in G of $\mathfrak{t}$ is just a finite extension of T , one easily sees that the fibre over each point of $Q_d^{reg}(\mathfrak{g})$ is finite. So the dimensions of the domain and the image are equal. □

§2.2 THE REDUCIBILITY OF $X_d(\mathfrak{g})$ (FOR d CLOSE TO ℓ)

The last proposition (1.5) tells us that $\overline{Q_d(\mathfrak{g})} = X_d(\mathfrak{g})$ if and only if the variety $X_d(\mathfrak{g})$ is irreducible. However, there are many sorts of examples one can give to show that $\overline{Q_d(\mathfrak{g})} \neq X_d(\mathfrak{g})$ in general. We will discuss a couple of these now.

The first method for finding examples is to find a family of abelian d-dimensional subalgebras whose dimension is bigger than the dimension of $Q_d(\mathfrak{g})$. Now if $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{g}$ of dimension p , then the Grassmannian $Gr_d(\mathfrak{a})$ is a $d(p - d)$-dimensional subvariety of $X_d(\mathfrak{g})$. (Indeed, conjugation by G generates a bigger family, but we can get results just by working in $\mathfrak{a}$.)

The determination of the largest possible value p_0 for p for each of the simple Lie algebras was made by Malcev [M]. For the classical simple Lie algebras, p_0 , like the dimension of $\mathfrak{g}$, is a quadratic polynomial in the rank ℓ . All this (together with Prop. 1.5) tells us that $\dim Q_\ell(\mathfrak{g}) \sim \ell^2$ while $\dim Gr_\ell(\mathfrak{a}_0) \sim \ell^3$, where $\mathfrak{a}_0$ is an abelian subalgebra of maximal dimension p_0 . (Here $f(\ell) \sim g(\ell)$ means that $f(\ell)$ and $g(\ell)$ are polynomials in ℓ of the same degree.)

This argument establishes

Proposition 2.1 For large ℓ, the variety $X_\ell(\mathfrak{g})$ has an irreducible component of dimension strictly larger than $\dim \overline{Q_\ell(\mathfrak{g})}$.

Example 2.2 Let $\mathfrak{g} = \mathfrak{sl}_8$, so $\ell = 7$. Malcev's formula for p_0 for $\mathfrak{g} = \mathfrak{sl}_{\ell+1}$ is $p_0 = [(\ell + 1)^2/4]$, so here $p_0 = 16$. We may write elements of $\mathfrak{g}$ as $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$, where A,B,C, and D are 4×4 matrices such that $\mathrm{trace}(A) + \mathrm{trace}(D) = 0$. With this notation, a choice for $\mathfrak{a}_0$ is

$$\mathfrak{a}_0 = \left\{ \left(\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right) \middle| \ B \text{ arbitrary} \right\} = \text{the nilradical of a maximal parabolic.}$$

Then $\dim Gr_7(\mathfrak{a}_0) = 7 \cdot 9 = 63$, while $\dim Q_7(\mathfrak{g}) = 56$.

Remark 2.3 For d close to ℓ the situation is similar. Specifically, if we fix e, and put $d = \ell - e$, then we get the same estimates as ℓ gets large. Note that this breaks down for d small - indeed, consider the case $d = 1$!

Now we turn to a more delicate method for finding abelian subalgebras outside of $\overline{Q_\ell(\mathfrak{g})}$. (This method will just work for the case $d = \ell$.) The idea is to show all the subalgebras of $\overline{Q_\ell(\mathfrak{g})}$ are algebraic, and then to exhibit non-algebraic ℓ-dimensional abelian subalgebras. I would like to thank B. Kostant for suggesting this to me.

To recall some basic notions about algebraic Lie algebras, we will work with an arbitrary algebraic group G' over k (still of char 0) with Lie algebra $\mathfrak{g}'$.

Definition 2.4 A Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}'$ is algebraic if $\mathfrak{a}$ is the Lie algebra $\mathcal{L}(A)$ of some algebraic subgroup A of G' .

This notion was developed by Chevalley. See [C] and [B].

In characteristic 0 , Lie algebras behave nicely since all maps are separable, and the mapping $H \to \mathcal{L}(H)$ gives a 1 to 1 functorial correspondence between connected closed subgroups of G' and algebraic Lie subalgebras of $\mathfrak{g}'$. The main point here is that $\mathcal{L}(A_1) \cap \mathcal{L}(A_2) = \mathcal{L}(A_1 \cap A_2)$ for any two closed subgroups A_1 and A_2 of G' .

In fact, the formation of Lie algebras commutes with various standard constructions, so that some examples of algebraic subalgebras are (1) the centralizer and normalizer of any subalgebra, (2) more generally, the transporter

$$\mathrm{trans}(\mathfrak{a}_1,\mathfrak{a}_2) = \{x \in \mathfrak{g}' \mid x \cdot \mathfrak{a}_1 \subseteq \mathfrak{a}_2\}$$

of any two subspaces $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of $\mathfrak{g}'$, and (3) the commutator $[\mathfrak{a}_1,\mathfrak{a}_2]$ of any two algebraic subalgebras of $\mathfrak{g}'$. Additionally, a subalgebra made up of nilpotent elements must be algebraic, because we can exponentiate (this is an algebraic map on nilpotents) to get the corresponding algebraic subgroup.

For any subset M of $\mathfrak{g}'$, there is a unique smallest algebraic subalgebra $\underline{a}(M)$ containing M. A simple argument using transporters shows that for any subalgebra $\mathfrak{a}$ one always has $[\underline{a}(\mathfrak{a}),\underline{a}(\mathfrak{a})] = [\mathfrak{a},\mathfrak{a}]$, so that it follows from examples (2) and (3) above that $[\mathfrak{a},\mathfrak{a}]$ is always an algebraic subalgebra. In particular, then, any semi-simple subalgebra is algebraic.

Since Jordan decomposition makes sense in any algebraic group and is functorial, we see that <u>an algebraic subalgebra must contain the semi-simple and nilpotent parts of its elements.</u> In fact if $\mathfrak{a}$ is an <u>abelian</u> algebraic subalgebra, then it is easy to verify that the subsets $\mathfrak{a}_s$ and $\mathfrak{a}_n$ of semi-simple and nilpotent elements of $\mathfrak{a}$ are linear subspaces and $\mathfrak{a} = \mathfrak{a}_s \oplus \mathfrak{a}_n$.

Now the condition of being algebraic is not a <u>closed</u> condition (i.e. the d-dimensional algebraic subalgebras do not form a closed subset of the Grassmannian $Gr_d(\mathfrak{g}')$), but the next proposition says that this property is preserved when the corresponding subgroups of G' form an algebraic family. I would like to thank my advisor, Steve Kleiman, for telling me how to prove this.

<u>Definition 2.5</u> Let Z be a variety and let S be a variety parameterizing a family of subvarieties of Z. I.e., assume we have a subset I of $Z \times S$ such that $\pi_2(I) = S$ and each fibre $\pi_2^{-1}(s)$, $s \in S$, is a subvariety of Z , where π_1 and π_2 are the projections of I to the two factors.

$$\begin{array}{ccccc} & & I & \hookrightarrow & Z \times S \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ Z & & & & S \end{array}$$

The family is <u>algebraic</u> if I is closed in $Z \times S$.

<u>Example 2.6</u> As an example of the definition (and this is the case we will be concerned with), suppose G' acts on Z and that W is a subvariety of Z . Then the family of translates of W under G' is an algebraic

family. Indeed, if H is the stabilizer

$$H = \{g \in G' \mid g \cdot W = W\} ,$$

then

$$I = \{(g \cdot w , gH/H) \mid w \in W\} \hookrightarrow Z \times G'/H .$$

So we have

$$I = \{(z, gH/H \mid g^{-1} \cdot z \in W , z \in Z\} ,$$

which is closed in $Z \times G'/H$ by the continuity of the G'-action.

Proposition 2.7 Let G' be a connected algebraic group over k with Lie algebra $\mathfrak{g}'$. Let S be a locally closed subset of the Grassmannian $Gr_d(\mathfrak{g}')$ such that S parameterizes a family of algebraic Lie subalgebras of $\mathfrak{g}'$. Assume that the corresponding family of algebraic subgroups of G' is an algebraic family. Then every point in $\overline{S}$ again represents an algebraic Lie subalgebra of $\mathfrak{g}'$.

Proof Note that right away we know that limit points (i.e. points in $\overline{S}$) represent Lie subalgebras, since (by the continuity of the bracket) being a subalgebra is a closed condition.

(a) For each point $L \in S$, let A_L be the connected algebraic subgroup of G' with Lie algebra L. The family of these subgroups is given by

$$I = \{(g,L) \mid g \in A_L\} \hookrightarrow G' \times S \, .$$

Now close up I in $G' \times Gr_d(\mathfrak{g}')$ and let π_1 and π_2 be the projections of $\bar{I}$ to the two factors.

$$\begin{array}{ccccc} & & \bar{I} & \hookrightarrow & G' \times Gr_d(\mathfrak{g}') \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ G' & & & & Gr_d(\mathfrak{g}') \end{array}$$

As we are assuming that I is algebraic, we know that $\bar{I}$ is unchanged over S (i.e. $\pi_2^{-1}(S) = I$). On the other hand, closing up the identity section of I over S, we see that $\pi_2(\bar{I}) = \bar{S}$.

For each $L \in \bar{S}$, $\pi_1 \cdot \pi_2^{-1}(L)$ is an algebraic subgroup of G'. Indeed, the continuity of the multiplication and inverse maps of G' insure that $\pi_1 \cdot \pi_2^{-1}(L)$ is a subgroup, and it is closed in G' as $Gr_d(\mathfrak{g}')$ is projective, hence proper over k.

(b) Now at points $L \in \bar{S}$ the dimension of the subgroup $\pi_1 \cdot \pi_2^{-1}(L)$ may jump up. However this problem is eliminated if S happens to be a curve (a one-dimensional irreducible variety). Because then, $\bar{I}$ is an irreducible variety of dimension $d + 1$, so the fact that $\pi_2^{-1}(L)$ is a proper subvariety of $\bar{I}$ of dimension at least d forces the dimension of $\pi_2^{-1}(L)$, and hence $\pi_1 \cdot \pi_2^{-1}(L)$, to be exactly d.

(c) Given $L \in \bar{S}$, we may find a curve in S whose closure contains L by proceeding as follows. Obviously we may assume that S is irreducible. By Bertini's Theorem, the intersection of an irreducible variety (of dimension greater than 1) with a general quadric hypersurface thru a fixed point is again irreducible. Of course as S is open dense in its closure, a general hypersurface section of $\bar{S}$ meets S. So taking successive general quadric hypersurface sections of $\bar{S}$ thru L, we cut out a sequence of irreducible subvarieties of $\bar{S}$ meeting S, such that the dimension drops by exactly one each time. Eventually, then, we get a curve C in $\bar{S}$ thru L with $S \cap C$ open dense in C, so that $S \cap C$ is a closed subset of $Gr_d(\mathfrak{g}')$ with $L \in \overline{S \cap C}$.

(d) So given $L \in \bar{S}$, replace S by the $S \cap C$ found in (c), and then apply (a) and (b). As formation of tangent spaces and the bracket structures is continuous, the d-dimensional algebraic subgroup $\pi_1 \cdot \pi_2^{-1}(L)$ has L as its Lie algebra. □

This concludes our general discussion of algebraic Lie subalgebras and we now return to the matter at hand.

Corollary 2.8 The elements of $\overline{Q_\ell(\mathfrak{g})}$ are all algebraic subalgebras.

Proof This is immediate from the Proposition in view of the conjugacy of maximal tori of G and Example 2.6. □

The following example indicates that it is easy to exhibit ℓ-dimensional abelian subalgebras which can't be algebraic because they don't contain the Jordan parts of all their elements.

Example 2.9 $\mathfrak{g} = \mathfrak{sl}_5$ and

$$= \left\{ \begin{pmatrix} a & 0 & 0 & b & c \\ & a & 0 & a & d \\ & & -4a & 0 & 0 \\ & & & a & 0 \\ & & & & a \end{pmatrix} \middle| \; a,b,c,d \in k \right\}$$

The Jordan parts of the element

$$x = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ & a & 0 & a & 0 \\ & & -4a & 0 & 0 \\ & & & a & 0 \\ & & & & a \end{pmatrix}$$

are

$$x_s = \begin{pmatrix} a & & & & \\ & a & & & \\ & & -4a & & \\ & & & a & \\ & & & & a \end{pmatrix} \quad \text{and} \quad x_n = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & a & 0 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix} .$$

§3.1 INTRODUCTION

Now we will look more closely at the irreducible component $\overline{Q_d(\mathfrak{g})}$ of the variety $X_d(\mathfrak{g})$ of d-dimensional abelian subalgebras of $\mathfrak{g}$. We have seen in §2.2 that, in general, the tori are not generic abelian subalgebras, i.e., that the closure of the variety $Q_d(\mathfrak{g})$ of d-dimensional tori is not all of $X_d(\mathfrak{g})$.

On the other hand, we can wonder about degenerate or specialized abelian subalgebras. Specifically, it makes sense to say that the elements of the closed G-orbits on the projective variety $X_d(\mathfrak{g})$ are the most degenerate abelian subalgebras of $\mathfrak{g}$. Also elements of $\overline{Q_d(\mathfrak{g})}$ are limits of tori, while those limits not in $Q_d(\mathfrak{g})$ are degenerations of tori. The main result of §3 is that, for $d \leq \ell$, the variety $\overline{Q_d(\mathfrak{g})}$ contains all the closed G-orbits of $X_d(\mathfrak{g})$, i.e. that the most degenerate d-dimensional abelian subalgebras are degenerate limits of tori.

It is useful to view all this in the context of embedded varieties. Recall the Plücker embedding

$$Gr_d(V) \hookrightarrow \mathbb{P}(\Lambda^d V)$$

of the Grassmannian of d-dimensional linear subspaces L of an affine linear space V maps each L to the line $\Lambda^d L$ of $\Lambda^d V$. If some algebraic group G' acts on V, then this projective embedding obviously respects the induced actions. Also, for any projective space $\mathbb{P}(V)$, we have the projection map $\pi : V - \{0\} \to \mathbb{P}(V)$. The <u>affine cone</u> over a subset Z of $\mathbb{P}(V)$ is the cone $\pi^{-1}(Z) \cup \{0\}$ in V. For instance the affine cone over $Gr_d(V)$ in $\mathbb{P}(\Lambda^d V)$ is just the set of totally decomposable tensors in $\Lambda^d V$.

In our situation, we have

$$X_d(\mathfrak{g}) \hookrightarrow Gr_d(\mathfrak{g}) \hookrightarrow \mathbb{P}(\Lambda^d \mathfrak{g}) ,$$

with G acting via its adjoint representation of $\mathfrak{g}$. With this projective embedding of $X_d(\mathfrak{g})$, we can now consider its linear span.

<u>Definition 1.1</u> Let $A_d(\mathfrak{g})$ denote the linear span in $\Lambda^d \mathfrak{g}$ of the affine cone over $X_d(\mathfrak{g})$ in $\mathbb{P}(\Lambda^d \mathfrak{g})$. (So $\mathbb{P}(A_d(\mathfrak{g}))$ is the linear span of $X_d(\mathfrak{g})$ in $\mathbb{P}(\Lambda^d \mathfrak{g})$.)

<u>Remarks 1.2</u> (1) The span of a G-invariant set is again obviously G-invariant, so $A_d(\mathfrak{g})$ is a finite

dimensional G-representation space, and the problem of finding the closed G-orbits in $X_d(\mathfrak{g})$ reduces to the <u>linear</u> problem of decomposing $A_d(\mathfrak{g})$ into irreducible $\mathfrak{g}$-representation spaces. We will recall in §3.2 Kostant's solution to the latter.

(2) The linear span of the affine cone over $Q_d(\mathfrak{g})$ in $\mathbb{P}(\Lambda^d\mathfrak{g})$ is clearly $\mathcal{U}(\mathfrak{g}) \cdot \Lambda^d\mathfrak{t}$, where $\mathfrak{t}$ is any maximal torus of $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. An obvious corollary of the result (Th. 5.1) that the closed G-orbits of $X_d(\mathfrak{g})$ lie in $\overline{Q_d(\mathfrak{g})}$ is the equality of the linear spans of $\overline{Q_d(\mathfrak{g})}$ and $X_d(\mathfrak{g})$ in $\mathbb{P}(V)$ whenever V is a G-space and $X_d(\mathfrak{g}) \xrightarrow{\varphi} \mathbb{P}(V)$ is a G-map such that the closed G-orbits in the span of $\varphi(X_d(\mathfrak{g}))$ all lie in $\varphi(X_d(\mathfrak{g}))$. This last requirement is satisfied by $X_d(\mathfrak{g}) \to \mathbb{P}(\Lambda^d\mathfrak{g})$ (§3.2), so that passing to affine cones, we will get (Cor. 5.5)

$$\mathcal{U}(\mathfrak{g}) \cdot \Lambda^d\mathfrak{t} = A_d(\mathfrak{g}) \text{ , for } d \le \ell \text{ .}$$

This equality of g-representation spaces has some applications, which will be discussed in §4. This was proved by King [Ki] for the case of $\mathfrak{g}$ a simple Lie algebra of exceptional type.

Finally, we will fix our root system notation and recall the nature of the highest weight line of an irreducible representation. Recall that each maximal torus $\mathfrak{t}$ of $\mathfrak{g}$ gives rise to a direct sum decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \Phi(\mathfrak{g},\mathfrak{t})} \mathfrak{g}^{\alpha}$$

into weight spaces for $\mathfrak{t}$ with $\Phi(\mathfrak{g},\mathfrak{t})$ a root system in the dual $\mathfrak{t}^{\vee}$ of $\mathfrak{t}$. Then the choice of Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{t}$ is equivalent to the choice of a positive system $\Phi^{+}(\mathfrak{g},\mathfrak{t})$, via the relation

$$\mathfrak{b} = \mathfrak{t} \oplus \sum_{\alpha \in \Phi^{+}(\mathfrak{g},\mathfrak{t})} \mathfrak{g}^{\alpha} .$$

Also the choice of $\Phi^{+}(\mathfrak{g},\mathfrak{t})$ is equivalent to the choice of a base $\Delta(\mathfrak{g},\mathfrak{t})$ of $\Phi(\mathfrak{g},\mathfrak{t})$. Given the latter, we define a partial order $\leq$ on $\mathfrak{t}^{\vee}$ by

$$\beta_1 \leq \beta_2 \Leftrightarrow \beta_2 - \beta_1 = \sum_{\alpha \in \Delta(\mathfrak{g},\mathfrak{t})} c_{\alpha}\alpha \quad \text{with each} \quad c_{\alpha} \geq 0 .$$

In particular, then Φ becomes a partially ordered set (poset, for short).

It is useful (see Lemma 4.9) to define the height $\mathrm{ht}(\varphi)$ of any element φ of the root lattice (i.e., the $\mathbb{Z}$-span of Φ in $\mathfrak{t}^{\vee}$) by

$$ht\left(\sum_{\alpha\in\Delta} c_\alpha \alpha\right) = \Sigma c_\alpha \cdot$$

Now suppose V is a representation space for G, hence also for $\mathfrak{g}$. Choose $\mathfrak{t} \subset \mathfrak{b}$, let $\mathfrak{m}$ be the nilpotent radical of $\mathfrak{b}$ and B the Borel subgroup of G with Lie algebra $\mathfrak{b}$. If $0 \neq v \in V$, then clearly the line $\langle v \rangle$ is stabilized by $\mathfrak{b}$ iff both $\mathfrak{m}$ kills v (as each root vector strictly increases the weight of any weight vector) and v is a weight vector for $\mathfrak{t}$. When v is such a vector, the standard theory tells us that the cyclic $\mathfrak{g}$-module $\mathcal{U}(\mathfrak{g}) \cdot v$ is indecomposable, that the subposet of weights of $\mathcal{U}(\mathfrak{g}) \cdot v$ has a largest element λ which is just the weight of v, and that the weight space $(\mathcal{U}(\mathfrak{g}) \cdot v)^\lambda$ is just $\langle v \rangle$. This implies that when $\mathcal{U}(\mathfrak{g}) \cdot v$ is irreducible (this is true automatically when V is finite dimensional, by complete reducibility of finite dimensional representations of a semi-simple Lie algebra), then $\langle v \rangle$ is the unique $\mathfrak{b}$-stable line in $\mathcal{U}(\mathfrak{g}) \cdot v$.

So in the case of V finite dimensional, $\langle v \rangle$ is the unique B-fixed point of $\mathbb{P}(\mathcal{U}(\mathfrak{g}) \cdot v)$, and as we vary the choice of B, the unique B-fixed points of $\mathbb{P}(\mathcal{U}(\mathfrak{g}) \cdot v)$ sweep out the unique closed G-orbit in $\mathbb{P}(\mathcal{U}(\mathfrak{g}) \cdot v)$. Indeed, they sweep an orbit by the conjugacy of Borel subgroups, the orbit is closed since it is an image of the projective variety G/B , and it is seen to be the unique closed orbit since any projective B-stable variety must have a B-fixed point. We call $G \cdot \langle v \rangle$ the <u>highest weight orbit</u> of $\mathcal{U}(\mathfrak{g}) \cdot v$ (the affine cone over $G \cdot \langle v \rangle$ is a "highest weight orbit" too, of course, but we won't need it).

This discussion proves the standard result

<u>Lemma 1.3</u> Let V be a finite dimensional G-representation space. Then the closed G-orbits in $\mathbb{P}(V)$ are precisely the highest weight orbits corresponding to the irreducible G-submodules of V . In particular, there are finitely many closed G-orbits in $\mathbb{P}(V)$ if and only if V decomposes into <u>non-isomorphic</u> irreducible G-submodules.

§3.2 THE CLOSED ORBITS ON $X_d(\mathfrak{g})$

Here, we state Kostant's result on the marvelous decomposition of $A_d(\mathfrak{g})$ (for any d) into irreducible $\mathfrak{g}$-representation spaces. Kostant assumed $k = \mathbb{C}$, but the results immediately follow for k algebraicly closed of characteristic 0 by the Lefschetz principle.

Theorem (Kostant [Ko2]). (1) The irreducible $\mathfrak{g}$- components of $A_d(\mathfrak{g})$ are all non-isomorphic as $\mathfrak{g}$-representation spaces. In particular, to enumerate the pieces, fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, and let $\{\mathfrak{a}_i\}_{i\in I}$ be the set of abelian d-dimensional ideals of $\mathfrak{b}$. Then I is finite and each cyclic $\mathfrak{g}$-module $\mathcal{U}(\mathfrak{g}) \cdot \Lambda^d\mathfrak{a}_i$ is irreducible with highest weight line, with respect to $\mathfrak{b}$, $\Lambda^d\mathfrak{a}_i$. The decomposition of $A_d(\mathfrak{g})$ is

$$A_d(\mathfrak{g}) = \bigoplus_{i\in I} \mathcal{U}(\mathfrak{g}) \cdot \Lambda^d\mathfrak{a}_i \, .$$

(2) $\mathbb{P}(A_d(\mathfrak{g})) \cap Gr_d(\mathfrak{g}) = X_d(\mathfrak{g})$.

Using Lemma 1.3, we immediately have

Corollary 2.1 The closed G-orbits in $\mathbb{P}(A_d(\mathfrak{g}))$ lie in $X_d(\mathfrak{g})$ and are precisely the G-orbits of the d-dimensional abelian ideals of a Borel subalgebra.

So the abelian ideals of $\mathfrak{b}$ are the most degenerate abelian subalgebra of $\mathfrak{a}$.

In §3.3 we will study degeneration of subspaces of $\mathfrak{g}$ along curves on the Grassmannian $Gr_d(\mathfrak{g})$. In §3.4 we will discuss abelian ideals of $\mathfrak{b}$ and find that certain of them are easily seen to be limits of tori (Prop. 4.4). Then in §3.5 the proof of the main result (Th. 5.1) proceeds by dealing rather explicitly with the types of simple Lie algebras, and then observing that the results (and the method, in fact) quite trivially pass to the semi-simple case.

§3.3 FIRST ORDER DEGENERATIONS ON $Gr_d(\mathfrak{g})$

One easy way to determine points in the closure of the G-invariant subset $Q_d(\mathfrak{g})$ is to close up U-orbits of points of $Q_d(\mathfrak{g})$, when U is an algebraic subgroup of G with $U \simeq \mathbb{G}_a$. Here $\mathbb{G}_a$ is the one-dimensional additive algebraic group, and $\mathbb{G}_a \simeq \mathbb{A}_k^1$ as varieties. For instance, if U is a one-dimensional group of unipotent elements in G , then we know $U \simeq \mathbb{G}_a$.

Recall the following geometric fact.

<u>Proposition</u> (see, for instance [H], I, 6.8). Let C be a non-singular curve, let P be a point of C , and let W be a projective variety. Then any morphism $C - \{P\} \xrightarrow{\Psi} W$ can be extended uniquely to a morphism $C \xrightarrow{\overline{\Psi}} W$.

<u>Remark</u> Of course such an extension does not exist in general when C has dimension greater than one, as then there are many tangent directions on C at P and different directions may lead to different points of W .

As $\mathbb{A}^1$ is $\mathbb{P}^1$ minus the point "∞" at infinity, the Prop. tells us we can define a unique limit point for $\mathbb{G}_a$ orbits on projective varieties.

Definition 3.1 Let the algebraic group U, with $U \simeq \mathbb{G}_a$, act (rationally, as always) on the projective variety W. Then for $w \in W$, define

$$\lim_{g \in U} g \cdot w = \overline{\Psi}(\infty) .$$

where $\overline{\Psi} : U \cup \{\infty\} \to W$ extends the orbit map

$$U \xrightarrow{\Psi} W \text{ by } g \mapsto g \cdot w .$$

Remarks 3.2 (1) We can be explicit about what these $\mathbb{G}_a$-orbits look like. The stabilizer of a point under and algebraic group action is always a closed subgroup, so the stabilizer of w above must be U itself, or just the identity. In the former case, the orbit is the single point w, so $\lim_{g \in U} g \cdot w = w$. In the latter case the orbit map is a bijective separable morphism of smooth varieties, hence, an isomorphism, so that $\lim_{g \in U} g \cdot w$ closes up the affine U-orbit.

(2) Let $\mathfrak{a}$ be a d-dimensional ideal of $\mathfrak{b}$, and U a subgroup of B with $U \simeq \mathbb{G}_a$. Then, on $Gr_d(\mathfrak{g})$,

$$\lim_{g \in U} g \cdot \mathfrak{a} = \mathfrak{a} .$$

This is obvious, because an algebraic group and its Lie algebra have the same invariant subspaces, so that $g \cdot \mathfrak{a} = \mathfrak{a}$ for all $g \in B$.

It is very easy to calculate limits under a unipotent group for the adjoint action, as the following lemma shows. In fact, for the Plücker embedding of the Grassmannian, the embedded curve $\overline{U \cdot L}$ is a very special one. So we first recall a geometric definition for curves embedded in $\mathbb{P}^n$.

Definition 3.3 A rational normal curve C in $\mathbb{P}^n$ is a curve of degree n which spans $\mathbb{P}^n$. If $C \hookrightarrow \mathbb{P}^m$ and if C is a rational normal curve in its span $\mathbb{P}^n$ in $\mathbb{P}^m$, then we will say that C is a rational normal curve of degree n in $\mathbb{P}^m$.

It is a geometric exercise to show that a rational normal curve C in $\mathbb{P}^n$ is the image of $\mathbb{P}^1$ under the embedding given by the complete linear system $|nP|$ of n points on $\mathbb{P}^1$, so that in particular, the rational normal curve in $\mathbb{P}^n$ is unique up to a linear automorphism of $\mathbb{P}^n$. Moreover, n is the least degree that a curve spanning $\mathbb{P}^n$ may have.

In homogeneous coordinates, the map $\mathbb{P}^1 \xrightarrow{|np|} \mathbb{P}^n$ is given (up to the action of $\mathbb{P}G\ell(n)$) by

$$[a,b] \mapsto [a^n, a^{n-1}b, \ldots, ab^{n-1}, b^n] .$$

So in particular

$$[1,b] \mapsto [1,b,\ldots,b^n] .$$

and this is the form in which we will be able to recognize rational normal curves. As usual, for any non-zero element z of $\mathfrak{g}$, we will let $\langle z\rangle$ denote the line in $\mathfrak{g}$ spanned by z, so that $\langle z\rangle$ is a point of $\mathbb{P}(\mathfrak{g})$.

<u>Lemma 3.4</u> Let U be a unipotent subgroup of G (i.e. U consists of unipotent elements) with $U \simeq \mathbb{G}_a$, and fix a non-zero element z of the Lie algebra $\mathcal{L}(U)$. Consider the action of U on $\mathbb{P}(\mathfrak{g})$ deduced from the adjoint action. Then, for a given $0 \neq v \in \mathfrak{g}$.

$$\lim_{g\in U} g \cdot \langle v\rangle = \langle (\mathrm{ad}\, z)^m \cdot v\rangle ,$$

where m is the least non-negative integer such that $(\mathrm{ad}\, z)^{m+1} \cdot v = 0$. When $m > 0$, then the closure $\overline{U \cdot \langle v\rangle} \simeq \mathbb{P}^1$ and $\overline{U \cdot \langle v\rangle}$ is embedded in $\mathbb{P}(\mathfrak{g})$ as a rational normal curve of degree m.

<u>Proof</u> The exponential map $\exp\colon \mathfrak{g} \to G$ is algebraic when restricted to the cone N of nilpotent elements in $\mathfrak{g}$ and gives an algebraic isomorphism of N with the unipotent variety of G. In particular, $\exp\colon \langle z\rangle = \mathcal{L}(U) \to U$ is an algebraic isomorphism.

By the functoriality of exp, the diagram below commutes for any rational representation $G \to \mathrm{Aut}\, V$:

$$\begin{array}{ccc} G & \longrightarrow & \operatorname{Aut} V \\ \exp \uparrow & & \uparrow \exp \\ \mathfrak{g} & \longrightarrow & \operatorname{End} V \end{array} ,$$

and we know that the right exponential map is given by familiar series

$$\exp(A) = e^A = 1 + A + \frac{A^2}{2!} + \ldots , \text{ for } A \in \operatorname{End} V .$$

For the adjoint representation $\operatorname{Ad}: G \to \operatorname{Aut} \mathfrak{g}$, then,

$$\operatorname{Ad}(\exp z) = \exp(\operatorname{ad} z) = e^{\operatorname{ad} z} .$$

Now with v and m as given, and with $t \in k$, we have

$$e^{\operatorname{ad}(tz)} \cdot v = v + t(\operatorname{ad} z) \cdot v + \frac{t^2}{2!}(\operatorname{ad} z)^2 \cdot v + \ldots + \frac{t^m}{m!}(\operatorname{ad} z)^m \cdot v ,$$

with each of the $(m + 1)$ terms non-zero when $t \neq 0$. (Note such a finite m exists because $\operatorname{ad} z$ is a nilpotent linear transformation.) So the orbit $U \cdot \langle v \rangle$ in $\mathbb{P}(\mathfrak{g})$ is

$$\{[1, t, t^2, \ldots, t^m, 0, \ldots, 0] \mid t \in k\} ,$$

where the homogeneous coordinates of $\mathbb{P}(\mathfrak{g})$ were chosen relative to an ordered basis of $\mathfrak{g}$ starting with

$$\{v, (\mathrm{ad}z) \cdot v, \ldots, \frac{(\mathrm{ad}z)^m}{m!} \cdot v\} .$$

So $\overline{U \cdot \langle v \rangle}$ is a rational normal curve of degree m and the closure was gotten by adding the point $\langle (\mathrm{ad}z)^m \cdot v \rangle$. □

This leads us to

Definition 3.5 Let U be a unipotent subgroup of G with $U \simeq \mathbb{G}_a$. If $0 \neq z \in \mathcal{L}(U)$ and $L \in Gr_d(\mathfrak{g})$, then define $\lim\limits_{t \to \infty} e^{tz} \cdot L$ to mean $\lim\limits_{g \in U} g \cdot L$. If also

$$e^{tz} \cdot L = \{v + tz \cdot v \mid v \in L\} \text{ , for all } t ,$$

and if $L \neq \lim\limits_{t \to \infty} e^{tz} \cdot L$, then call $\overline{U \cdot L}$ a first order degeneration of L to $\lim\limits_{t \to \infty} e^{tz} \cdot L$.

Remark 3.6 According to the above definition, $\overline{U \cdot L}$ is obviously a first order degeneration if $(\mathrm{ad}z)^2|_L \equiv 0$ while $(\mathrm{ad}z)|_L \not\equiv 0$. In fact, then for each point $\langle v \rangle$ of L , $\overline{U \cdot \langle v \rangle}$ is a linear space (either a point or a line). However, as we are interested in how things look on the Grassmannian, the definition had to be more general.

Indeed, consider $\mathfrak{g} = \mathfrak{sl}_4$ and let

$L = \langle X_\alpha, X_{\alpha+\beta+\gamma} \rangle$, where $\alpha = t_1 - t_2, \beta = t_2 - t_3$, and $\gamma = t_3 - t_4$

(cf. Example 4.5 for notation). Put $z = X_\beta + X_\gamma$. Then

$$e^{tz} \cdot X_\alpha = X_\alpha - tX_{\alpha+\beta} + \frac{t^2}{2} X_{\alpha+\beta+\gamma}$$

$$e^{tz} \cdot X_{\alpha+\beta+\gamma} = X_{\alpha+\beta+\gamma} .$$

So under e^{tz}, X_α moves on a conic, but

$$e^{tz} \cdot L = \langle X_\alpha - tX_{\alpha+\beta}, X_{\alpha+\beta+\gamma} \rangle$$

and the degeneration is first order.

§3.4 THE ABELIAN IDEALS OF

Now we turn to study the abelian ideals of a Borel subalgebra $\mathfrak{b}$, since these are the subalgebras to which we want to degenerate tori.

Lemma 4.1 [Ko2]. Each abelian ideal of $\mathfrak{b}$ is a span of root vectors, relative to the root system $\Phi(\mathfrak{g},\mathfrak{t})$ resulting from any choice of a maximal torus $\mathfrak{t}$ in $\mathfrak{b}$.

Proof Let $\mathfrak{a}$ be an ideal of $\mathfrak{b}$. Then in particular $\mathfrak{a}$ is $\mathfrak{t}$-stable, so $\mathfrak{a}$ is the span of $\mathfrak{a} \cap \mathfrak{t}$ and some root spaces $\mathfrak{g}^{\alpha}$, with $\alpha \in \Phi(\mathfrak{g},\mathfrak{t})$. If $\mathfrak{a}$ is also abelian, then it follows that $\mathfrak{a} \cap \mathfrak{t} = \{0\}$. Indeed, suppose some $0 \neq h \in \mathfrak{a} \cap \mathfrak{t}$. Then $\beta(h) \neq 0$ for some root β . So $[h,X_\beta] = \beta(h)X_\beta \in \mathfrak{a}$. But this is absurd since h and X_β do not commute. □

Notation 4.2 Let $\mathfrak{a}$ be a subalgebra of $\mathfrak{b}$ which is a span of root vectors. Then $R(\mathfrak{a})$ will denote the subset of Φ such that

$$\mathfrak{a} = \langle X_\alpha \mid \alpha \in R(\mathfrak{a})\rangle .$$

Remark 4.3 Just as the nilpotent radical $\mathfrak{n}$ of $\mathfrak{b}$ is the span of all the positive root vectors relative to any choice of $\mathfrak{t} \subset \mathfrak{b}$, the lemma shows the abelian ideals of $\mathfrak{b}$ are distinguished subspaces of $\mathfrak{m}$ which are spans of certain subsets of positive root vectors for any choice of $\mathfrak{t} \subset \mathfrak{b}$.

Using the methods of the last section, we can immediately show that certain of the abelian ideals of are limits of tori. In fact, the next proposition says more.

From now on we fix a choice of maximal torus $\mathfrak{t}$ in a Borel subalgebra $\mathfrak{b}$, and form the resulting partially ordered root system $\Phi = \Phi(\mathfrak{g},\mathfrak{t})$, with base $\Delta = \Delta(\mathfrak{g},\mathfrak{t})$.

<u>Proposition 4.4</u> [Kostant] If the abelian subalgebra $\mathfrak{a}$ of $\mathfrak{b}$ is a span of root vectors $X_{\alpha_1},\ldots,X_{\alpha_d}$ such that the roots $\alpha_1,\ldots,\alpha_d$ are linearly independent in $\mathfrak{t}^\vee$, then $\mathfrak{a}$ is a limit of d-dimensional tori (i.e. $\mathfrak{a} \in \overline{Q_d(\mathfrak{g})}$) via a first order degeneration on $Gr_d(\mathfrak{g})$.

<u>Proof</u> Put $z = \sum_{i=1}^{d} X_{\alpha_i}$. We will degenerate by the subgroup generated by $\exp(z)$.

Restricted to $\mathfrak{t}$, $\exp(tz) = \mathrm{id} + t\,\mathrm{ad}z$, for $t \in k$. I.e., for $h \in \mathfrak{t}$,

$$\exp(tz)\cdot h = e^{\mathrm{ad}(tz)}\cdot h = h - t\sum_{i=1}^{d}\alpha_i(h)X_{\alpha_i},$$

since the commutativity of the X_{α_i} causes the higher powers of $(\mathrm{ad}z)$ to die on h. So the degeneration of subspaces of $\mathfrak{t}$ by $\exp(tz)$ is first order.

Now set

$$\mathfrak{t}_0 = \{h \in \mathfrak{t} \mid \alpha_i(h) = 0 \text{ for all } i = 1 \text{ to } d\} .$$

Then

$$(*) \quad \lim_{t\to\infty} \exp(tz)\cdot h = \begin{cases} h & \text{if } h \in \mathfrak{t}_0 \\ -\Sigma\, \alpha_i(h) X_{\alpha_i} & \text{if } h \notin \mathfrak{t}_0 \end{cases} .$$

In particular, pick a complement $\mathfrak{t}_1$ to $\mathfrak{t}_0$ in $\mathfrak{t}$, so $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1$. Then $\mathfrak{t}_1$ is a d-dimensional torus, and the restrictions of the α_i to $\mathfrak{t}_1$ are still linearly independent. So the above limit calculation $(*)$ implies that each $X_{\alpha_i} \in \lim_{t\to\infty} \exp(tz)\cdot\mathfrak{t}_1$. Now the inclusion $\mathfrak{a} \subset \lim_{t\to\infty} \exp(tz)\cdot\mathfrak{t}_1$ must be an equality since the reverse inclusion follows automatically from $(*)$ just because $\mathfrak{t}_1 \cap \mathfrak{t}_0 = \{0\}$. □

<u>Example (of Prop.) 4.5</u> Let $\mathfrak{g} = \mathfrak{sl}_{\ell+1}$, and let $\mathfrak{t}$ and $\mathfrak{b}$ be the diagonal matrices and the upper triangular matrices, respectively. Let t_i be the linear functional on $\mathfrak{t}$ which just picks out the ith diagonal entry. Then $\Phi^+ = \{t_i - t_j \mid 1 \le i<j \le \ell + 1\}$ and $X_{t_i-t_j}$ is the standard matrix $e_{i,j}$.

Suppose $\ell = 3$, $d = 2$, and $\alpha_1 = t_1 - t_3$, $\alpha_2 = t_1 - t_4$. Then

$$\mathfrak{t}_0 = \left\{ \begin{pmatrix} a & & & \\ & -3a & & \\ & & a & \\ & & & a \end{pmatrix} \middle| \ a \in k \right\},$$

and we may choose

$$\mathfrak{t}_1 = \left\{ \begin{pmatrix} a & & & \\ & 0 & & \\ & & b & \\ & & & c \end{pmatrix} \middle| \ \begin{matrix} a + b + c = 0, \\ a , b , c \in k \end{matrix} \right\},$$

for instance. Now

$$\exp(tz) \cdot \begin{pmatrix} a & & & \\ & 0 & & \\ & & b & \\ & & & c \end{pmatrix} = \begin{pmatrix} 1 & 0 & t & t \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \begin{pmatrix} a & & & \\ & 0 & & \\ & & b & \\ & & & c \end{pmatrix} \begin{pmatrix} 1 & 0 & -t & -t \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 & (b-a)t & (c-a)t \\ & 0 & 0 & 0 \\ & & b & 0 \\ & & & c \end{pmatrix} .$$

As $a = b = c$ does not occur, the limit as $t \to \infty$ is

$$\begin{pmatrix} 0 & 0 & (b-a) & (c-a) \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} .$$

So the limit of $\mathfrak{t}_1$ is $\mathfrak{a} = \langle X_{\alpha_1}, X_{\alpha_2} \rangle$.

Remarks 4.6 (1) Kostant proved the proposition by using the linear independence of the α_i to form a dual basis $H_1,\ldots,H_d$ of a subspace of $\mathfrak{t}$, i.e. $\alpha_i(H_j) = \delta_{i,j}$. Then, directly using the action of G on $\Lambda^d\mathfrak{g}$, we find

$$\exp(tz) \cdot (H_1 \wedge\ldots\wedge H_d) = (H_1 - tX_{\alpha_1}) \wedge\ldots\wedge (H_d - tX_{\alpha_d}) ,$$

so that the limit as $t \to \infty$ is $\pm X_{\alpha_1} \wedge \ldots \wedge X_{\alpha_d}$.

(2) King [Ki] proved the result (cf. Remarks 1.2(2)) that the linear span $\mathcal{U}(\mathfrak{g}) \cdot \Lambda^d\mathfrak{t}$ of the d-dimensional tori must contain $\mathfrak{a}$. He did this by first showing that one may choose linearly independent elements $H_1,\ldots,H_d$ of $\mathfrak{t}$ such that the determinant of the $d \times d$ matrix $[\alpha_i(H_j)]$ is non-zero. Then he calculated

$$(X_{\alpha_d}\cdots X_{\alpha_1}) \cdot (H_1 \wedge\ldots\wedge H_d) = (-1)^d \det[\alpha_i(H_j)] X_{\alpha_1} \wedge\ldots\wedge X_{\alpha_d} .$$

where $X_{\alpha_d}\cdots X_{\alpha_1} \in \mathcal{U}^d(\mathfrak{g})$.

(3) Certainly the roots corresponding to an abelian ideal $\mathfrak{a}$ of $\mathfrak{b}$ need not be linearly independent. (See Example 4.7 immediately following and the end of Example 4.10.)

(4) Much of the proof of Prop. 4.4, carries thru when we drop the assumption that $\alpha_1,\ldots,\alpha_d$ are linearly independent. Indeed $\exp(tz)$ is unchanged and the limit calculation (*) in the proof is still valid.

The difference is that now $\dim\mathfrak{t}_0 \geq (\ell - d)$, so that $\dim\mathfrak{t} \leq d$. In fact, it follows immediately from the proof that if there are exactly r independent linear relations among the α_i, then $\dim\mathfrak{t}_1 = (d - r)$, and $\lim_{t\to\infty} \exp(tz)\cdot\mathfrak{t}_1$ is the $(d - r)$-dimensional subspace $\mathfrak{a}'$ of $\mathfrak{a}$ given by

$$\mathfrak{a}' = \{\Sigma c_i X_{\alpha_i} \mid \Sigma p_i c_i = 0 \text{ if } \Sigma p_i \alpha_i \equiv 0 \text{ on } \mathfrak{t},\ c_i \in k\}.$$

Moreover, if $\mathfrak{t}'$ is any d-dimensional torus in $\mathfrak{t}$ containing a complement to $\mathfrak{t}_0$, then $\lim_{t\to\infty} \exp(tz)\cdot\mathfrak{t}' = (\mathfrak{t}' \cap \mathfrak{t}_0) \oplus \mathfrak{a}'$. It turns out that we can perform a couple of <u>more</u> first order degenerations on $(\mathfrak{t}' \cap \mathfrak{t}_0) \oplus \mathfrak{a}'$ which leave $\mathfrak{a}'$ in $\mathfrak{a}$ and carry $(\mathfrak{t}' \cap \mathfrak{t}_0)$ into $\mathfrak{a}$ in such a way that the final limit is $\mathfrak{a}$. This is the philosophy of the proof of Th. 5.1. (Actually, we will partition $R(\mathfrak{a})$ into subsets of independent roots, and then degenerate to the corresponding subspaces of separately.)

Example 4.7 $\mathfrak{g} = \mathfrak{sl}_5$, $d = \ell = 4$, and

$$R(\mathfrak{a}) = \{t_1 - t_4, t_1 - t_5, t_2 - t_4, t_2 - t_5\} ,$$

where we keep that notation of Example 4.5. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ denote these roots in the order in which they were listed. Then they have the single relation $\alpha_1 + \alpha_4 = \alpha_2 + \alpha_3$, and

$$\mathfrak{t}_o = \left\{ \begin{pmatrix} a & & & & \\ & a & & & \\ & & -4a & & \\ & & & a & \\ & & & & a \end{pmatrix} \middle| \ a \in k \right\} .$$

When $z = \sum_{i=1}^{4} X_{\alpha_i}$ as usual, we get

$$\lim_{t\to\infty} \exp(tz) \cdot \mathfrak{t} = \mathfrak{t}_o \oplus \{ \sum_{i=1}^{4} c_i X_{\alpha_i} \mid c_i \in k \text{ and } c_1 + c_4 = c_2 + c_3 \} .$$

So

$$\mathfrak{a}' = \left\{ \begin{pmatrix} 0 & 0 & 0 & w & x \\ & 0 & 0 & y & z \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix} \middle| \ w + z = x + y \right\} .$$

To get $\mathfrak{a}$ as a limit of tori, consider $\beta = t_2 - t_3$. Then $[X_\beta, \mathfrak{a}'] = 0$ and $\beta|_{\mathfrak{t}_o} \not\equiv 0$, so that e^{tX_β} leaves $\mathfrak{a}'$ stable and

$$\lim_{t\to\infty} e^{tX_\beta} \cdot (\mathfrak{t}_o \oplus \mathfrak{a}') = \langle X_\beta \rangle \oplus \mathfrak{a}' .$$

Finally, consider $\gamma = t_3 - t_4$ and degenerate by e^{tX_γ}. Again, $\mathfrak{a}'$ is left stable, so we see

$$\lim_{t\to\infty} e^{tX_\gamma} \cdot (\langle X_\beta\rangle \oplus \mathfrak{a}') = \langle X_{\beta+\gamma}\rangle \oplus \mathfrak{a}' = \mathfrak{a}.$$

Note that the last two degenerations could not be replaced by the single degeneration $e^{tX_{\beta+\gamma}}$, since $\beta + \gamma \in R(\mathfrak{a})$ so that $\beta + \gamma$ dies on $\mathfrak{t}_0$. So to degenerate $\mathfrak{t}_0$ to $\langle X_{\beta+\gamma}\rangle$, we had to "travel" from the 0-weight space of $\mathfrak{g}$ to the $(\beta + \gamma)$-weight space by way of the intermediate β-weight space (or, just as well, we could have first degenerated $\mathfrak{t}_0 \oplus \mathfrak{a}'$ by e^{tX_γ}).

To generalize this method, it is better to replace z in our very first degeneration by e^{tz}: instead of $z = \sum_{i=1}^{4} X_{\alpha_i}$, put $z = X_{\alpha_1} + X_{\alpha_2} + X_{\alpha_4}$ ($\langle X_{\alpha_1}, X_{\alpha_2}, X_{\alpha_4}\rangle$ is the largest sub-$\mathfrak{b}$-ideal of $\mathfrak{a}$ spanned by independent root vectors.) Then degenerate by e^{tX_β} and e^{tX_γ} just as before.

Actually, in the proof of Th. 5.1, we will partition $R(\mathfrak{a})$ slightly differently (since then it seems to be easier to write everything down).

From the proposition, it is now clear that we need to understand the set of roots $R(\mathfrak{a})$ for each abelian ideal $\mathfrak{a}$. Actually we will end up pretty much transferring the whole problem to the root system considered as a partially ordered set.

We will be using the following notions from order theory. The books [A] and [Bi] and the thesis [W] are good references.

Definition 4.8 Let $\{S, \leq\}$ be a partially ordered set (a poset).

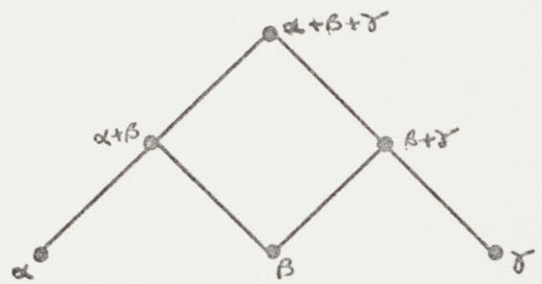
(1) A subset I of S is an upper (respectively, lower) ideal of S if for all $x \in I$ and $y \in S$, we have $x \leq y \Rightarrow y \in I$ (respectively, $x \geq y \Rightarrow y \in I$).

(2) A chain in S is a subset in which every two elements are comparable. We say x covers y in S, for $x, y \in S$, if $x > y$ and there exists no $z \in S$ such that $x > z > y$.

(3) The Hasse diagram of S is a diagram made up of dots and lines which specifies all the elements and relations of S. Specifically, a dot is drawn for each element of S, with each element placed higher than the ones it covers. Next a line is drawn from x down to y whenever x covers y.

(4) The poset S is <u>graded of degree n</u> if all maximal chains of S have n elements. Then we can define the <u>degree</u> $\deg(x)$ of any x to be its position from the bottom in any maximal chain thru x (with minimal elements being assigned degree 1 , etc.).

For example, the poset of positive roots of $\mathfrak{sl}_4$ is graded of degree 3 and has Hasse diagram



Note that the term "rank" is usually used in place of "degree", but we will use "degree" to avoid later confusion with the rank of the root system.

<u>Lemma 4.9</u> (1) Let $\mathfrak{a}$ be an ideal of $\mathfrak{b}$ with $\mathfrak{a} \subset \mathfrak{m}$ = nilpotent radical of $\mathfrak{b}$. Then $R(\mathfrak{a})$ is an upper ideal of Φ^+ .

(2) The poset Φ^+ is graded, and the degree of an element $\varphi = \sum_{\alpha\in\Delta} c_\alpha \alpha$ is just its height $ht(\varphi) = \sum_{\alpha\in\Delta} c_\alpha$.

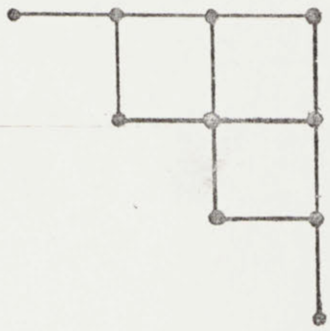
<u>Proof</u> (1) This is obvious, since $[\mathfrak{g}^\alpha,\mathfrak{g}^\beta] \subseteq \mathfrak{g}^{\alpha+\beta}$ for any α , $\beta \in \Phi$.

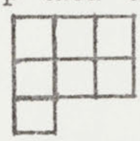
(2) I would like to thank Dave Vogan for telling me this proof. It suffices to show that φ_1 covers $\varphi_2 \Rightarrow \varphi_1 - \varphi_2 \in \Delta$. Let $(,)$ be the Killing form from $\mathfrak{t}$ transferred to $\mathfrak{t}^\vee$. Recall that for any 2 non-proportional roots α and β, $(\alpha,\beta) > 0 \Rightarrow \alpha - \beta \in \Phi$, while $(\alpha,\beta) < 0 \Rightarrow \alpha + \beta \in \Phi$.

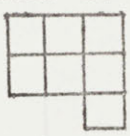
Let φ and $\varphi + \beta$ be positive roots with $\beta \in \Phi^+$, but $\beta \notin \Delta$. Since $(\beta,\beta) > 0$, there exists a simple root α such that $(\beta,\alpha) > 0$. Then β not simple $\Rightarrow \beta - \alpha$ is a root. Now if $(\varphi + \beta,\alpha) > 0$ then $\varphi + \beta - \alpha \in \Phi$ so that $\varphi < \varphi + \beta - \alpha < \varphi + \beta$ is a chain in Φ^+. Otherwise, if $(\varphi + \beta,\alpha) \leq 0$, then $(\varphi,\alpha) < 0$ so that $\varphi + \alpha \in \Phi$ and $\varphi < \varphi + \alpha < \varphi + \beta$ is a chain in Φ^+. □

<u>Example 4.10</u> For $\mathfrak{g} = \mathfrak{sl}_{\ell+1}$, the upper ideals of Φ^+ have a familiar pictorial representation (which is indicative of the general case). With our standard choice of $\mathfrak{t} \subset \mathfrak{b}$ (see Example 4.5), the matrix entries strictly above the diagonal correspond to positive root vectors, and hence to roots. Indeed, if we draw in horizontal and vertical lines, then we get the Hasse diagram for Φ^+ (granted, drawn at a slightly strange angle).

For instance, for $\mathfrak{sl}_5$ we get



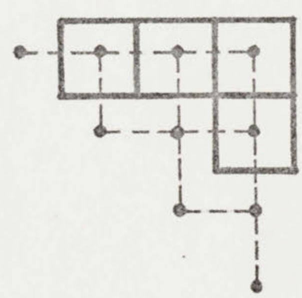
Now recall Ferrer's diagrams are block diagrams representing non-increasing integer partitions. They are usually drawn justified to the top and to the left, so that, say, (3,3,1) is drawn  .

However we will justify them to the top and to the right, so they look like  .

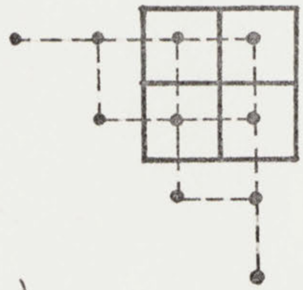
The point is that ideals $\mathfrak{a}$ of $\mathfrak{b}$, with $\mathfrak{a} \subset \mathfrak{m}$, correspond precisely to these Ferrer's diagrams drawn on the Hasse diagram for Φ^+ , with each box of the Ferrer's diagram enclosing the nodes Φ^+ corresponding to the roots in $R(\mathfrak{a})$. In particular, then, the number of boxes in the Ferrer's diagram is equal to the dimension of $\mathfrak{a}$.

Thus, in $\mathfrak{sl}_5$,

$\mathfrak{a} = \langle X_{t_1-t_3}, X_{t_1-t_4}, X_{t_1-t_5}, X_{t_2-t_5} \rangle \leftrightarrow$



and

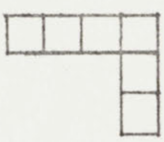
$$\mathfrak{a} = \langle X_{t_1-t_4}, X_{t_1-t_5}, X_{t_2-t_4}, X_{t_2-t_5} \rangle \leftrightarrow$$  .

(The latter is the ideal of Example 4.7.)

One can check quite easily that, for $\mathfrak{g} = \mathfrak{sl}_{\ell+1}$,

$d \leq \ell$,

(1) every d-dimensional ideal $\mathfrak{a}$ of $\mathfrak{b}$ with $\mathfrak{a} \subset \mathfrak{m}$ is abelian, and

(2) for an abelian d-dimensional ideal $\mathfrak{a}$ of $\mathfrak{b}$, $R(\mathfrak{a})$ consists of linearly independent roots iff the corresponding Ferrer's diagram is "L-shaped", i.e. looks like  .

§3.5 PROOF OF THE THEOREM

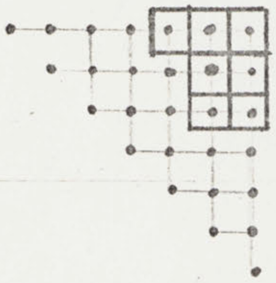
This section is devoted to proving

Theorem 5.1 Let G be a connected semi-simple algebraic group with Lie algebra $\mathfrak{g}$, of rank ℓ. Then for $d \leq \ell$, all the d-dimensional abelian ideals $\mathfrak{a}$ of any Borel subalgebra $\mathfrak{b}$ are limits in $Gr_d(\mathfrak{g})$ of d-dimensional tori of $\mathfrak{g}$, (i.e., such $\mathfrak{a}$ lie in $\overline{Q_d(\mathfrak{g})}$). Moreover, these limits can be arrived at thru a sequence of order one degenerations on the Grassmannian.

The proof is in 3 steps: the case where $\mathfrak{g}$ is a classical simple Lie algebra, the case where $\mathfrak{g}$ is an exceptional simple Lie algebra, and then passage to the case of $\mathfrak{g}$ semi-simple.

Proposition 5.2 The assertion of Th. 5.1, is true when $\mathfrak{g}$ is a classical simple Lie algebra, i.e. when $\mathfrak{g}$ is simple of type $A_\ell(\ell \geq 1)$, $B_\ell(\ell \geq 2)$, $C_\ell(\ell \geq 3)$, or $D_\ell(\ell \geq 4)$.

Idea of proof. (see also example 4.7) We will describe the method for $\mathfrak{g} = \mathfrak{sl}_{\ell+1}$. Recall the Ferrer's diagram representation of an abelian ideal $\mathfrak{a}$ of $\mathfrak{b}$ (example 4.10). For example,



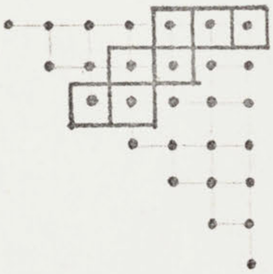
corresponds to a 7-dimensional ideal $\mathfrak{a}_0$ for $\mathfrak{g} = \mathfrak{sl}_8$. Each element of $R(\mathfrak{a})$ corresponds to a box of the Ferrer's diagram. Now let $R_i(\mathfrak{a})$ denote the ith row of $R(\mathfrak{a})$, so that $R(\mathfrak{a}) = R_1(\mathfrak{a}) \cup R_2(\mathfrak{a}) \cup \ldots$ is a partition of $R(\mathfrak{a})$.

Next inductively form a new block configuration $\widetilde{R}(\mathfrak{a})$ by starting at the top and working down as follows.

(1) Put $\widetilde{R}_1(\mathfrak{a}) = R_1(\mathfrak{a})$.

(2) Having defined $\widetilde{R}_{i-1}(\mathfrak{a})$, slide the ith row of $R(\mathfrak{a})$ left horizontally until it rightmost block is directly under the leftmost block of $\widetilde{R}_{i-1}(\mathfrak{a})$. Call this new ith row $\widetilde{R}_i(\mathfrak{a})$.

(3) Put $\widetilde{R}(\mathfrak{a}) = \widetilde{R}_1(\mathfrak{a}) \cup \widetilde{R}_2(\mathfrak{a}) \cup \ldots$. For instance, in our example in $\mathfrak{sl}_8$, $\widetilde{R}(\mathfrak{a}_0)$ is given by



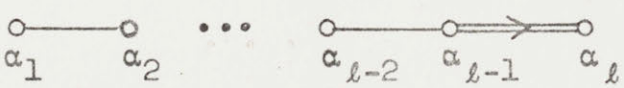
The point of this procedure is that $\widetilde{R}(\mathfrak{a})$ is a set of linearly independent roots and the corresponding root vectors all commute (actually we will only need the latter within each row of $\widetilde{R}(\mathfrak{a})$). So we apply Prop. 4.4 for the root set $\widetilde{R}(\mathfrak{a})$, and then it turns out we can perform obvious first order degenerations which move the resulting subalgebra over to $\mathfrak{a}$, as in Example 4.7 . Actually, in the proof of 5.1 we will do these degenerations row by row.

In the proof, then, we need to generalize the notions of the row decomposition (we will call it a <u>layer</u> decomposition) and of "sliding left" (a lowering operator on poset) to the other classical cases.

<u>Proof of Prop. 5.2</u> Each of the classical simple root systems has an almost canonical ordering of its simple roots. Fix this ordering in the usual way, as indicated by the following Dynkin diagrams. Here $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ is the base of Φ corresponding to $\mathfrak{b}$. Also included is the expression for the highest root λ .

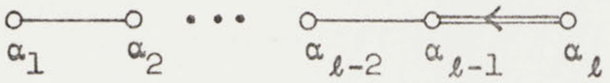
$A_\ell (\ell \geq 1)$ 

$\lambda = \alpha_1 + \ldots + \alpha_\ell$

$B_\ell (\ell \geq 2)$ 

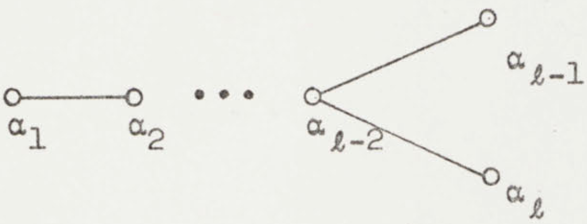
$\lambda = \alpha_1 + 2(\alpha_2 + \ldots + \alpha_\ell)$

$C_\ell (\ell \geq 3)$



$\lambda = 2(\alpha_1 + \ldots + \alpha_{\ell-1}) + \alpha_\ell$

$D_\ell (\ell \geq 4)$

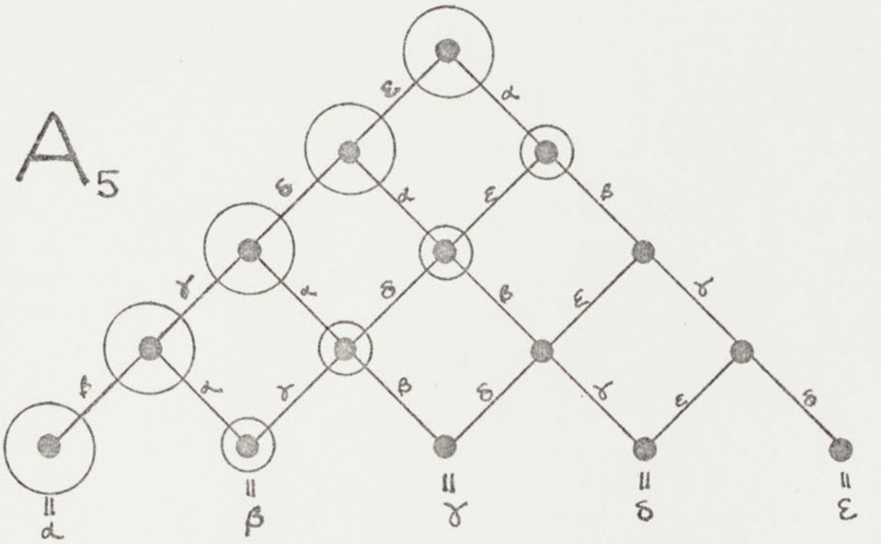


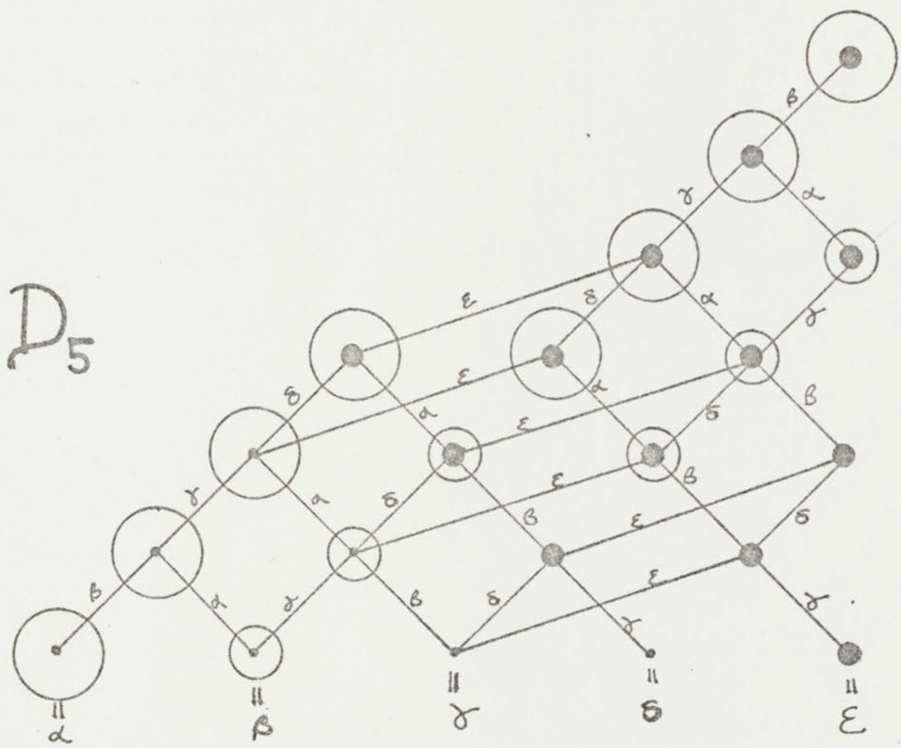
$\lambda = \alpha_1 + 2(\alpha_2 + \ldots + \alpha_{\ell-2}) + \alpha_{\ell-1} + \alpha_\ell$.

First we will partition the set Φ^+ of positive roots into <u>layers</u> Λ_i , for $i = 1$ to ℓ , as follows. Each root $\varphi \in \Phi^+$ can be uniquely written as $\varphi = \sum_{i=1}^{\ell} c_i \alpha_i$ with each c_i a non-negative integer (this is what it means for Δ to be a base). The <u>support</u> of φ , which is denoted by supp φ , is the set of simple roots α_i for which $c_i \neq 0$. Define the <u>layers</u> Λ_i inductively for $i = 1$ to ℓ by

$$\Lambda_i = \{\varphi \in \Phi^+ \mid \varphi \not\in \bigcup_{j<i} \Lambda_j \text{ , and } \alpha_i \in \operatorname{supp} \varphi\} \text{ .}$$

Obviously these "layers" form a partition of Φ^+ . The following diagrams indicate the layer decompositions for the four types.





KEY:

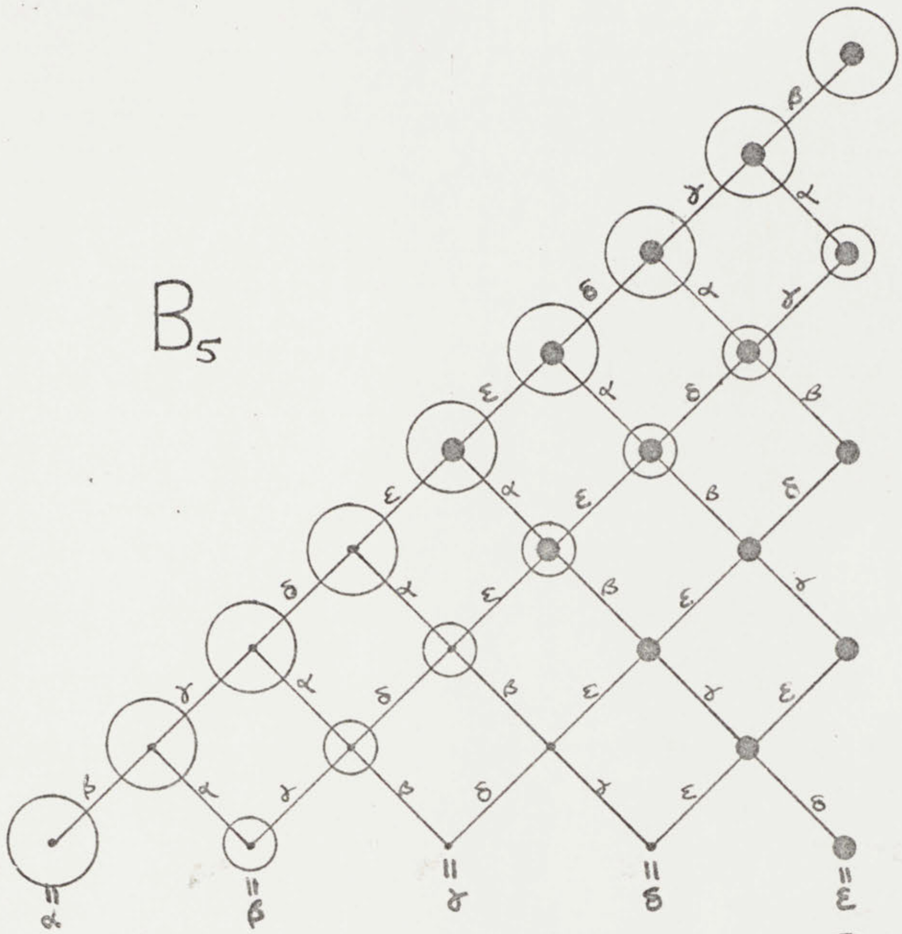
◯ surrounds roots in Λ_1

○ surrounds roots in Λ_2

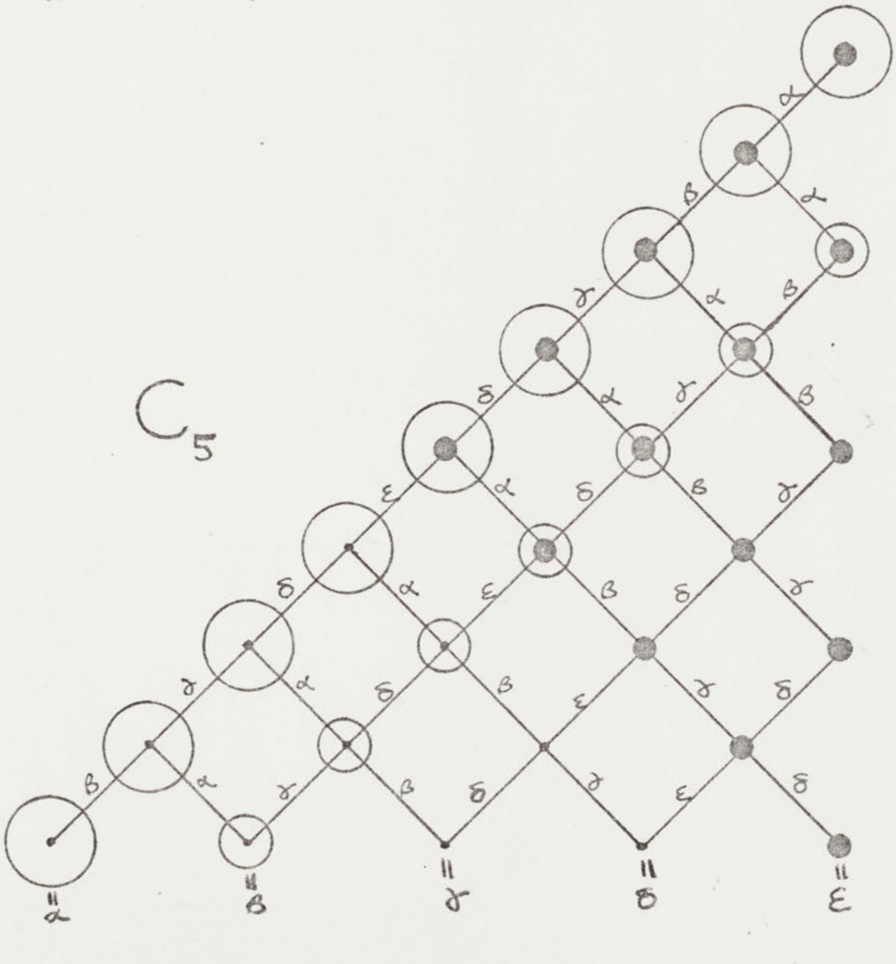
● denotes roots in $\Lambda^* = \bigcup_i \Lambda_i^*$

$\alpha = \alpha_1, \ \beta = \alpha_2, \ \gamma = \alpha_3, \ \delta = \alpha_4, \ \varepsilon = \alpha_5$

B_5



C_5



These layers arise in the following way. Consider the filtration of the root system Φ given by

$$\Phi = \Phi_1 \supset \Phi_2 \supset \dots \supset \Phi_\ell \text{ , where}$$

Φ_i = subsystem of Φ generated by the simple roots $\alpha_i,\dots,\alpha_\ell$.

So Φ_i is a root system of rank $(\ell - i + 1)$ whose Dynkin diagram is just the diagram of Φ with the first $(i - 1)$ nodes (and the lines attached to them) removed. Looking at the four classical Dynkin diagrams, we see immediately that each Φ_i is irreducible and of the same type as Φ (include the redundant forms D_3 , B_2 , etc.), except when $i = \ell - 2$ and Φ is of type D_ℓ (in which case, Φ_i is $A_1 \times A_1$) . Clearly, the complement of Φ^+_{i+1} in Φ^+_i is Λ_i .

What we are interested in is the poset structure of the Λ_i . Obviously

$$\Lambda_i = \{\varphi \in \Phi^+ \mid \varphi \notin \bigcup_{j<i} \Lambda_j \text{ and } \varphi \geq \alpha_i\} ,$$

so that Λ_i is an upper ideal of Φ^+_i . And $\Lambda_i \hookrightarrow \Phi_i \hookrightarrow \Phi^+$ are inclusions of <u>graded</u> subposets. When Φ_i

is an irreducible root system, Φ_i^+ has a <u>highest root</u> <u>λ_i</u>, which is thus the largest element of Λ_i. In fact, in types A_ℓ, B_ℓ and C_ℓ, all the Λ_i are totally ordered.

One easily verifies that for types B_ℓ, C_ℓ, and D_ℓ, the poset Λ_1 is just "two copies of the Dynkin diagram stuck together at the ends". For example, in D_5, Λ_1 is  which is two copies of .

(This is clearer if one considers the additive structure.) Correspondingly, for A_ℓ, Λ_1 is just one copy of the Dyknin Diagram (with diagram turned upside down compared to the previous cases). Note that this description of Λ_1 says that the only ℓ-dimensional upper ideal of Λ_1 is "the top copy of the Dynkin diagram". Actually, the "top copy" is where we want to work, so define

$$\Lambda_i^* = \left\{ \varphi \in \Lambda_i \;\middle|\; \begin{array}{l} ht(\lambda_i)-ht(\varphi) \leq (\ell-i+1)-1 \text{ if } \Phi \text{ type } A_\ell, B_\ell, C_\ell, \\ ht(\lambda_i)-ht(\varphi) \leq (\ell-i+1)-2 \text{ if } \Phi \text{ type } D_\ell \end{array} \right\}.$$

Then Λ_i^* is the "top copy".

Now the consecutive differences of elements in a given Λ_i^* are different simple roots and supp $\lambda_i = \{\alpha_i,\ldots,\alpha_\ell\}$, so we see the roots in a Λ_i^* are <u>linearly independent</u>. Also, each Λ_i^* is a set of roots with the property:

(1) $$\varphi_1,\varphi_2 \in \Lambda_i^* \Rightarrow \varphi_1 + \varphi_2 \notin \Phi ,$$

(so that the corresponding root vectors commute).
Indeed, in types A_ℓ, B_ℓ, D_ℓ, the highest root contains α_1 just once so there even Λ_1 has this property (1). For type C_ℓ , property (1) follows from considering the height function, as $\varphi_1,\varphi_2 \in \Lambda_i^* \Rightarrow ht(\varphi_1) + ht(\varphi_2) > ht(\lambda_i)$. (For type C_ℓ , $\varphi_1 = \alpha_1 + \ldots + \alpha_{\ell-1}$ and $\varphi_2 = \alpha_1 + \ldots + \alpha_\ell$ are two roots in Λ_1 such that $\varphi_1 + \varphi_2 = \lambda_1$.)

For each i (except $i = \ell$ in type D_ℓ), we have an inclusion of graded posets $\Lambda_{i-1} \hookrightarrow \Lambda_i$, by $\varphi \to \varphi + \alpha_i$. We can also define a lowering operator (graded, of degree -1)

$$\theta : (\Lambda_i^* \text{ with its minimal elements deleted}) \to \Lambda_i^* ,$$

by putting $\theta(\varphi)$ = the element of Λ_i^* which φ covers in the partial order. This is defined everywhere except at $\varphi = \lambda_i - \alpha_{i+1} - \cdots - \alpha_{\ell-2}$ ("fork in $D_{\ell+i-1}$") for type D_ℓ, so there we put $\theta(\varphi) = \lambda_i - \alpha_{i+1} - \cdots - \alpha_{\ell-1}$. These operators θ on the layers are compatible with the layer inclusions, i.e.

$$\varphi \in \Lambda_i^*\ ,\ \varphi \text{ not minimal in } \Lambda_i^* \Rightarrow \theta(\varphi) - \alpha_i = \theta(\varphi - \alpha_i)\ .$$

We will be using one more fact about the Λ_i^* (which is obvious from the Dynkin diagram description): if φ_1 and φ_2 are in Λ_i^* with $\varphi_1 > \varphi_2$ then $\varphi_1 - \varphi_2$ is an element of Φ^+.

Now we can proceed with the degenerations. Let $\mathfrak{a}$ be an abelian d-dimensional $(d \leq \ell)$ ideal of $\mathfrak{b}$, so that $R(\mathfrak{a})$ is an upper ideal of Φ^+. We first want to replace $R(\mathfrak{a})$ by a set of linearly independent roots $\widetilde{R}(\mathfrak{a})$.

Put $R_i(\mathfrak{a}) = R(\mathfrak{a}) \cap \Lambda_i$ and r_i = cardinality of $R_i(\mathfrak{a})$. Easily $R_i(\mathfrak{a}) \subset \Lambda_i^*$ and $R_i(\mathfrak{a})$ is an upper ideal of Λ_i. In particular, then, $R_i(\mathfrak{a})$ is a set of independent roots such that the corresponding root vectors commute. For future use, put $S_i(\mathfrak{a}) = \bigcup_{j \leq i} R_j(\mathfrak{a})$ and $s_i = r_1 + \cdots + r_i$.

If $R(\mathfrak{a}) = R_1(\mathfrak{a})$, then we are done by Prop. 4.4. So assume not. Then it follows that, in type D_ℓ, at most <u>one</u> of the two minimal elements of Λ_1^* lies in $R_1(\mathfrak{a})$. Even more, it follows, since $R(\mathfrak{a})$ is an upper ideal of Φ^+ of cardinality less than or equal to ℓ , that $R(\mathfrak{a})$ and the set $\widetilde{R}(\mathfrak{a})$ which we are about to construct all lie in just one of the root systems {root system generated by $\alpha_1,\ldots,\alpha_{\ell-2},\alpha_{\ell-1}$} or {root system generated by $\alpha_1,\ldots,\alpha_{\ell-2},\alpha_\ell$} . The point is that it is unnecessary in what follows to make special arguments for type D_ℓ when we want to choose least elements, etc.- the bad cases just don't arise.

Put $\widetilde{R}_1(\mathfrak{a}) = R_1(\mathfrak{a})$ and let μ_1 = least element of $R_1(\mathfrak{a})$. Now we want to "slide down" $R_2(\mathfrak{a})$ along Λ_2 . Specifically, if Φ is of type A_ℓ, B_ℓ , or D_ℓ , then put

$$\widetilde{R}_2(\mathfrak{a}) = \{\mu_1 - \alpha_1, \theta(\mu_1 - \alpha_1), \ldots, \theta^{r_2-1}(\mu_2 - \alpha_1)\} .$$

If Φ is of type C_ℓ , then put

$$\widetilde{R}_2(\mathfrak{a}) = \{\theta(\mu_1 - \alpha_1), \ldots, \theta^{r_2}(\mu_1 - \alpha_1)\} .$$

Now inductively define $\widetilde{R}_{i+1}(\mathfrak{a})$, $i \geq 2$, for each i such that $r_{i+1} \neq 0$ as follows. If Φ is type A_ℓ , then inductively define

$$\mu_i = \text{least element of } \widetilde{R}_i(\mathfrak{a})$$

and $\widetilde{R}_{i+1}(\mathfrak{a}) = \{\mu_i - \alpha_i \, , \, \theta(\mu_i - \alpha_i) \, , \, \dots \, , \, \theta^r(\mu_i - \alpha_i)\}$

$$\text{where } r = r_{i+1} - 1 \, .$$

If Φ is type B_ℓ, C_ℓ, or D_ℓ, on the other hand, inductively define

$$\mu_i = \text{least element of } \widetilde{R}_i(\mathfrak{a})$$

and $\widetilde{R}_{i+1}(\mathfrak{a}) = \{\theta(\mu_i - \alpha_1) \, , \, \dots \, , \, \theta^r(\mu_i - \alpha_i)\}$

$$\text{where } r = r_{i+1} \, .$$

Looking at differences between consecutive elements, we see that $\widetilde{R}(\mathfrak{a}) = \widetilde{R}_1(\mathfrak{a}) \cup \widetilde{R}_2(\mathfrak{a}) \cup \dots$ is a set of <u>independent</u> roots. Since each $\widetilde{R}_i(\mathfrak{a}) \subset \Lambda_i^*$ (clear from height conditions), the $\widetilde{R}_i(\mathfrak{a})$ are root sets whose corresponding root vectors commute.

Choose a complement $\mathfrak{t}_1$ to

$$\{h \in \mathfrak{t} \mid \alpha(h) = 0 \text{ for all } \alpha \in \widetilde{R}(\mathfrak{a})\} \, .$$

So the roots in $\widetilde{R}(\mathfrak{a})$ are linearly independent on $\mathfrak{t}_1$. Put $z = \sum_{\alpha \in R_1(\mathfrak{a})} X_\alpha$. Then, as in Prop. 4.4,

$$\lim_{t\to\infty} e^{tz} \cdot \mathfrak{t}_1 = \mathfrak{t}_2 \oplus \langle X_\alpha \rangle_{\alpha \in R_1(\mathfrak{a})} \text{ , where}$$

$$\mathfrak{t}_2 = \{h \in \mathfrak{t}_1 \mid \alpha(h) = 0 \quad \text{for all} \quad \alpha \in \widetilde{R}_1(\mathfrak{a})\} .$$

Now, starting with $i = 2$, perform the 2 steps below, and then repeat them for $i = 3$ and so on.

Step 1. Put $z = \sum_{\alpha \in \widetilde{R}_i(\mathfrak{a})} X_\alpha$. Then

$$\lim_{t\to\infty} e^{tz} \cdot (\mathfrak{t}_i \oplus \langle X_\alpha \rangle_{\alpha \in S_{i-1}(\mathfrak{a})}) = \mathfrak{t}_{i+1} \oplus \langle X_\alpha \rangle_{\alpha \in \widetilde{R}_i(\mathfrak{a})} \oplus \langle X_\alpha \rangle_{\alpha \in S_{i-1}(\mathfrak{a})} .$$

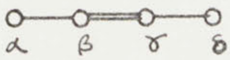
where $\mathfrak{t}_{i+1} = \{h \in \mathfrak{t}_i \mid \alpha(h) = 0 \quad \text{for all} \quad \alpha \in \widetilde{R}_i(\mathfrak{a})\}$.

Step 2. Let $\{\varphi_1, \ldots, \varphi_{r_i}\}$ be the elements of $\widetilde{R}_i(\mathfrak{a})$ listed in decreasing order (i.e., in the poset). Similarly, let $\{\beta_1, \ldots, \beta_{r_i}\}$ be elements of $R_i(\mathfrak{a})$ listed in decreasing order (so $\beta_1 = \lambda_i$, for instance). Perform the successive degenerations $\lim_{t\to\infty} e^{tX_z}$, $z = X_{\beta_1 - \varphi_1}$, then for $z = X_{\beta_2 - \varphi_2}$ etc., on the result of Step 1. The effect of each degeneration is just to move X_{φ_i} to X_{β_i} (in particular, the $\beta_i - \varphi_i$ die on $\mathfrak{t}_{i+1}$). So the final abelian subalgebra is $\mathfrak{t}_{i+1} \oplus \langle X_\alpha \rangle_{\alpha \in S_i(\mathfrak{a})}$. □

<u>Proposition 5.3</u> The assertion of Th. 5.1 is true when $\mathfrak{g}$ is an exceptional simple Lie algebra.

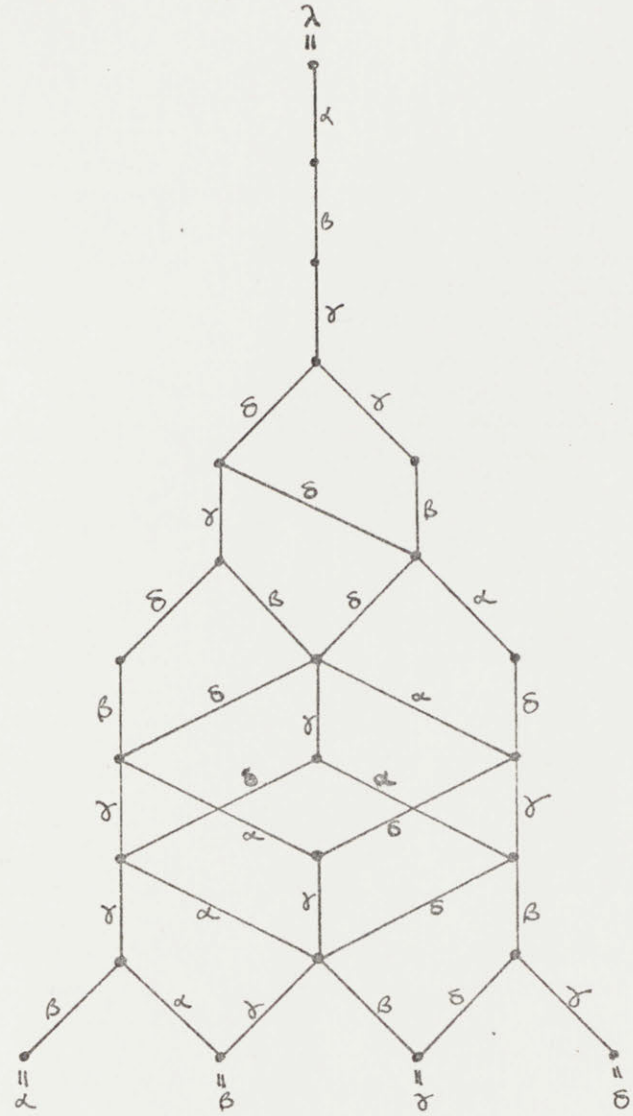
<u>Proof</u> There are five simple Lie algebras of exceptional type, namely G_2, F_4, E_6, E_7, and E_8. With the aid of the Hasse diagrams for the posets of positive roots (see next page) we can easily list the upper ideals of Φ^+ with d elements, for $d \leq \ell$. (Considering the grading on Φ^+, its easy to see that all of these correspond to <u>abelian</u> ideals of $\mathfrak{b}$.) All we need is the upper part of Φ^+, in fact just the part within ℓ degrees of the highest root. So for E_6, E_7, and E_8, we will just draw this part.

F_4



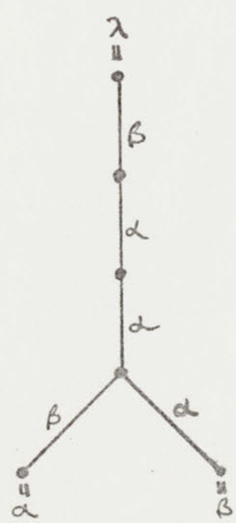
highest root λ

$= 2\alpha + 3\beta + 4\gamma + 2\delta$

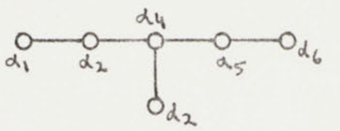


G_2

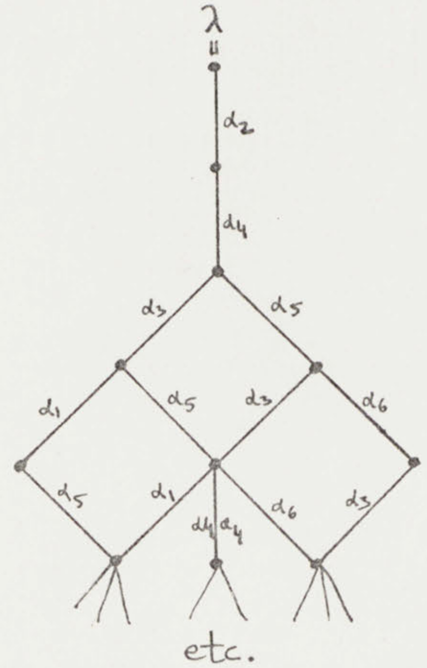
$\lambda = 3\alpha + 2\beta$



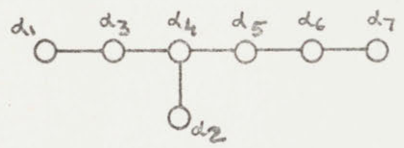
E_6



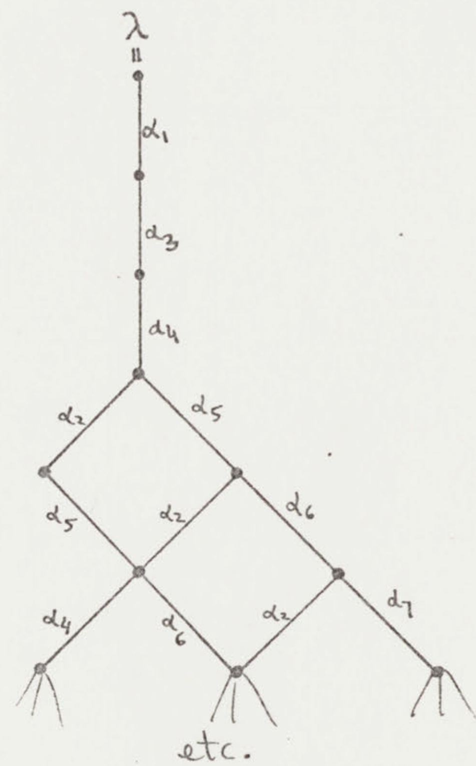
$$\lambda = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_4 + \alpha_5$$



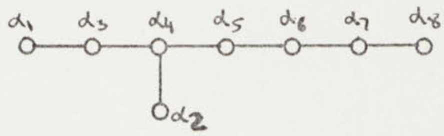
E_7



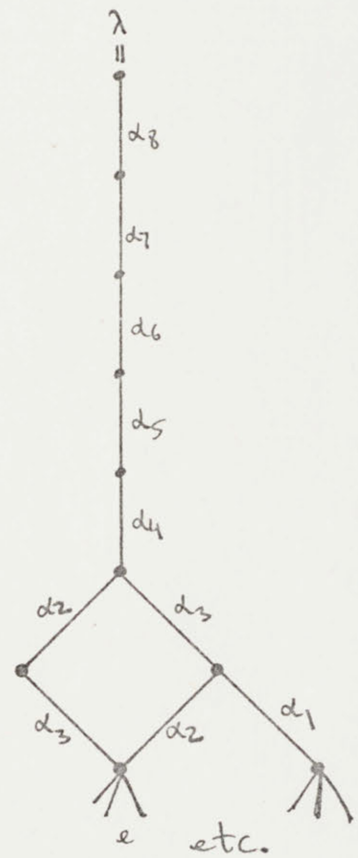
$$\lambda = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$$



E_8



$$\lambda = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$$



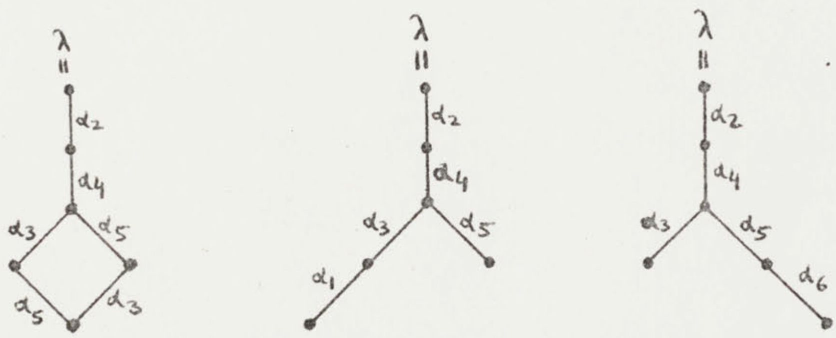
When the upper ideals consist of independent roots, then Prop. 4.4 applies and we conclude that the corresponding abelian ideal $\mathfrak{a}$ is a limit of tori by a single first order degeneration. It is easy to recognize which root sets are independent by looking at the differences of consecutive roots and recalling that the highest root λ involves all the single roots (i.e., support$(\lambda) = \Delta$) .

Consider first the case $d = \ell$. The following diagrams indicate the ℓ-element upper ideals of Φ^+ for each of the five types.

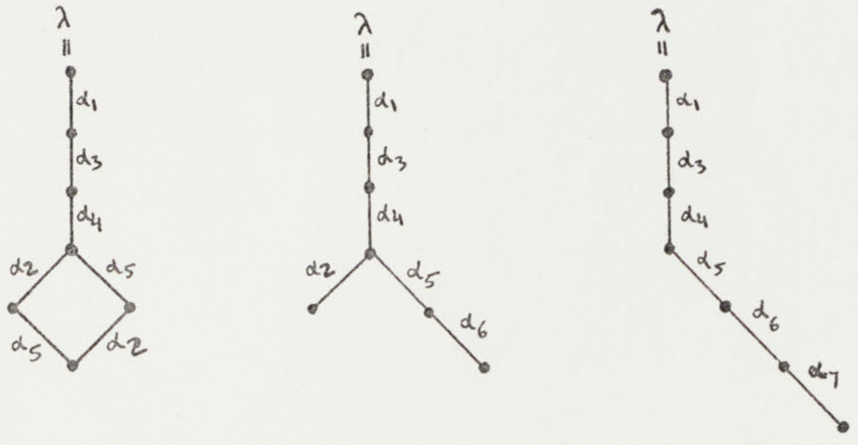
G_2

F_4

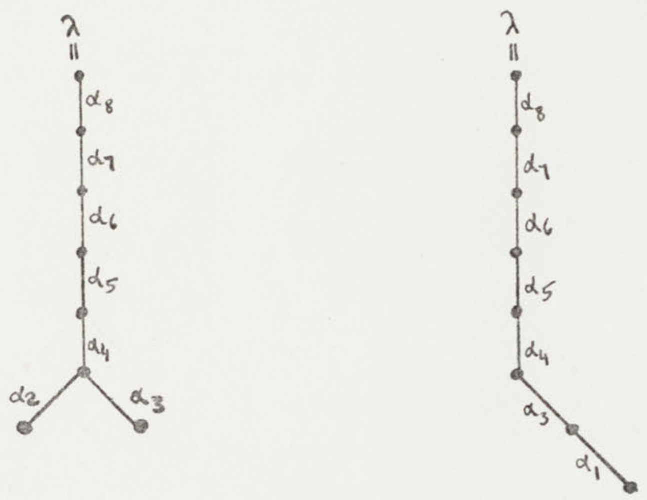
E_6



E_7



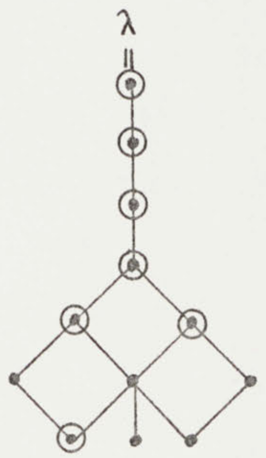
E_8



So in only two cases, the first diagrams for E_6 and E_7, do we have to deal with dependent roots. We proceed just as in the proof of Prop.5.2.

(1) For the ideal in E_6, call it I, form a new subset I' of Φ^+ by just replacing the lowest element

$\mu = \lambda - \alpha_2 - \alpha_4 - \alpha_5 - \alpha_3$ of I by $\mu - \alpha_1$. So the elements of I' are the circled roots in the following diagram.



Now the roots in I' are independent and the root vectors commute, so degenerate tori (Prop. 4.4) to get the abelian subalgebra $\mathfrak{a}'$ with $R(\mathfrak{a}') = I'$. Then

$$\lim_{t \to \infty} e^{tz} \cdot \mathfrak{a}' = \mathfrak{a}, \; z = X_{\alpha_1},$$

where $\mathfrak{a}$ is the abelian ideal with $R(\mathfrak{a}) = I$.

(2). For the ideal in E_7, call it J, do the same thing. Form a new set J' by replacing the lowest element μ of J by $\mu - \alpha_6$. Apply Prop. 4.4 to J', then degenerate by e^{tz}, for $z = X_{\alpha_6}$. □

<u>Proof of Th. 5.1</u> So we know the theorem for $\mathfrak{g}$ simple. Now the semi-simple Lie algebra $\mathfrak{g}$ has a unique decomposition (up to order) $\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i$, into a Lie algebra direct sum of simple subalgebras (the $\mathfrak{g}_i$ are just the simple ideals of $\mathfrak{g}$). Then the Borel subalgebra $\mathfrak{b}$ decomposes into $\mathfrak{b} = \bigoplus_{i=1}^{r} \mathfrak{b}_i$, where $\mathfrak{b}_i = \mathfrak{g} \cap \mathfrak{g}_i$.

So if $\mathfrak{a}$ is an abelian d-dimensional ideal of $\mathfrak{b}$, then

$$\mathfrak{a} = [\mathfrak{b}, \mathfrak{a}] = [\oplus \mathfrak{b}_i, \mathfrak{a}] = \oplus [\mathfrak{b}_i, \mathfrak{a}] \subseteq \oplus (\mathfrak{g}_i \cap \mathfrak{a}) .$$

This forces

$$\mathfrak{a} = \oplus \mathfrak{a}_i, \text{ where } \mathfrak{a}_i = \mathfrak{g}_i \cap \mathfrak{a}.$$

Next let $\mathfrak{t}$ be a maximal torus of $\mathfrak{b}$, so $\mathfrak{t} = \oplus \mathfrak{t}_i$, where $\mathfrak{t}_i = \mathfrak{t} \cap \mathfrak{g}_i$ is a maximal torus of $\mathfrak{g}_i$.

Now G has simple algebraic subgroups $G_1, \dots, G_r$ with Lie algebras $\mathfrak{g}_1, \dots, \mathfrak{g}_r$ such that $G_1 \times \dots \times G_r \to G$ is an isogeny (surjective with finite kernel). By last two propositions, we can, for each i, choose a subtorus $\mathfrak{t}_i'$ of $\mathfrak{t}_i$ such that $\overline{G_i \cdot \mathfrak{t}_i'}$ contains $\mathfrak{a}_i$.

Now

$$(G_1 \times \ldots \times G_r) \cdot (\mathfrak{t}_1' \oplus \ldots \oplus \mathfrak{t}_r') = \oplus\, G_i \cdot \mathfrak{t}_i' .$$

But closure of $\oplus G_i \cdot \mathfrak{t}_i'$ in $Gr_d(\mathfrak{g})$ must contain $\oplus\, \overline{G_i \cdot \mathfrak{t}_i'}$, hence must contain $\mathfrak{a}$.

And the whole process is still one of order one degenerations. □

<u>Corollary 5.4</u> For $d \leq \ell$, $X_d(\mathfrak{g})$ is connected.

<u>Proof</u> Any irreducible component of $X_d(\mathfrak{g})$ is a closed G-invariant subvariety of $X_d(\mathfrak{g})$ hence meets $\overline{Q_d(\mathfrak{g})}$ at some closed G-orbit. □

<u>Corollary 5.5</u> For $d \leq \ell$, the linear spans of $Q_d(\mathfrak{g})$ and $X_d(\mathfrak{g})$ in $\mathbb{P}(\Lambda^d \mathfrak{g})$ are equal. Passing to affine cones, this means

$$\mathcal{U}(\mathfrak{g}) \cdot \Lambda^d \mathfrak{t} = A_d(\mathfrak{g}) .$$

<u>Proof</u> This follows immediately from the theorem in view of the fact (Cor. 2.1) that $\mathbb{P}(A_d(\mathfrak{g}))$ is spanned by the closed orbits in $X_d(\mathfrak{g})$. □

§4.1 APPLICATIONS OF $\mathcal{U}(\mathfrak{g}) \cdot \Lambda^d \mathfrak{t} = A_d(\mathfrak{g})$, $d \leq \ell$

Fix a maximal torus T of G with Lie algebra $\mathfrak{g}$, and let W denote the Weyl group $W(G,T) = N(T)/T$ of G with respect to T . For any G-representation space V , the action of G on V gives an action of $N(T)$, and hence of W , on the space of T-invariants in V (which is the space of $\mathfrak{t}$ -invariants, i.e. the zero weight space V^0). One can try to locate the irreducible representations of W on the zero-weight spaces of various irreducible V .

<u>Example 1.1</u> Suppose $G = SL_n$, so that W is the symmetric group S_n on n letters. Then we know from the representation theory of finite groups that the number of distinct irreducible finite dimensional representations (over k) of S_n is equal to the number of conjugacy classes in S_n , which of course is given by the partition function $p(n)$. There is a nice family of $p(n)$ irreducible representations of SL_n such that the action of S_n on each zero-weight space is irreducible and all the irreducible representations of S_n occur, namely (as observed in [G] and [Ko 3]) the irreducible pieces of

$\overset{n}{\otimes}\,\mathbb{C}^n$, where SL_n acts on $\mathbb{C}^n$ in the standard way (the first fundamental representation). In fact, $(\overset{n}{\otimes}\,\mathbb{C}^n)^o$ is just the regular representation of S_n .

We can now ask about the zero-weight spaces of the irreducible pieces in $A_d(\mathfrak{g})$, $d \leq \ell$. As explained in [Ki], we have

<u>Proposition 1.2</u> 1) [Solomon] $\Lambda^d\mathfrak{t}$ is an irreducible representation space for W .
2) The W-module $\Lambda^d\mathfrak{t}$ occurs in V^o for each irreducible piece V of $A_d(\mathfrak{g})$.

<u>Proof</u>. 1) This is proven in [So].
2) As $\mathcal{U}(\mathfrak{g}) \cdot \Lambda^d\mathfrak{t} = A_d(\mathfrak{g})$, the projection of $\Lambda^d\mathfrak{t}$ to V must be non-zero. Here we are projecting $A_d(\mathfrak{g})$ to V via the unique decomposition of $A_d(\mathfrak{g})$ into irreducible pieces. This projection commutes with the action of W , so the W-module $\Lambda^d\mathfrak{t}$ appears in V^o . □

In particular when $d = \ell$, we get the line $\cdot\, \Lambda^\ell\mathfrak{t}$, and this case can be connected up with the theory of co-adjoint orbits for $\mathfrak{g}$ discussed in [Ko 4] as follows.

First we recall the situation considered there. Let $\mathfrak{t}^\vee$ be the dual of $\mathfrak{t}$ and let $d\colon \mathfrak{g}^\vee \to \Lambda^2\mathfrak{g}^\vee$ be the exterior derivative map. This map extends uniquely to a ring homomorphism

$$\gamma\colon S(\mathfrak{g}^{\vee}) \to \Lambda^{e}(\mathfrak{g}^{\vee}) ,$$

where $S(\mathfrak{g}^{\vee})$ is the symmetric algebra on $\mathfrak{g}^{\vee}$, and $\Lambda^{e}(\mathfrak{g}^{\vee})$ is the commutative algebra formed by the even dimensional pieces of the exterior algebra $\Lambda(\mathfrak{g}^{\vee})$ on $\mathfrak{g}^{\vee}$. Note that γ doubles the degree, i.e.

$$\gamma\colon S^{i}(\mathfrak{g}^{\vee}) \to \Lambda^{2i}(\mathfrak{g}^{\vee}) .$$

Now the dimension of the coadjoint orbit $G \cdot w$, $w \in \mathfrak{g}^{\vee}$, is equal to $2o(w)$ where $o(w)$ is the largest integer i such that

$$(dw)^{i} \neq 0 \text{ in } \Lambda(\mathfrak{g}^{\vee}) ,$$

by Prop. 1.3 [Ko 4]. This holds for any complex Lie algebra (actually the result Kostant gives is more general), but of course the theory simplifies for $\mathfrak{g}$ semi-simple. Indeed, then the coadjoint and adjoint representations are isomorphic, and we know that the maximum value $o(w)$ assumes is $o(w) = \dim \mathfrak{g} - \ell = 2r$ (these are the <u>regular</u> w for the coadjoint action) where r is the number of positive roots for $\mathfrak{g}$.

Consider the subspace E of $\Lambda^{2r}(\mathfrak{g}^{\vee})$ spanned by the $(dw)^r$ for $w \in \mathfrak{g}^{\vee}$. It follows quite easily that, as $\mathfrak{g}$-representation spaces

$$E \simeq \mathcal{U}(\mathfrak{g}) \cdot \Lambda^{\ell}\mathfrak{t} \simeq A_{\ell}(\mathfrak{g}) ,$$

(so that, in particular, we have a description of the highest weight vectors of E). To see this, first note that

$$E = \gamma(S^r(\mathfrak{g}^{\vee})) ,$$

since $(dw)^r = \gamma(w^r)$ and the elements w^r span $S^r(\mathfrak{g}^{\vee})$. The latter also implies that $S^r(\mathfrak{g}^{\vee}) = \mathcal{U}(\mathfrak{g}) \cdot S^r(\mathfrak{t}^{\vee})$, where $\mathfrak{t}^{\vee}$ is the dual to $\mathfrak{t}$ via the killing form $(,)$.

Next, let $\{e_{\varphi} \mid \varphi \in \Phi\}$ be a set of root vectors of $\mathfrak{g}$ normalized so that

$$(e_{\varphi}, e_{\psi}) = \begin{cases} 1 & \text{if } \varphi = -\psi \\ 0 & \text{otherwise.} \end{cases}$$

Also for $z \in \mathfrak{g}$, let $\overline{z}$ denote the killing form dual $(z,-)$ in $\mathfrak{g}^{\vee}$. Computing $d\colon \mathfrak{g}^{\vee} \to \Lambda^2(\mathfrak{g}^{\vee})$ we easily get

$$d(\overline{h}) = - \sum_{\varphi \in \Phi^+} \varphi(h)\overline{e}_{\varphi} \wedge \overline{e}_{-\varphi} , \ h \in \mathfrak{t} ,$$

so that

$$d(\bar{h}^r) = (-1)^r r! \prod_{\varphi \in \Phi^+} \varphi(h) \bar{e}_\varphi \wedge \bar{e}_{-\varphi} .$$

So via the natural identification $\Lambda^{2r}(\mathfrak{g}^\vee) \simeq \Lambda^\ell(\mathfrak{g})$, we have $d(\bar{h}^r) \in \Lambda^\ell \mathfrak{t}$. Thus $\gamma(S^r(\mathfrak{t})) \subset \Lambda^\ell \mathfrak{t}$ and $E = \mathfrak{U}(\mathfrak{g}) \cdot \Lambda^\ell \mathfrak{t}$.

To see why the space E is interesting, consider the map

$$\Gamma: \Lambda^e(\mathfrak{g}) \to S(\mathfrak{g})$$

dual to γ (with $S(\mathfrak{g}^\vee)^\vee$ identified with $S(\mathfrak{g})$, etc.). Γ is defined intrinsically in [Ko 4]. As γ and Γ are dual linear transformations, we certainly know that (1) for $v \in S(\mathfrak{g}^\vee)$, $\gamma v = 0 \Leftrightarrow f(v) = 0$ for all $f \in \operatorname{Im}\Gamma$ and (2) there is a natural map $S(\mathfrak{g}^\vee)/\ker \gamma \xrightarrow{\sim} (\operatorname{Im} \Gamma)^\vee$, so that $\operatorname{Im} \gamma \simeq (\operatorname{Im} \Gamma)^\vee$ over $\mathfrak{g}$.

Thus, putting $R^i(\mathfrak{g}) = \Gamma(\Lambda^{2i}(\mathfrak{g})) \subset S^i(\mathfrak{g})$ and recalling $E = \gamma(S^r(\mathfrak{g}^\vee))$, we have established that $R^r(\mathfrak{g})^\vee \simeq A_\ell(\mathfrak{g})$ as $\mathfrak{g}$-spaces, and $R^r(\mathfrak{g})$ is a space of polynomials of degree r in $S(\mathfrak{g})$ such that, for $w \in \mathfrak{g}^\vee$, w is not regular iff all $f \in R^r(\mathfrak{g})$ vanish at w .

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