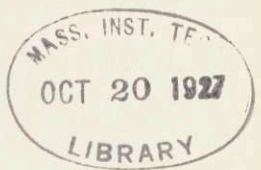


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MATHEMATICAL THEORY OF THE PROCESS
OF CONSOLIDATION OF MUD DEPOSITS

by

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I. INTRODUCTION

The most important mud deposits are formed at the mouths of large rivers. Here the decrease in the velocity of their streams, and the presence of certain salts in the ocean water, which act as electrolytes when the grains are of colloidal size, cause the sedimentation of the suspended matter. In time deltas are formed there. Other important deposits occur on the beds and sites of natural streams and lakes. This process of sedimentation goes on continuously, and large areas of land take the place of previous water surfaces. As time goes on, these mud deposits, besides increasing in size, become more or less consolidated and often serve as a foundation for large and important cities.

Therefore one can easily see the importance of a careful study of the behavior of such mud deposits under varying conditions. By local observations and by a single laboratory experiment, which will be described later, we can determine the rate at which sedimentation takes place, whether it is uniform or not, the specific weight, and other physical characteristics of the deposit: the coefficient of permeability, etc. Having these data, we are in a position to fully determine and predict the state of stress, water-content, and settlement at any time.

The knowledge of the stress conditions in a mud deposit is important in so far as it explains practically

all phenomena connected with foundation construction on soft grounds.

As to the structure of the sediment, two cases will be considered, namely: that of a sediment of homogeneous material and that in which there exists within the sediment a thin layer of less permeable material.

Three aspects of the process of consolidation of such mud seposits will be considered. (a). Consolidation due to the own weight of the material; (b). consolidation due to the weight of a top fill of very permeable material or its equivalent - evaporation; and (c). consolidation by drainage. In all cases the bottom surface of the sediment will be assumed horizontal and impermeable.

The theory developed will hold not only for mud deposits but also for clays and fine-grained materials in general, provided that no air is present in the voids of the material.

Free use will be made of Fourier's Series and Integrals in the attempts to solve the differential equations. The application of Heaviside's Operational Method of solving differential equations will be illustrated in one case.

II. GENERAL LAWS AND DEFINITIONS.

1. Definitions.

Volume of Voids (n) is the ratio between the total volume of voids (N) and the total volume of material (V):

$$n = \frac{N}{V}$$

Voids Ratio (ϵ) is the ratio between the total volume of voids and the total volume of solid matter (V_s):

$$\epsilon = \frac{N}{V_s} = \frac{N}{V-N} = \frac{n}{1-n}$$

The voids-ratio then, is a measure of the water content per unit of solid matter.

Granular Pressure or Stress (p) is the intensity of the pressure acting between the grains of the material at a given point and in a specified direction.

Hydrodynamic Pressure or Stress (w) is the excess of the intensity of the water pressure over the hydrostatic pressure at a given point, and acts with the same intensity in all directions.

Reduced Dimensions (x, y, z) are dimensions (or distances) equivalent to a volume of voids (or voids-ratio) equal to zero. We are forced to use reduced dimensions because if there is a change in either or both of the above defined pressures, in a given mass of mud, there will be, as a consequence, a change in its water content which in turn will cause a change in its dimensions. Hence the true dimensions are as variable as the internal stresses

themselves, while the reduced dimensions are unaltered by any one of the above changes.

2. Darcy's Law:

This law states that if water is admitted through a layer of granular material of cross-sectional area (A) and thickness l , then the quantity of water Q , percolating through any section of the layer, perpendicular to its cross section, per unit of time, is given by

$$Q = i k A \quad (1)$$

In this formula, i is the hydraulic gradient which, for steady flow of water, is equal to the ratio $\frac{h}{l}$ (Fig. 1), h being the hydraulic head, and k is the coefficient of permeability of the material. From the above formula we see

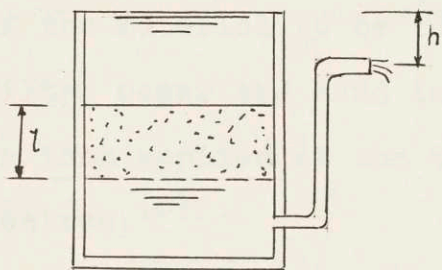


Fig. 1

that the coefficient of permeability is equal to the velocity of percolation per unit of time, under a hydraulic head equal to the thickness of the layer, i.e., under a hydraulic gradient equal

to unity. If, however, the flow is not steady, and if we call w the hydraulic head at a section distant s from any convenient reference line, then

$$i = \frac{\partial w}{\partial s}$$

at the section under con-

sideration.

$$\text{Hence, } Q = K \frac{\partial w}{\partial s} A \quad (2)$$

This law is strictly true for laminary flow of

water such as is generally that through fine-grained materials. It has been verified by many experiments (1).

3. Laboratory Experiment.

With this experiment we aim to obtain two curves

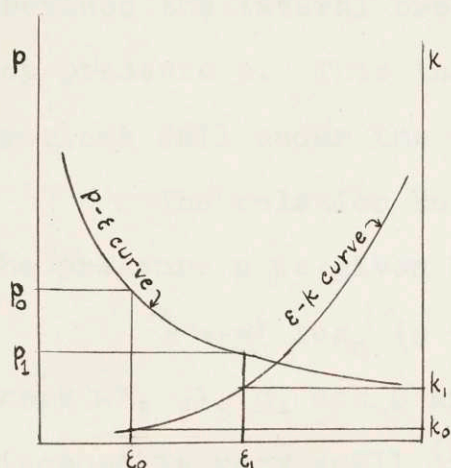


Fig. 2.

ves showing the variation of water-content with pressure and the variation in the coefficient of permeability with water-content.

The curves representing these variations are shown in Fig.2.

The apparatus used to obtain the $p - \epsilon$ curve consists of a container in which a layer

of the material to be tested is placed and then covered by filter paper and sand immersed in water. The pressure p is then applied at the top and varied through the range desired. (2)

(1). For formulae for the coefficient of permeability, k , for sands and clays, and for a discussion regarding the validity of the law of Darcy, see:

Principles of Soil Mechanics, by Dr. Charles

Terzaghi, Engineering News Record, Nov. 14, 1925.

Cf. also Terzaghi's "Erdbaumechanik", Chapter IV.

(2). Cf. Principles of Soil Mechanics by C. Terzaghi, Engineering News Record, Nov. 19, 1925 and Nov. 26, 1925.

Also Terzaghi's "Erdbaumechanik", Fig. 13, p. 83 and p. 87, equation (24).

The pressure-moisture curve thus obtained is valid for the case of linear flow of water. If, however, there exists a flow of water in more than one direction, an apparatus similar to the above one can be made having lateral filters, provided the lateral pressures are known functions of the top pressure p . This is seldom the case and most of the problems fall under the one dimensional case.

The relation between the water-content (ξ) and the pressure p is given by the following equation⁽²⁾

$$\xi = -\alpha' \log_e (p + p_1) - \beta' (p + p_1) + C$$

where α' , β' , p_1 and C are constants.

Since β' is very small ⁽³⁾ the above equation may be written

$$\xi = -\alpha \log_e (p + \beta) + C$$

$$\text{and } \frac{d\xi}{dp} = -\frac{\alpha}{p+\beta} \quad (3)$$

which is the equation of the tangent to the curve.

The ratio $a = -\frac{d\xi}{dp}$ = modulus of compression, may be taken as constant and equal to the average of its extreme values if the range through which p varies is not very large. ⁽⁴⁾

The ratio $\frac{k}{a}$ (where k is the coefficient of permeability) was found to be almost constant for materials with plastic consistency, ⁽⁵⁾, and will be so considered in what follows.

(3). "Erdbaumechanik", p. 162

Hence we can write:

$$a = - \frac{d\xi}{dp} = \frac{\alpha}{p+\beta} \quad (4)$$

$$\text{and } \frac{k}{a} = c = \text{constant.} \quad (5)$$

The constants α and β should be determined for the range through which the pressure p ranges in the actual problem dealt with. For very small pressures, β is very large, while for large values of p it is very small.

4. General Equation of Stresses in a Mass of Granular Material with Water-Filled Voids.

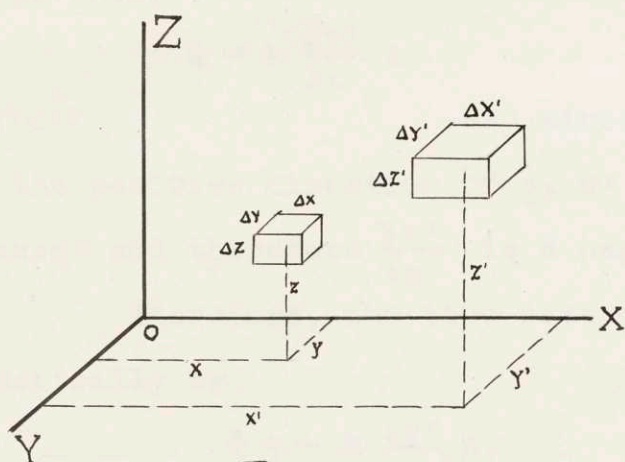


Fig. 3

Let a mass of mud be referred to a system of rectangular coordinates. Let O be its origin and let (x', y', z') be the coordinates of the center of an elementary prism whose

sides are $\Delta x'$, $\Delta y'$, $\Delta z'$. Assume the flow of water to be in the positive directions of the coordinate axes X , Y and Z , and let w' be the hydrodynamic pressure at the center of this elementary prism; i.e., at point (x', y', z') .

(4). C. Terzaghi: Eng. News-Record, Nov. 26, 1926, p. 874. Also "Erdbaumechanik" p. 141.

(5). "Erdbaumechanik" Fig. 22, p. 121. Also pp. 126-7.

The hydrodynamic pressure intensities at the faces of this prism - which is assumed to be very small and finally approach the limiting value zero - will therefore be:

$$\begin{aligned} \text{Plane YZ: left face } w' - \frac{1}{2} \frac{\partial w'}{\partial x'} \Delta x'; & \text{ right face: } w' + \frac{1}{2} \frac{\partial w'}{\partial x'} \Delta x'. \\ \text{" XZ: front " } w' - \frac{1}{2} \frac{\partial w'}{\partial y'} \Delta y'; & \text{ back " } w' + \frac{1}{2} \frac{\partial w'}{\partial y'} \Delta y'. \\ \text{" XY: lower " } w' - \frac{1}{2} \frac{\partial w'}{\partial z'} \Delta z'; & \text{ upper " } w' + \frac{1}{2} \frac{\partial w'}{\partial z'} \Delta z'. \end{aligned}$$

Now, according to the law of Darcy, the quantity of water (Q), percolating normally through a plane surface whose sectional area is A , per unit of time, in a direction s , is given by

$$Q = k \frac{\partial w'}{\partial s} A,$$

and since the flow of water is in the positive direction of s , w' must decrease as s increases and therefore $\frac{\partial w'}{\partial s}$ is a negative quantity.

Therefore, the time rate of percolation is given numerically by

$$Q = -k \frac{\partial w'}{\partial s} A.$$

Applying this relation to parallel opposite faces of the elementary prism, we get the following values of Q for the faces parallel to the YZ plane:

$$\begin{aligned} \text{left face: } -k \Delta y' \Delta z' \frac{\partial}{\partial x'} \left(w' - \frac{1}{2} \frac{\partial w'}{\partial x'} \Delta x' \right) &= -k \Delta y' \Delta z' \left(\frac{\partial w'}{\partial x'} - \frac{1}{2} \frac{\partial^2 w'}{\partial x'^2} \Delta x' \right) \\ \text{right " } -k \Delta y' \Delta z' \frac{\partial}{\partial x'} \left(w' + \frac{1}{2} \frac{\partial w'}{\partial x'} \Delta x' \right) &= -k \Delta y' \Delta z' \left(\frac{\partial w'}{\partial x'} + \frac{1}{2} \frac{\partial^2 w'}{\partial x'^2} \Delta x' \right) \end{aligned}$$

Subtracting the second expression from the first, we find the difference between inflow and outflow per unit of time to be

$$k \Delta x' \Delta y' \Delta z' \frac{\partial^2 w'}{\partial x'^2};$$

and similar expressions for the other pairs of parallel faces.

Therefore the difference between the total inflow and the total outflow of water in the elementary prism per unit time is

$$k \Delta x' \Delta y' \Delta z' \left(\frac{\partial^2 w'}{\partial x'^2} + \frac{\partial^2 w'}{\partial y'^2} + \frac{\partial^2 w'}{\partial z'^2} \right) \quad (6)$$

This difference should be equal to the time rate of change (an increase in this case) in water content of the elementary prism.

At this point, attention is called to the fact that, in the above differentiations, k has been considered as a constant, and equal to the average value of the coefficient of permeability. Thus, to a continuous change in the granular pressure at a given point from p_0 to p_1 say, there is a corresponding change in water content per unit of solid matter of $\varepsilon_1 - \varepsilon_0$, and [therefore the coefficient of permeability varies continuously] from a value k_0 to a value k_1 . If the changes in pressures are relatively small, we are justified in assuming k constant, and having a value intermediate between k_0 and k_1 .

If, however, due account is to be taken of the variation in the values of k , we should proceed as follows: Let k be the value of the coefficient of permeability at the center of the elementary prism; then its value at the two faces parallel to the YZ plane are

$$k - \frac{1}{2} \frac{\partial k}{\partial x'} \Delta x' \quad \text{and} \quad k + \frac{1}{2} \frac{\partial k}{\partial x'} \Delta x'.$$

Hence the time rate of percolation at these two faces is

given by

$$-(k - \frac{1}{2} \frac{\partial k}{\partial x'} \Delta x') \frac{\partial}{\partial x'} (w' - \frac{1}{2} \frac{\partial w'}{\partial x'} \Delta x') \Delta y' \Delta z'$$

and
$$-(k + \frac{1}{2} \frac{\partial k}{\partial x'} \Delta x') \frac{\partial}{\partial x'} (w' + \frac{1}{2} \frac{\partial w'}{\partial x'} \Delta x') \Delta y' \Delta z'.$$

Differentiating and subtracting the second expression from the first we find

$$\Delta x' \Delta y' \Delta z' (k \frac{\partial^2 w'}{\partial x'^2} + \frac{\partial k}{\partial x'} \frac{\partial w'}{\partial x'})$$

as the difference between the inflow and outflow of water per unit of time. Therefore the difference between total inflow and outflow per unit of time in the prism is expressed by

$$\Delta x' \Delta y' \Delta z' \left[k \left(\frac{\partial^2 w'}{\partial x'^2} + \frac{\partial^2 w'}{\partial y'^2} + \frac{\partial^2 w'}{\partial z'^2} \right) + \frac{\partial k}{\partial x'} \frac{\partial w'}{\partial x'} + \frac{\partial k}{\partial y'} \frac{\partial w'}{\partial y'} + \frac{\partial k}{\partial z'} \frac{\partial w'}{\partial z'} \right] \quad (7)$$

In order to find an expression for the time rate of change in water content, we have to introduce reduced dimensions. Let the new (reduced) dimensions of our elementary prism be Δx , Δy and Δz , and let (x, y, z) be the coördinates (reduced) of its center, where the hydrodynamic pressure is now w instead of w' . These transformations do not change the value of w numerically, for it is still equal to w' but referred to a new system of coördinates. w is then equivalent to the temperature (or potential) difference in the case of flow of heat (or electricity) through an isotropic body.

The expressions (6) and (7) for the time rate of change in water content, in terms of the reduced dimensions are:

$$k \Delta x \Delta y \Delta z \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \quad (8)$$

$$\text{and } \Delta x \Delta y \Delta z \left[k \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \frac{\partial k}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial k}{\partial y} \frac{\partial w}{\partial y} + \frac{\partial k}{\partial z} \frac{\partial w}{\partial z} \right]. \quad (9)$$

Now we have seen that ϵ measures the water content per unit volume of solid matter, and if we let $\Delta\epsilon$ be the change in water content per unit of volume of solid matter in the time element Δt , then the change in water content of the elementary prism $\Delta x \Delta y \Delta z$ per unit of time will be $\Delta x \Delta y \Delta z \frac{\partial \epsilon}{\partial t}$ [10]

$$\text{But } \frac{\partial \epsilon}{\partial t} = \frac{\partial \epsilon}{\partial p} \frac{\partial p}{\partial t} = - a \frac{\partial p}{\partial t}$$

$$\text{Therefore (10) becomes } -\Delta x \Delta y \Delta z a \frac{\partial p}{\partial t} \quad (11)$$

Equating (8) and (9) to (11) we obtain:

$$\frac{\partial p}{\partial t} = - \frac{k}{a} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \quad (12)$$

if k is taken as constant, and

$$\frac{\partial p}{\partial t} = - \frac{k}{a} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - \frac{1}{a} \left(\frac{\partial k}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial k}{\partial y} \frac{\partial w}{\partial y} + \frac{\partial k}{\partial z} \frac{\partial w}{\partial z} \right) \quad (13)$$

if due account is taken of the variation in the coefficient of permeability.

These are the fundamental differential equations for the stress distribution in a mass of granular material.

Equation (7) is similar to Fourier's equation of heat conduction, the only difference being in having p instead of w in the left-hand member.

In applying the differential equation (either (12) or (13)) to mud deposits, two distinct stages must be considered. The first stage comprises the lapse of time

between the sedimentation of the first and last layers of material, that is, when material is still being deposited. After this stage no more material is added to the deposit, and its consolidation takes place either under the own weight of the solid matter or under the influence of an external load or evaporation. This is the second stage.

In the first stage, since the (reduced) dimensions of the deposit change continuously with the time, it is evident that t is not an independent variable, while in the second stage the time and space variables are completely independent of each other.

We will designate by t' the time in the first stage and by t that in the second stage.

a cylinder of the material with cross-sectional area S and height h increases continuously up to a value h' (Fig. 1) and then to represent the initial conditions.



Fig. 1 shows the case of sedimentation on an inclined plane. The material will be deposited in the direction of the slope of the plane. If the slope is small, the material will be deposited in the direction of the slope, but if the slope is large, the material will be deposited in the direction of the normal to the plane.

III. Problem I. Mud Deposit of Homogeneous Material - Uniform Rate of Sedimentation - Bottom Surface Impermeable and Horizontal. Consolidation due to the own weight of the deposit.

1. Formation of mud deposits.

Fig. 4 shows a deposit of homogeneous material of total thickness H . As the true thickness at any time t' varies from $H'=0$ to a final value H , the reduced thickness varies from $h'=0$ to $h'=h$. We choose the bottom surface as the origin of coordinates. Since the conditions in Δ horizontal plane are the same at all points, there will be flow of water in only one direction, and we can take a cylinder of the material ^{of} unit cross-sectional area whose height increases continuously up to a value H (reduced = h) to represent the actual conditions.

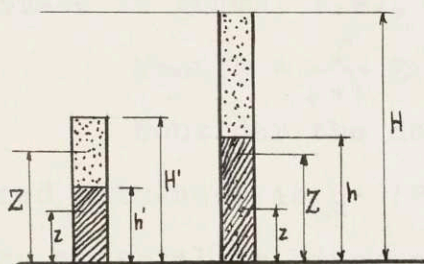


Fig. 4.

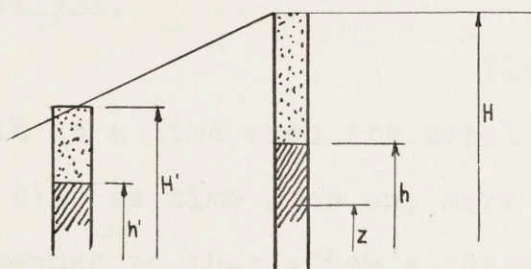


Fig. 5.

Fig. 5 illustrates the case of sedimentation in an inclined plane. Here there will be flow of water in two directions, but if the slope of the plane is small, as is usually the case, the lateral flow may be disregarded and

the problem treated under the case of linear flow of water.

Before attempting to solve the problem illustrated by Fig. 4, we will show why there is flow of water in the vertical direction and how it affects the state of internal stresses.

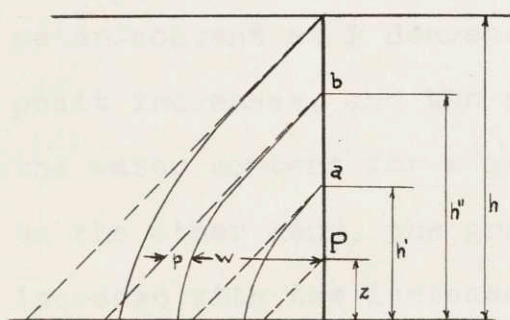


Fig. 6.

Let $\gamma + 1$ be the specific weight of the deposited material when dry, then γ will be its specific weight under water.

Let g be the quantity (grams) of dry substance deposited per unit surface per unit time, and q the corresponding quantity under water which is different from g since it is expressed in grams, i.e., as weight.

$$\text{Then } q = \frac{\gamma}{\gamma + 1} g. \quad (14).$$

Consider the deposit at a time when its total reduced thickness is h' (Fig. 6). As time goes on, more and more material is being sedimented so that after a certain lapse of time the top surface of the deposit is located at a height h'' (reduced) from the bottom. Pick out an arbitrary point at a distance z (reduced) from the bottom. As the top surface of the deposit increased from a position a to a position b the total weight of the material (or the total pressure) at P increased from $\gamma(h'-z)$ to $\gamma(h''-z)$. Part of this increase in the total pressure at

P is taken up by the solid particles, and part by the water. Therefore the hydrodynamic pressure increases continuously. But since a difference in hydrodynamic pressure implies flow of water (in the same way as a difference in temperature (or potential) implies flow of heat (or electricity)), we will have water coming out continuously from the top surface. As a consequence the water-content at P decreases as the thickness of the deposit increases, and the smaller is z the smaller will be the water content for a given thickness of the deposit. On the other hand, the granular pressure (p) at P must increase with the increase in the thickness of the deposit (as can be seen from pressure-moisture curves in Section II) and also the smaller is z for a given thickness, the greater will be the granular pressure p.

We will now pass to the solution of our problem, first by applying the differential equation for k constant and then for k variable.

The differential equations (12) and (13) for the case of linear flow of water become:

$$\frac{\partial p}{\partial t} = - \frac{k}{a} \frac{\partial^2 w}{\partial z^2} = - c \frac{\partial^2 w}{\partial z^2} \quad (15)$$

and

$$\frac{\partial p}{\partial t} = - c \frac{\partial^2 w}{\partial z^2} - \frac{1}{a} \frac{\partial k}{\partial z} \frac{\partial w}{\partial z} \quad (16)$$

2. Solution of Equation (15).

(a). First stage: Since time is not an independent variable during the first stage of the process of consolidation, we write (15) as follows:

$$\frac{\partial p}{\partial t'} = -c \frac{\partial^2 w}{\partial z^2} \quad (15a)$$

Let the total reduced thickness of the deposit be \underline{h}' and let \underline{t}'_1 be the time required for the deposit to reach a height \underline{h}' . The volume of material sedimented per unit area and unit of time is $\frac{q}{\gamma}$ and this is also the thickness of the layer of material deposited per unit of time. Therefore in a time interval \underline{t}'_1 , the thickness of material deposited is $\frac{q}{\gamma} \underline{t}'_1$, and this must be equal to \underline{h}' .

$$\underline{t}'_1 = \frac{\gamma}{q} \underline{h}'.$$

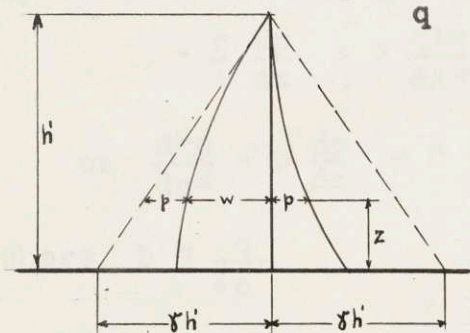


Fig. 7.

Consider now any point at a height z from the bottom such that $z \leq h'$.

The time required for the layer at height z to come into existence is $t'_2 = \frac{\gamma}{q} z$.

Of course no flow of water through the layer located at a distance z from the bottom could possibly exist before the layer itself had come into existence. Since the variable time in the differential equation (15a) concerns the interval from which water begins to percolate through the layer under consideration (i.e., from which the layer has come into existence) up to any value t' , we must have $t' = 0$ for a time $t'_2 = \frac{\gamma}{q} z$, and therefore

$$t' = \frac{\gamma}{q} (h' - z) \quad (17)$$

The total downward pressure acting at a height z from the bottom is $\gamma(h' - z)$, and this must be balanced by

the granular and hydrodynamic pressures in order to maintain equilibrium; hence

$$p + w = \gamma(h - z) \quad (18)$$

$$\text{or } \frac{\partial w}{\partial z} = -\gamma - \frac{\partial p}{\partial z} \quad (19)$$

$$\text{and } \frac{\partial^2 w}{\partial z^2} = -\frac{\partial^2 p}{\partial z^2} \quad (20)$$

From (17) and (19) we obtain the relation

$$\frac{\partial p}{\partial t} = -\frac{q}{\delta} \frac{\partial p}{\partial z}.$$

By substitution in (15a) we get

$$-\frac{q}{\delta} \frac{dp}{dz} = c \frac{d^2 p}{dz^2}$$

$$\text{or } \frac{d^2 p}{dz^2} + b \frac{dp}{dz} = 0 \quad (21)$$

$$\text{where } b = \frac{q}{\delta c} \quad (22)$$

$$\text{The solution of (21) is } p = A + B e^{-bz} \quad (23)$$

where A and B are constants of integration.

To determine A and B we know that p and w are zero for $z = h'$ (Fig. 7) and since the bottom surface is assumed to be impermeable, we must have $\frac{dw}{dz} = 0$ for $z = 0$, that is, the curve of hydrodynamic pressure must be perpendicular to the bottom surface. To $\frac{dw}{dz} = 0$ corresponds $\frac{dp}{dz} = -\gamma$ and therefore the boundary conditions are

$$z = h' : p = 0, w = 0 \quad (24)$$

$$z = 0 : \frac{dw}{dz} = 0, \frac{dp}{dz} = -\gamma \quad (25)$$

From conditions (24) and (25) we find

$$A = \frac{\gamma}{b} \text{ and } B = -\frac{\gamma}{b} e^{-b h'}$$

$$\text{Therefore } p = \frac{\delta}{b} (e^{-bz} - e^{-bh'}) \quad (26)$$

The hydrodynamic pressures is then found from (18).

The pressure distribution when $h' = h$, i.e., at the time when sedimentation is assumed to stop is

$$p = \frac{\delta}{b} (e^{-bz} - e^{-bh}) \quad (27)$$

$$\text{and } w = \gamma (h-z) - \frac{\delta}{b} (e^{-bz} - e^{-bh}) \quad (28)$$

b. Second Stage. We will consider here the effect of the own weight of the material on the process of consolidation. In this stage the time is independent of z and therefore from (18) we get

$$\frac{\partial p}{\partial t} = - \frac{\partial w}{\partial t}$$

Substituting in (15) we have the following differential equation:

$$\frac{\partial w}{\partial t} = c \frac{\partial^2 w}{\partial z^2} \quad (29)$$

which is identical with Fourier's equation for the linear flow of heat, provided w and c are made to correspond to the temperature difference and the diffusivity respectively⁽⁶⁾.

If we make use of this analogy, our problem will be equivalent to that of non-steady flow of heat through a plate of isotropic material of thickness h , having one

(6) For a thermodynamic analogy of this problem, see Principles of Soil Mechanics, by C. Terzaghi, Engineering News Record Nov. 26, 1925, and also Terzaghi's "Erdbaumechanik", pp. 142-143.

bounding plane impermeable to heat and the other at zero temperature.

We will solve (29) by Fourier's method (7); we will find a Fourier series development for w which will satisfy (29) and also the boundary conditions.

~~The boundary conditions as they stand ($w=0$ for $z=h$ and $\frac{\partial w}{\partial z}=0$ for $z=0$) cannot be applied directly to determine the constant terms in the series development of w , but~~ The property that the bottom surface ($z=0$) is impermeable gives rise to the following interpretation (9).

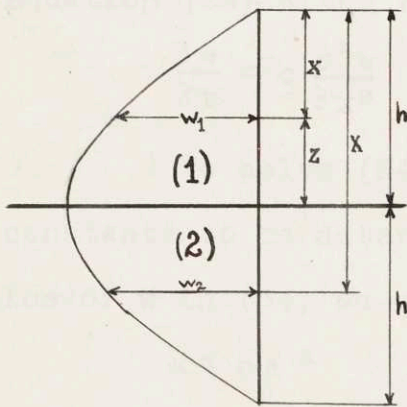


Fig. 8.

Suppose that we have a plate of thickness equal to $2h$ (Fig. 8), provided with a plane of separation at its center which is impermeable to heat. Let both halves of this plate have identical temperature distributions as shown in

Fig. 8. Then ^{either half (1) or (2) may} without changing the be removed]

temperature distribution on the other half, since there is no flow of heat through the central plane. Making the transformation

$$z = h - x \quad (30)$$

the boundary conditions will be

$$\left. \begin{aligned} w &= 0 \text{ for } x = 0 \\ \text{and } w &= 0 \text{ for } x = 2h. \end{aligned} \right\} \quad (31)$$

(7). Cf. Byerly's Fourier's Series and Spherical Harmonics and also Ingersoll and Zobel's Mathematical Theory of Heat Conduction. pp. 44-65.

(8). Cf. Ingersoll-Zobel's Math. Theory of Heat Conduction p. 72 and p. 107.

In order to simplify our equations we will count the time so that $t = 0$ corresponds to the time when sedimentation no longer occurs.

The initial distribution of the hydrodynamic pressure (or temperature) throughout the layer of thickness $2h$ is, according to equations (18) and (26)

$$w_1 = f_1(x) = \gamma x - \frac{\gamma}{b} e^{-bx} (e^{bx} - 1) \text{ for } t = 0 \text{ and} \\ 0 \leq x \leq h \quad (32)$$

$$w_2 = f_2(x) = \gamma(2h-x) - \frac{\gamma}{b} e^{-b(2h-x)} [e^{b(2h-x)} - 1] \\ \text{for } t = 0 \text{ and } h \leq x \leq 2h \quad (33)$$

Equation (29) after the transformation (30), becomes:

$$\frac{\partial w}{\partial t} = c \frac{\partial^2 w}{\partial x^2} \quad (34)$$

To solve (34) let $w = e^{\alpha t + \beta x}$ where α and β are constants to be determined. Substituting this assumed value of w in (34) we obtain the relation

$$\alpha = c \beta^2$$

and in order to have our solution in terms of trigonometric functions (instead of hyperbolic) we set

$$\beta^2 = -\lambda^2 \text{ or } \beta = \pm i\lambda \text{ where } i = \sqrt{-1}$$

Therefore $w = e^{-c\lambda^2 t} \pm i\lambda x$ is a solution of (34).

Remembering that $e^{\pm iy} = \cos y \pm i \sin y$, we shall have

$$w = e^{-c\lambda^2 t} \cos \lambda x \quad (35)$$

$$\text{and } w = e^{-c\lambda^2 t} \sin \lambda x \quad (36)$$

as particular (or partial) solutions of equation (34).

Now solution (36) will satisfy both of the boundary conditions (31) for all values of t provided that we take

$$\lambda = \frac{n\pi}{2h}$$

where n is an integer ($n = 1, 2, 3, \dots$)

$$\text{Therefore } w = e^{-\frac{c n^2 \pi^2 t}{4h^2}} \sin \frac{n\pi}{2h} x \quad (37)$$

is a particular solution of (34).

If, in (37), we assign to n any arbitrary integral positive value, the equation thus obtained will satisfy both (31) and (34) and will therefore be a particular solution. Consequently there are an infinite number of particular solutions satisfying (31) and (34). Any of these particular solutions when multiplied by a constant is also a solution, and therefore the general solution of equation

$$(34) \text{ is given by } w = \sum_{n=1}^{n=\infty} a_n e^{-\frac{c n^2 \pi^2 t}{4h^2}} \sin \frac{n\pi}{2h} x \quad (38)$$

where a_n is constant, i.e., it assumes constant values for all integral values of n .

To complete the solution it remains only to determine a_n as a function of n . To do this we make use of the fact that the initial distribution of hydrodynamic pressure is known (equations (32) and (33)), that is, for $t = 0$, $w = \varphi(x)$ where $\varphi(x)$ stands for both $f_1(x)$ and $f_2(x)$.

$$\text{Therefore } \varphi(x) = \sum_{n=1}^{n=\infty} a_n \sin \frac{n\pi}{2h} x$$

From this expression we obtain

$$a_n = \frac{1}{h} \int_0^{2h} \varphi(x) \sin \frac{n\pi}{2h} x \, dx.$$

Substituting in this expression the two values of (x) given by (32) and (33) we get

$$a_n = \frac{\gamma}{h} \left\{ \int_0^h \left[x - \frac{e^{-bh}}{b} (e^{bx} - 1) \right] \sin \frac{n\pi}{2h} x \, dx + \int_h^{2h} \left[2h-x - \frac{e^{-bh}}{b} [e^{b(2h-x)} - 1] \right] \sin \frac{n\pi}{2h} x \, dx \right\}.$$

Integrating and simplifying we obtain

$$\begin{aligned} \frac{h}{\gamma} a_n &= \frac{4h^2}{n\pi} \left(\cos n\pi - \cos \frac{n\pi}{2} \right) + \frac{4h^2}{n^2\pi^2} \left(2 \sin \frac{n\pi}{2} - n\pi \cos \frac{n\pi}{2} + \right. \\ &\quad \left. + n\pi \cos n\pi \right) - \frac{2he^{-bh}}{n\pi b} (\cos n\pi - 1) - \left[\frac{e^{-bh}}{b} \right]^x \\ &\quad \times \left[\frac{e^{bh} (2b \sin \frac{n\pi}{2}) - \frac{n\pi}{2h} (\cos n\pi - 1)}{b^2 + \frac{n^2\pi^2}{4h^2}} \right]. \end{aligned}$$

The right-hand member of this equation vanishes for all even values of n and therefore only odd values of n should be considered. When n is odd we have:

$$\frac{h}{\gamma} a_n = 8h^2 \left[\frac{\sin \frac{n\pi}{2}}{n^2 \pi^2} - \frac{\sin \frac{n\pi}{2} - \frac{2hb}{n\pi} e^{-bh}}{4h^2 b^2 + n^2 \pi^2} \right]$$

$$\text{or } a_n = \frac{16\gamma h^2 b}{\pi} \cdot \frac{\frac{2hb}{n\pi} \sin \frac{n\pi}{2} + e^{-bh}}{n(4h^2 b^2 + n^2 \pi^2)} \quad (39)$$

$$\begin{aligned} \text{Therefore} \\ w &= \frac{16\gamma h^2 b}{\pi} \sum_{n=1,3,5,\dots} \left\{ \left[\frac{\frac{2hb}{n} \sin \frac{n\pi}{2} + e^{-bh}}{n(4h^2 b^2 + n^2 \pi^2)} \right]^x \right. \\ &\quad \left. \times \left[e^{-\frac{cn^2 \pi^2}{4hs}} t \sin \frac{n\pi}{2h} (h-z) \right] \right\} \quad (40) \end{aligned}$$

is the equation giving the value of the hydrodynamic pressure for any positive values of t and z. Knowing w we can compute p from equation (18).

Equation (40) is seen to satisfy the differential equation (34) and both of the boundary conditions:

$w=p=0$ for $z = 0$ and $\frac{\partial w}{\partial z} = 0$ for $z = h$ and for all values of t . When $t = \infty$ equation (40) gives $w = 0$, $p = \gamma(h-z)$ as should be the case.

The series represented by equation (40) converges very rapidly (as will be shown later in applying it to a concrete problem), so that an approximate solution may be obtained by setting $n = 1$, and disregarding the subsequent terms in the series. Doing this we have

$$w = \frac{16\delta h^2 b}{\pi} \frac{\frac{2h_0 b}{\pi} + e^{-bh}}{4h^2 b^2 + \pi^2} e^{-\frac{c\pi^2}{4h^2} t} \sin \frac{\pi}{2h}(h-z) \quad (41)$$

We will now attempt a solution for the case in which due account is taken of the variation of the coefficient of permeability k , and the modulus of compression a .

3. Solution of Equation (16).

(a). First Stage: From equation (3) we have

$$a = - \frac{dt}{dp} = \frac{\alpha}{p+\beta}$$

$$\text{and } k = ac = \frac{c\alpha}{p+\beta}$$

$$\frac{1}{a} \frac{dk}{dz} = \frac{c}{a} \frac{da}{dz} = - \frac{c}{p+\beta} \frac{dp}{dz}$$

$$t' = \frac{\gamma}{q} (h-z) \text{ and } w = \gamma(h-z) - p$$

$$\text{Therefore } \frac{dp}{dt'} = - \frac{q}{\delta} \frac{dp}{dz} \text{ and } \frac{dw}{dz} = -\gamma - \frac{dp}{dz}, \frac{d^2 w}{dz^2} = - \frac{d^2 p}{dz^2}$$

Substituting in (16) we get

$$-\frac{q}{\gamma} \frac{dp}{dz} = c \left[\frac{d^2p}{dz^2} - \frac{1}{p+\beta} \frac{dp}{dz} \left(\gamma + \frac{dp}{dz} \right) \right]$$

$$\text{or } c \frac{d^2p}{dz^2} + \left(\frac{q}{\gamma} - \frac{c\gamma}{p+\beta} \right) \frac{dp}{dz} - \frac{c}{p+\beta} \left(\frac{dp}{dz} \right)^2 = 0 \quad (42)$$

This differential equation is of a type seldom encountered, and therefore we will give all the steps required for its solution.

First, let $m = \frac{dp}{dz}$, then $\frac{d^2p}{dz^2} = m \frac{dm}{dp}$ and we get

$$cm \frac{dm}{dp} + \left(\frac{q}{\gamma} - \frac{c\gamma}{p+\beta} \right) m - \frac{c}{p+\beta} m^2 = 0$$

$$\text{or } \frac{dm}{dp} + \left(\frac{q}{\gamma c} - \frac{\gamma}{p+\beta} \right) - \frac{m}{p+\beta} = 0$$

Let now $m = m_0 + u$ where m_0 is a constant at our disposal to which we will give such a value as to simplify our differential equation. Substituting, we have:

$$\frac{du}{dp} - \frac{1}{p+\beta} u = \frac{\gamma}{p+\beta} - \frac{q}{\gamma c} + \frac{m_0}{p+\beta}$$

$$\text{Take } m_0 = -\gamma$$

$$\text{Then } \frac{du}{dp} = \frac{1}{p+\beta} u = -\frac{q}{\gamma c}$$

Let now $u = xy$ where both x and y are independent variables.

$$\text{Therefore } \left(\frac{dy}{dp} - \frac{1}{p+\beta} y \right) + \frac{1}{x} \left(y \frac{dx}{dp} + \frac{q}{\gamma c} \right) = 0$$

$$\text{Now let } y \text{ be such that } \frac{dy}{dp} - \frac{1}{p+\beta} y = 0$$

Its solution is $y = p + \beta$.

With this value of y the differential equation becomes

$$(p+\beta) \frac{dx}{dp} = - \frac{q}{\gamma c}$$

which has

$$x = - \frac{q}{\gamma c} \log R (p+\beta),$$

as a solution, R being a constant of integration.

$$\therefore u = xy = - \frac{q}{\gamma c} (p+\beta) \log R (p+\beta)$$

$$\text{and } \frac{dp}{dz} = m = m_0 + u = - \gamma - \frac{q}{\gamma c} (p+\beta) \log R (p+\beta)$$

$$\text{or } \frac{dp}{dz} + \frac{q}{\gamma c} (p+\beta) \log R (p+\beta) + \gamma = 0 \quad (43)$$

The boundary conditions are, as before,

$$z = h: p = w = 0$$

$$z = 0: \frac{dw}{dz} = 0 \quad \text{or} \quad \frac{dp}{dz} = -\gamma$$

If we call p_1 the value of p at the bottom of the deposit ($z = 0$) where $\frac{dp}{dz} = -\gamma$ we will have from (43) the following condition:

$$\frac{q}{\gamma c} (p_1 + \beta) \log R (p_1 + \beta) = 0$$

Since β is essentially positive and R must be finite, we must have

$$R(p_1 + \beta) = 1 \quad (44)$$

Let $r = \log R (p+\beta)$. Then the solution of (43) is $\frac{1}{A} \int \frac{dr}{\frac{r}{A} + e^{-r}} = - \frac{q}{\gamma c} (z + S)$ (45)

where S is a constant of integration and

$$A = \frac{R\gamma^2 c}{q}$$

In order to evaluate the integral in (45), expand e^{-r} in a power series, and then the integral will take the form

$$\frac{1}{A} \int \frac{d r}{\frac{r}{A} + 1 - r + \frac{r^2}{2!} - \frac{r^3}{3!} + \dots}$$

If r is small we can disregard the terms in the series development containing r in powers higher than the second. With this assumption only an approximate solution can be obtained but such a solution is not expected to differ appreciably from the exact solution, since for small values of z (where the variation of p is greater) r is very small, it being zero when $z = 0$ on account of (44) if (44) is still to hold true after the above assumption is made. (This will be shown later to be the case).

Equation (45) then becomes:

$$\frac{1}{A} \int \frac{d r}{1 - \left(1 - \frac{1}{A}\right) r + \frac{r^2}{2}} = - \frac{q}{\gamma^2 c} (z + S)$$

$$\text{or } \frac{1}{A} \left[\frac{1}{\sqrt{\left(1 - \frac{1}{A}\right)^2 - 2}} \log \frac{r - \left(1 - \frac{1}{A}\right) - \sqrt{\left(1 - \frac{1}{A}\right)^2 - 2}}{r - \left(1 - \frac{1}{A}\right) + \sqrt{\left(1 - \frac{1}{A}\right)^2 - 2}} \right] =$$

$$= - \frac{q}{\gamma^2 c} (z + S) \quad (A)$$

$$\text{if } \left(1 - \frac{1}{A}\right)^2 < 2$$

$$\text{and } \frac{1}{A} \left[\frac{2}{\sqrt{2 - (1 - \frac{1}{A})^2}} \tan^{-1} \frac{r + (1 - \frac{1}{A})}{\sqrt{2 - (1 - \frac{1}{A})^2}} \right] = - \frac{q}{\gamma c} (z+S) \quad (B)$$

$$\text{if } (1 - \frac{1}{A})^2 < 2.$$

Since integrals of logarithmic functions are in general logarithmic functions we will assume solution (A) to hold.

Substituting in (A) the values of r and A and introducing the boundary conditions, $\frac{dp}{dz} = -\gamma$ for $z = 0$ where $p = p_1$, and $p = 0$ for $z = h$, we get:

$$\left[\log R (p_1 + \beta) \right]^2 - 2 \left(1 - \frac{q}{R\gamma^2 c} \right) \log R (p_1 + \beta) - \frac{2}{R(p_1 + \beta)} = -2$$

One solution of this equation is $R(p_1 + \beta) = 1$ which is in accordance with equation (44).

The equation of the pressure distribution is

$$\frac{\log R (p+\beta) - \left(1 - \frac{q}{R\gamma^2 c} \right) - \sqrt{\left(1 - \frac{q}{R\gamma^2 c} \right)^2 - 2}}{\log R (p+\beta) - \left(1 - \frac{q}{R\gamma^2 c} \right) + \sqrt{\left(1 - \frac{q}{R\gamma^2 c} \right)^2 - 2}} = \left[e^{-R\sqrt{\left(1 - \frac{q}{R\gamma^2 c} \right)^2 - 2} z} \right] \left[e^{-R\sqrt{\left(1 - \frac{q}{R\gamma^2 c} \right)^2 - 2} S} \right] \quad (46)$$

The constant p_1 is determined from (44) and R and S from the following equations:

$$\frac{\log R\beta - (1 - \frac{q}{R\gamma^2 c}) - \sqrt{(1 - \frac{q}{R\gamma^2 c})^2 - 2}}{\log R\beta - (1 - \frac{q}{R\gamma^2 c}) + \sqrt{(1 - \frac{q}{R\gamma^2 c})^2 - 2}} =$$

$$= \frac{(1 - \frac{q}{R\gamma^2 c})^2 + \sqrt{(1 - \frac{q}{R\gamma^2 c})^2 - 2}}{(1 - \frac{q}{R\gamma^2 c}) - \sqrt{(1 - \frac{q}{R\gamma^2 c})^2 - 2}} e^{-R\gamma\sqrt{(1 - \frac{q}{R\gamma^2 c})^2 - 2} h}$$

----- (47)

$$e^{-R\gamma\sqrt{(1 - \frac{q}{R\gamma^2 c})^2 - 2} s} = \frac{2}{\left[- (1 - \frac{q}{R\gamma^2 c}) + \sqrt{(1 - \frac{q}{R\gamma^2 c})^2 - 2} \right]^2}$$

----- (48)

By working out the solution in terms of trigonometric functions, (solution (B)), and determining the constants R, S and p_1 , it was found that the condition expressed by equation (44) does not hold, and therefore it does not represent the solution of our problem.

Equation (47) can be solved only by trials. In order to make the solution of equation (47) easier we will work out a less accurate solution by setting $e^{-r} = 1 - r$ which is not much in error for small values of r. The equation thus obtained will be less complex, and will show approximately the values of R to be tried in equation (47).

We have

$$\frac{1}{A} \int \frac{d r}{\frac{r}{A} + 1 - r} = - \frac{q}{\gamma c} (z + S) \quad (49)$$

Integrating and simplifying we get

$$\left(1 - \frac{q}{R\gamma^2 c}\right) \log R(p + \beta) = 1 - e^{-\left(\frac{q}{\gamma^2 c} - R\gamma\right)(z + S)}.$$

Introducing the boundary conditions, we have

$$\left(1 - \frac{q}{R\gamma^2 c}\right) \log R(p_1 + \beta) + \frac{1}{R(p_1 + \beta)} = 1,$$

one solution of which is $R(p_1 + \beta) = 1$, and this is also in accordance with (44).

Also $S = 0$,

$$\left(1 - \frac{q}{R\gamma^2 c}\right) \log R\beta = 1 - e^{-\left(\frac{q}{\gamma^2 c} - R\gamma\right)h}, \quad (50)$$

$$\text{and } \left(1 - \frac{q}{R\gamma^2 c}\right) \log R(p + \beta) = 1 - e^{\left(\frac{q}{\gamma^2 c} - R\gamma\right)z} \quad (51)$$

R can be found from (50) and then values of p for several values of z are given by (51).

(b). Second Stage: In this stage, since t is independent of z , the differential equation is partial instead of ordinary and if it can be solved at all the resulting equation will be so complex as to make the analysis worthless. As a matter of fact, a comparison between the results obtained by applying equations (27) and (46) to a concrete problem, which will be given later, conclusively shows that they differ but slightly, and therefore equation (27) is accurate enough for any practical purpose. The same would, of course, be true for the second stage of consolidation. Consequently, there is no need of trying to solve equation (16) for the second stage.

So far we have determined the pressure distribution

with reference to the reduced depth of the deposit. We will now show how it can be determined with reference to the actual or true thickness of the deposit, and also its settlement at any time.

4. Actual Pressure Distribution and Settlements.

Let \underline{H} be the actual thickness of the deposit which corresponds to the reduced thickness h . Let also \underline{Z} be the distance of a point in the actual deposit measured from the bottom surface, and corresponding to the reduced distance z . Then

$$H = \int_0^h (1 + \xi) dz \quad (52)$$

$$\text{and } Z = \int_0^z (1 + \xi) dz \quad (53)$$

where, as before, ξ is the voids-ratio and measures the water-content per unit of solid matter.

In general we know the true depth of the deposit and what we want to find is its reduced thickness. If ξ is known for several depths, then a curve can be plotted and the integrals (52) and (53) calculated graphically.

Now it was pointed out that there is a definite relation between the water content and the granular pressure for any given material, and this is given by

$$\xi = -\alpha_1 \log(p + \beta) - \alpha_2 (p + \beta) + C_1$$

or, since α_2 is very small, we may write

$$\xi = -\alpha \log(p + \beta) + C_2.$$

Now \underline{p} can be expressed as a function of \underline{z} : For the first stage of consolidation it is expressed by equation (27) and for the second stage by equation (40) where $p = \gamma(h - z) - w$, the variable \underline{t} being kept constant during the integration since this is performed with respect to \underline{z} only.

Therefore

$$Z = \int_0^z [-\alpha \log (p+\beta) + C] dz \quad (54)$$

where $C = 1 + C_2$.

Having thus obtained Z as a function of z , and the pressure distribution in terms of z , the latter will be determined in terms of Z by simply changing the ordinates by the transformation (54). In practice, however, the variation of $\underline{\xi}$ with \underline{z} is small and practically uniform so that an average value of ξ may be taken and introduced in equation (53). Let ξ_a be the average value of ξ , then

$$Z = (1 + \xi_a) z \quad (55)$$

which is a linear transformation and is equivalent to

$$Z = \frac{H}{h} z \quad (55a)$$

To compute the settlement of the top surface of the deposit, let H_0 and ξ_0 respectively, be the true depth and the voids-ratio at time $t = 0$, i.e., just after sedimentation has stopped. Let H_1 and ξ_1 be the corresponding quantities at any subsequent time $t = t_1$. If we denote

by s the settlement of the top surface at any time t , we will have $s_1 = H_0 - H_1$ as the total settlement of the top surface of the deposit in the time interval t_1 .

Now the time rate of change in water-content per unit of solid matter is $\frac{\partial \epsilon}{\partial t}$ which is a negative quantity since ϵ decreases with the time and therefore the time rate of change in water-content in a layer of thickness dz located at distance z above the bottom surface is

$$- \frac{\partial \epsilon}{\partial t} dz = a \frac{\partial p}{\partial t} dz.$$

Hence the rate at which the top surface of the deposit is settling at any specified time $t = t_1$ is

$$\left(\frac{ds}{dt}\right)_{t=t_1} = \int_0^h a \left(\frac{\partial p}{\partial t}\right)_{t=t_1} dz \quad (56)$$

Therefore the total settlement which takes place in a time interval from $t = 0$ to $t = t_1$ is

$$s_1 = \int_0^{t_1} \int_0^h a \left(\frac{\partial p}{\partial t}\right) dz dt = \int_0^{t_1} \left(\frac{ds}{dt}\right) dt \quad (57)$$

and that from a time $t = t_1$ to a time $t = t_2$ is

$$s^1 = \int_{t_1}^{t_2} \int_0^h a \left(\frac{\partial p}{\partial t}\right) dz dt$$

Introducing the value of $\frac{\partial p}{\partial t} (= -\frac{\partial w}{\partial t})$ from equation (40) in equations (56) and (57) and keeping a constant during the integrations, we get:

$$\frac{ds}{dt} = 8 a q h \sum_{n=1,3,\dots} \frac{\frac{2hb}{n\pi} \sin \frac{n\pi}{2} + e^{-bh}}{4h^2 b^2 + n^2 \pi^2} e^{-\frac{cn^2 \pi^2}{4h^2} t} \quad (56a)$$

$$\text{and } s = \int_0^t \frac{ds}{dt} dt = \frac{32a^2 qh^3}{k \pi^2} \frac{2hk \sin \frac{n}{2} + e^{-bh}}{n^2(4h^2b^2+n^2 s)} \times \left(1 - e^{-\frac{c n^2 \pi^2}{4 h^2}}\right) \quad (57a)$$

as the rate of settlement and total settlement of the top of the deposit at any time, t .

At this point it is well to point out the fact that the coefficient of permeability k , as used in the previous equations is in terms of the reduced dimensions, while in performing the experiment already described, its value was obtained in terms of the true dimensions of the layer of material under test. In applying the law of Darcy, let $Q' = k' i' A'$ refer to the true dimensions of the layer and $Q = k i A$ to the reduced dimensions. Then, since $Q = Q'$ and $A = A'$ we must have

$$k' i' = k i$$

$$\text{or } k = k' \frac{i'}{i} = k' \frac{\frac{h}{I}}{\frac{h(1+\epsilon)}{I}} = \frac{k'}{1+\epsilon} \quad (58)$$

according to Fig. 1, where I is the true thickness of the layer and h the hydraulic head.

If the average value of ϵ is again introduced, we shall have

$$k = \frac{h}{H} k' \quad (58a)$$

We are now in a position to apply our formulae to a specific problem, but before we do so we will investigate the meaning of some ^{of} the previous equations.

Equation (27) shows that the smaller \underline{b} ($b = \frac{qa}{\gamma k}$) is, the greater will be \underline{p} for a given value of \underline{z} . Hence, of two mud deposits for which \underline{a} , $\underline{\gamma}$ and \underline{k} are the same, the one for which \underline{q} is smaller (q measures the rate at which solid matter is being sedimented) will be in a more consolidated state than the other. This fact deserves some consideration for frequently, at the same basin, for some reason or other, sedimentation is much more intensive in some particular locations than in others, and the fact that the material deposited is the same throughout the basin would lead to the erroneous conclusion that the state of internal stresses should be the same throughout the basin. Also, other things being equal, the greater is the coefficient of permeability, the more consolidated will be the deposit.

Equation (26) shows that the state of internal stress is independent of the hydrostatic head under which the deposit is being formed, and merely depends on the thickness of the deposit. This is evident because the excess in hydrostatic pressure, i.e., the hydrodynamic pressure is what produces the flow of water within the material.

Equation (40) shows that the time rate of change of the hydrodynamic stress, $(\frac{\partial w}{\partial t})$, decreases with the time in the same rate as the hydrodynamic stress itself. The slope $(\frac{\partial w}{\partial z})$ of the $w - z$ curve for a given value of \underline{z} is

continuously decreasing as time goes on, and therefore the quantity of water percolating through a given section of the deposit per unit time decreases with the time. This means that the variation of p with the time is greater at the beginning of the process of consolidation than for large values of t , i.e., the consolidation is more effective at the beginning of the (second) stage.

5. Example.

We shall now take up the problem of determining the state of internal stress and settlements of a delta deposit advancing at a given constant speed towards the ocean as is given on p. 177 of Terzaghi's "Erdbaumechanik". The consolidation of the deposit is due only to its own weight. The deposit is advancing towards the ocean at a rate of 1 m. per year and its true depth is 50 m.; the plane on which sedimentation takes place has a slope of 1 to 10 and therefore the time required for the formation of a layer of 50 m. is 500 years. The specific weight of the material sedimented is 2.7 grams per cubic centimeter, and this gives $\gamma = 2.7 - 1 = 1.7$ grams per cubic centimeter. The average value of the voids-ratio is 1.0 and therefore the reduced thickness of the deposit is 25 m. The rate of sedimentation is then $\frac{10}{2} = 5$ cubic centimeter per year and per unit of area. Therefore:

$$q = 5 \times 1.7 = 8.5 \text{ gr/cm}^2 \text{ per year.}$$

The true average value of the coefficient of

permeability is 1.63 cm./year and therefore its reduced value is

$$k = \frac{1.63}{1 + \epsilon_a} = \frac{1.63}{2} = 0.815 \text{ cm./year.}$$

The average value of the modulus of compression is

$$a = 0.00024 \text{ cm.}^5/\text{gr.}$$

Having these informations we are in a position to determine the pressure distribution throughout the deposit.

(a). First Stage. Let $h' = 10 \text{ m} = 1,000 \text{ cm.}$

$$b = \frac{q a}{\delta k} = \frac{8.5 \times 0.00024}{1.7 \times 0.815} = 0.001472,$$

$$c = \frac{k}{a} = 3,395,$$

$$\frac{\delta}{b} = \frac{1.7}{0.001472} = 1,153,$$

$$e^{-bh'} = 0.2305,$$

$$\delta h = 1.7 \times 2,500 = 4,250 \text{ gr./cm.}^3.$$

From equation (26) we obtain the following values of \underline{p} and \underline{w} for $z = 0$, and $z = 500 \text{ cms.}$, respectively:

$$p = 1,153 (1 - 0.2305) = 887 \text{ gr./cm.}^3$$

$$w = 813 \text{ gr./cm.}^2$$

$$p = 1,153 (0.4790 - 0.2305) = 287 \text{ gr./cm.}^3$$

$$w = 563.$$

At the end of the first stage of consolidation, ($h = 2,500$ cms.) we have the following pressure distribution:

z cms.	0	500	1,000	1,500	2,000	2,500
p gr./cm ²	1,125	525	237	110	30	0
w gr./cm ²	3,125	2,875	2,313	1,590	820	0

b. Second Stage. In this case we will compute the pressures for $t = 500, 1,000$ and $5,000$ years, the time being measured from the beginning of the stage, i.e., after the sediment has reached a height equal to h .

Equation (40) may be written in the form:

$$w = \frac{16 \delta h^2 b}{\pi} \sum_{n=1,3,5,\dots} N_n e^{-\frac{cn^2\pi^2}{4h^2}} \sin \frac{n\pi}{2h} (h-z)$$

where
$$N_n = \frac{\frac{2hb}{n\pi} \sin \frac{n\pi}{2} + e^{-bh}}{n(4h^2b^2 + n^2\pi^2)}$$

$$\frac{16 \delta h^2 b}{\pi} = 79,700.$$

The results may now be tabulated as follows:

n	N_n	$e^{-\frac{cn^2\pi^2}{4h^2} t}$			$\sin \frac{n\pi}{2h} (h-z)$			
		t=500	t=1,000	t=5,000	z=0	$z=\frac{1}{4} h$	$z=\frac{1}{2} h$	$z=\frac{3}{4} h$
1	+0.0370	0.512	0.262	0.00122	+1	+0.924	+0.707	+0.383
3	-0.00264	0.00238	0.000061	-	-1	-0.216	+0.707	+0.924
5	+0.000820	-	-	-	+1	-0.383	-0.707	+0.924

From this table we can compute the values of

$$N_n e^{-\frac{cn^2\pi^2}{4h^2} t} \sin \frac{n\pi}{2h} (h-z)$$

which are given in the following table.

Values of z	t = 500		t = 1,000		t = 5,000	
	n = 1	n = 3	n = 1	n = 3	n = 1	n = 3
0	+0.0190	+6.28 x 10 ⁻⁶	+0.00969	+1.61 x 10 ⁻⁸	+0.0000451	-
$\frac{1}{4}h$	+0.0175	+1.36 x 10 ⁻⁶	+0.00895	+0.348 x 10 ⁻⁸	+0.0000416	-
$\frac{1}{2}h$	+0.0136	-4.44 x 10 ⁻⁶	+0.00685	-1.14 x 10 ⁻⁸	+0.0000319	-
$\frac{3}{4}h$	+0.00726	-5.80 x 10 ⁻⁶	+0.00371	-1.49 x 10 ⁻⁸	+0.0000173	-

Values of z	Values of w in gr./cm. ²			Values of p in gr./cm. ²		
	t=500	t=1,000	t=5,000	t=500	t=1,000	t=5,000
0	1,510	771	3.6	2,740	3,479	4,246
$\frac{1}{4}$ h	1,395	713	3.3	1,803	2,483	3,194
$\frac{1}{2}$ h	1,083	546	2.5	1,042	1,579	2,123
$\frac{3}{4}$ h	579	296	1.4	484	767	1,061

The times 500, 1000 and 5,000 years correspond respectively to points in the deposit at the distances 500, 1,000 and 5,000 m. from the coast. From the above table we see that at a point 5 km. from the coast the deposit is practically consolidated. It should be remembered that so far as we have studied only the process of consolidation of the deposit under the action of its own weight, and evaporation together with other irregular phenomena occurring at the top surface of the deposit play an important part during the process of consolidation as will be shown later.

No data are available to compute the water content throughout the deposit, but, at least for the first stage of consolidation, we can take

$$Z = 2z$$

by and this single transformation obtain the actual pressure curves.

The results given in the above table are shown graphically in Fig. 9.

(c). Settlements and Rates of Settlements.

These are found from equations (56a) and (57a) which were obtained for the case of $a = \text{constant}$.

If, however, there is an appreciable variation in the values of a , we have, as already pointed out,

$$a = \frac{\alpha}{p + \beta}$$

where α and β are constants. Then

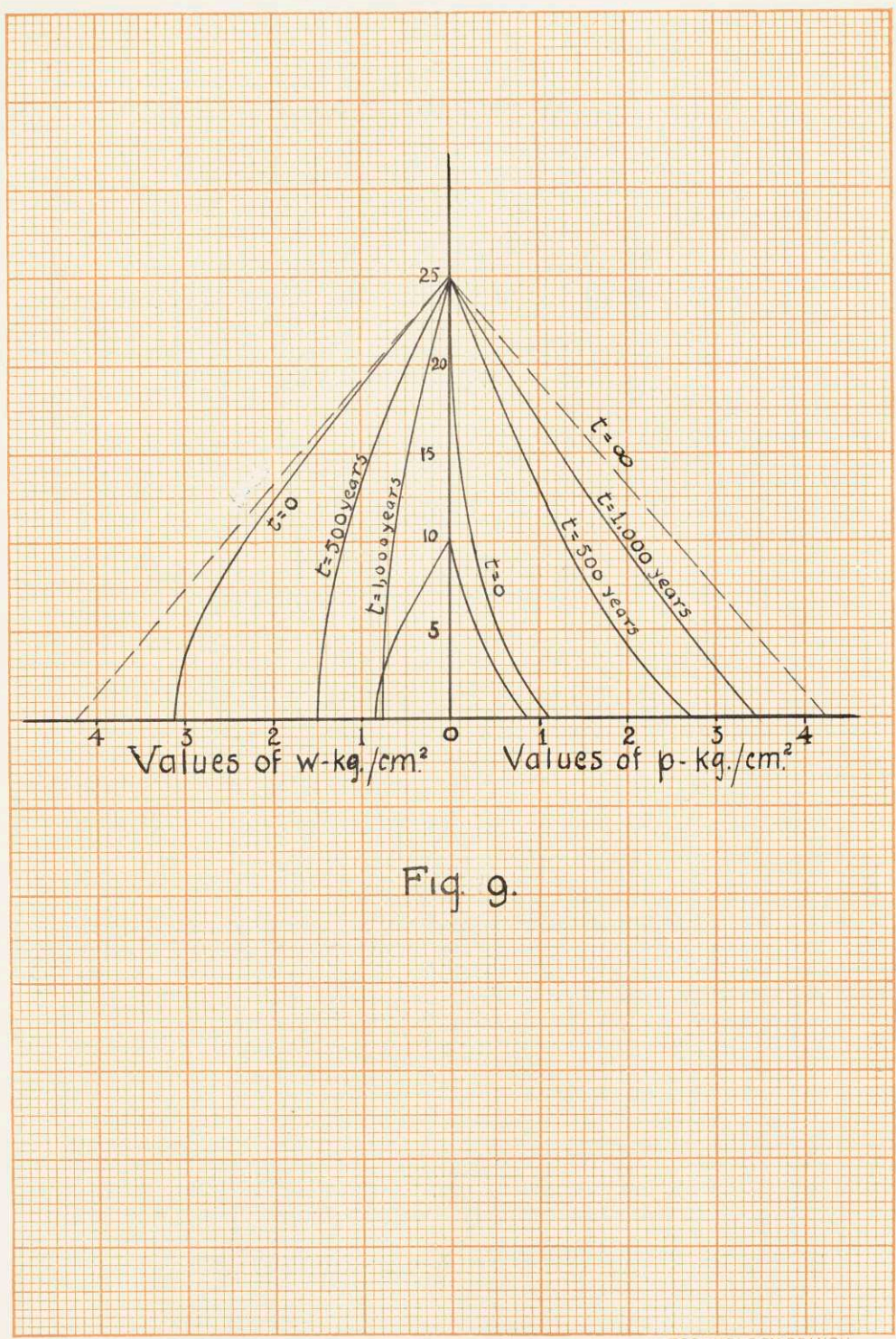


Fig 9.

Fig 9

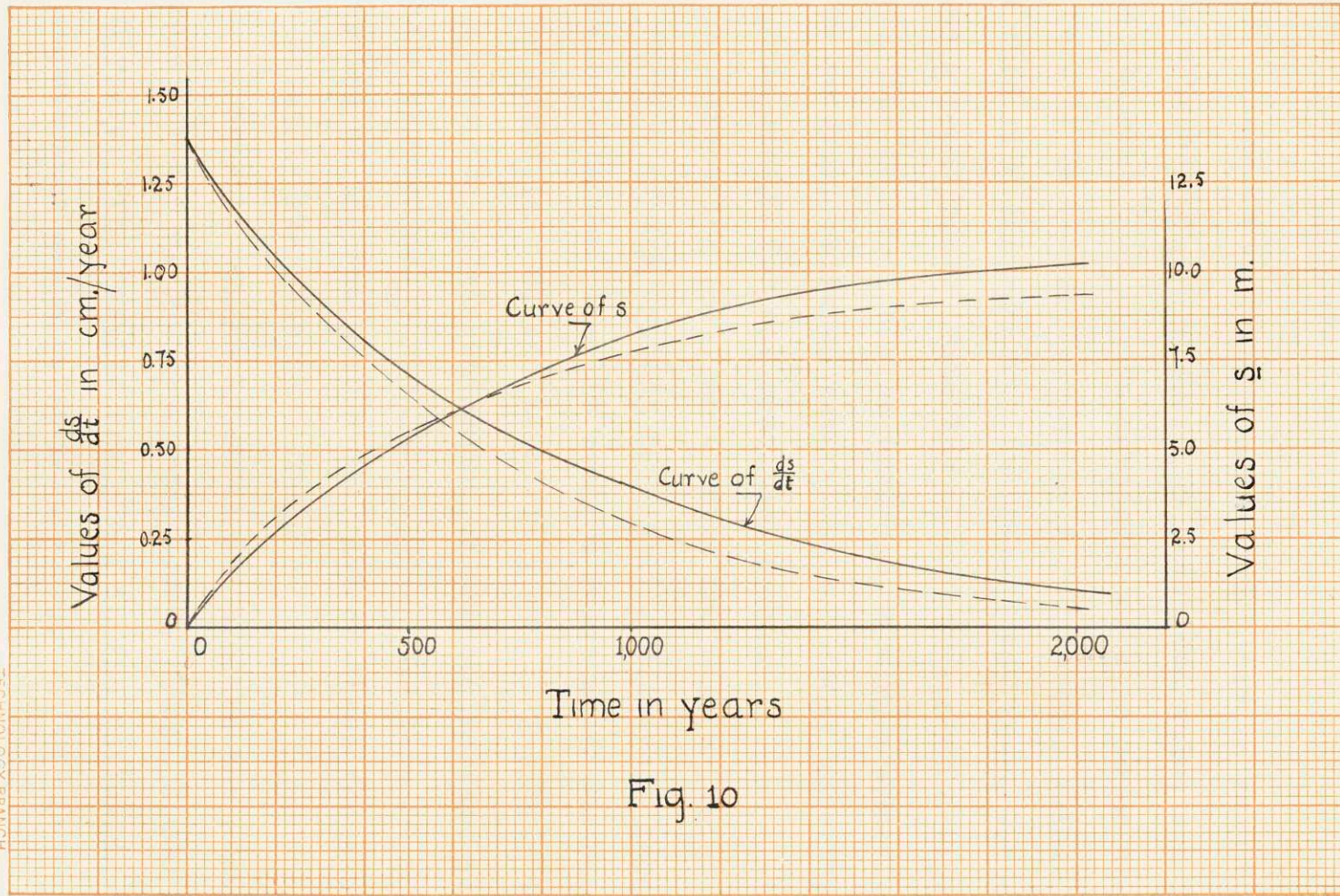


Fig. 10

$$\frac{ds}{dt} = \int_0^h \frac{\alpha}{p + \beta} \frac{\partial p}{\partial t} dz$$

and

$$s = \int_0^t \frac{ds}{dt} dt.$$

The above integrations would lead to such complex expressions as to be of no practical value and therefore it seems that if the variation of a is to be accounted for, it is better to compute a for several values of t and then find the rates of settlement for these values of t and a from (56a). Then the total settlement during any given time interval can be computed by simple additions. It is however, believed that equations (56a) and (57a) are accurate enough for any practical purpose.

Equation (56a) shows that the rate at which the top surface of the deposit is settling decreases exponentially with the time, it being a maximum at the beginning of the stage ($t = 0$) and zero for $t = \infty$. Equation (57a) shows that the total settlement of the top surface increases exponentially with the time and is zero at the beginning of the stage ($t = 0$). All of these facts have been actually observed in Nature.

We will now apply (56a) and (57a) to the previous example for $t = 0, 50, 100, 500, 1,000, 2,000, 3,000$ and $5,000$ years. The results are given in the following table, and represented graphically in Fig. 10.

Values of t years	0	50	100	500	1,000	2,000	3,000	5,000	∞
$\frac{ds}{dt}$ in cm./year	1.35	1.26	1.20	0.71	0.40	0.105	0.0276	0.00184	0
s in m.	0	0.611	1.29	5.27	8.30	10.25	10.82	11.00	11.05

Both equations (56a) and (57a) depend directly on \underline{a} since $\underline{k} = \underline{c} \underline{a}$ or, in other words, the constants \underline{b} and \underline{c} are not affected by the variation in \underline{a} . Now \underline{a} decreases with the time and therefore the curve of $\frac{ds}{dt}$ should be lower for large values of \underline{t} , but this does not indicate that the curve of s would be lowered in the same proportion, for it depends also on the previous values of \underline{a} . Thus, if for $t = 2,000$ years the value of \underline{a} were one-half of that used in computing the above table, we would have curves as those shown ^{dotted} in Fig. 10.

We can compute Z in terms of z and find the actual pressure distribution for given values of \underline{t} . Since we do not know the values of α and β for the material of the deposit in question, we will use equation (55a) instead of equation (54). The results are as follows:

Values of t , years	0	50	100	500	1,000	5,000
Ratio Z/z or H/h	2	1.976	1.948	1.779	1.668	1.560

(d). Solution by Applying Equations (46) and (51).

In order to obtain the pressure distribution we need the value of β . This is not known. We will therefore proceed to find the variation of the pressures p with β . The computations are very tiresome, and will not be given here. The results obtained by applying equation (51) are given in the following table, where p_1 is the

granular pressure at the bottom surface of the deposit.

R	β	p_1	Maximum Error in%
0.00010	8,820	1,180	0.72
20	3,770	1,230	3.31
25	2,750	1,250	-
30	2,060	1,270	8.46
36	1,480	1,295	-
40	1,200	1,300	16.8
50	685	1,315	18.7

The last column indicates the maximum error introduced by placing $e^{-r} = 1-r$. This of course does not mean that the results obtained deviate from the exact ones by the same amounts, since r is not a constant but a variable, no fixed relation existing between the two errors in question.

The above results are illustrative in that they show how little p_1 depends on β . β was made to vary from 8,820 to 685 while the corresponding values of p were found to be 1,180 and 1,315.

We turn now to the more accurate solution represented by equation (46). As already pointed out, this equation represents the solution of our problem only when

$$\left(1 - \frac{q}{R s_c}\right)^2 > 2.$$

This shows that R in this case must be less than about 0.000359. For this value of R the equation does

not hold, the pressure curve being a triangle. p_1 was computed from equation (46) for two values of R and the results are as follows:

R	β	p_1
0.00010	8,820	1,180
0.00030	1,960	1,370

The maximum errors introduced by placing $e^{-r} = 1 - r + \frac{r^2}{2}$ are in both cases very small.

The pressures for other values of z were computed the results being shown in Fig. 11.

Now the values of β are in general, much smaller than those for which p_1 was computed, so that it seems that there would be wider discrepancies than those shown by the preceding curves. But, on the other hand, in order to compare the results of the two theories, α and β must be such as to make the average value of a , as computed from

$$a = \frac{\alpha}{p + \beta}$$

compatible with the average value of 0.00024. The value of a (0.00024) was computed from a range in voids-ratio of from 1.2 to 0.8⁽⁹⁾ which correspond respectively to the voids-ratio at the top surface

(9). K. Terzaghi's "Erdbaumechanik" p. 178, See also p.141.

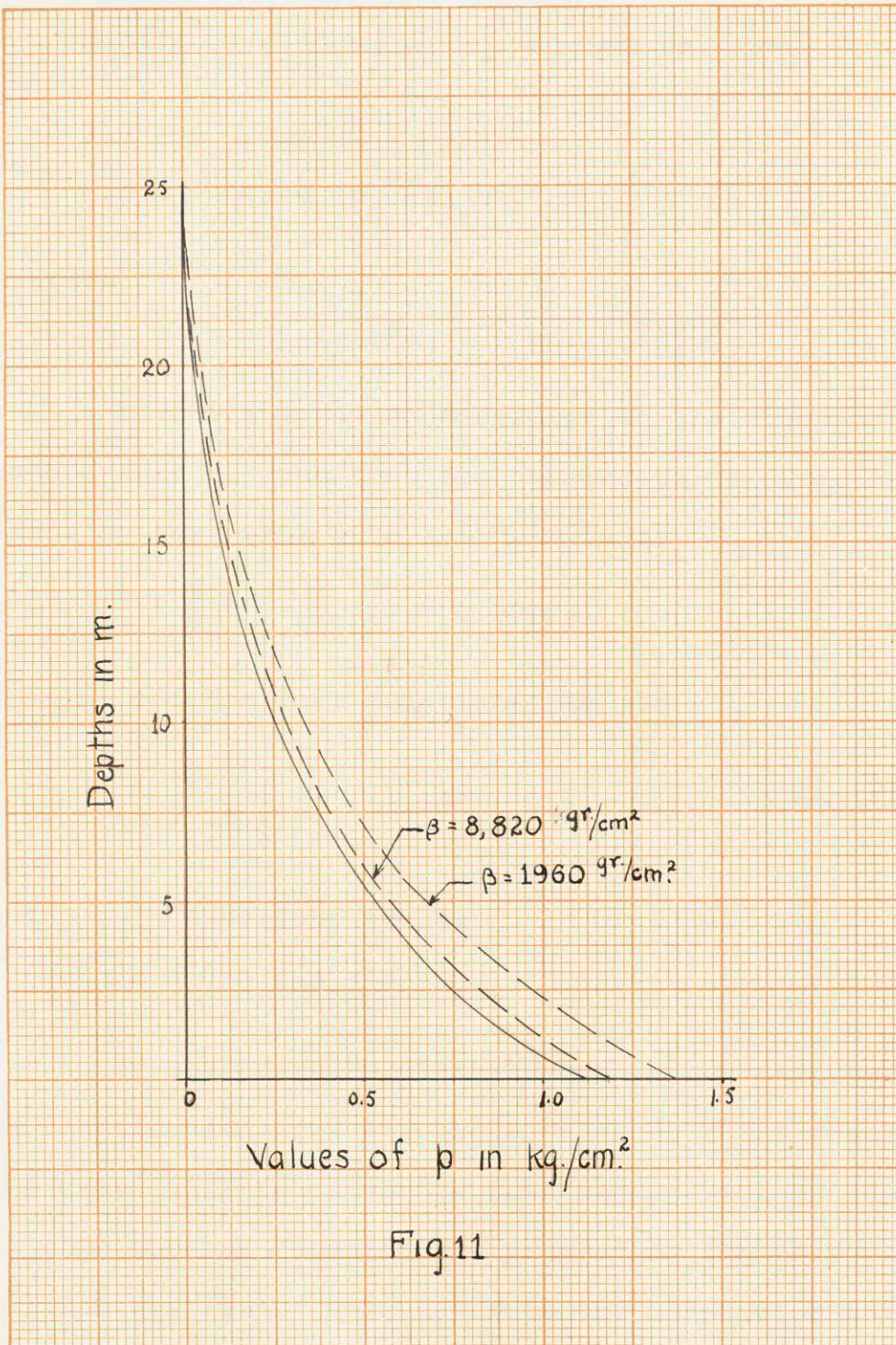


Fig.11

where $p = 0$ and the bottom surface where $p = p_1$.

Hence

$$\frac{1.2-0.8}{p_1-0} = 0.00024 \text{ or } p_1 = 1,668 \text{ gr./cm}^2.$$

Also $1.2 = -\alpha \log(0 + \beta) + C$

$$0.8 = -\alpha \log(p_1 + \beta) + C$$

Therefore

$$= \frac{0.4}{\log(1+\frac{p_1}{\beta})} = \frac{0.4}{\log(1+\frac{1,668}{\beta})}$$

If then, \underline{a} varies according to the above law, α and β must be such that the average value of \underline{a} , as p is made to vary from 0 to 1,668, will be

$$\frac{1}{2} \left(\frac{\alpha}{\beta} + \frac{\alpha}{1,668+\beta} \right) = 0.00024.$$

Combining the two last equations, we get:

$$\frac{0.4}{\log(1+\frac{1,668}{\beta})} (1,668 + 2\beta) - 0.8 - 0.00048 \beta^2 = 0.$$

Solving this equation, we get:

$$\beta = 3,730 \text{ and } \alpha = 1.06.$$

If the pressure distribution is now computed from equation (46) for the above value of β the resulting $p - z$ curve will be closer to that represented by equation (27) than the one shown in the preceding figure for which $\beta = 1,960 \text{ gr./cm.}^2$. This leads to the conclusion that the theory developed for \underline{a} constant is far more accurate than one could ever expect. Hence, the differential equation

IV Problem 2 - Mud Deposit under the Influence of a very permeable Fill, placed on top of it at the Beginning of the Second Stage of Consolidation.

1. Determination of stresses.

The consolidation due to the own weight of the material has already been considered in the preceding section. The combined effect of the own weight of the material and that of the top fill will be ascertained by properly combining the solution obtained for the two cases separately. Therefore in order to study the behavior of the mud deposit under the influence of the top fill alone, we disregard the weight of the material and proceed as follows, after neglecting the time required for depositing the fill, and also the resistance of the fill against percolation.

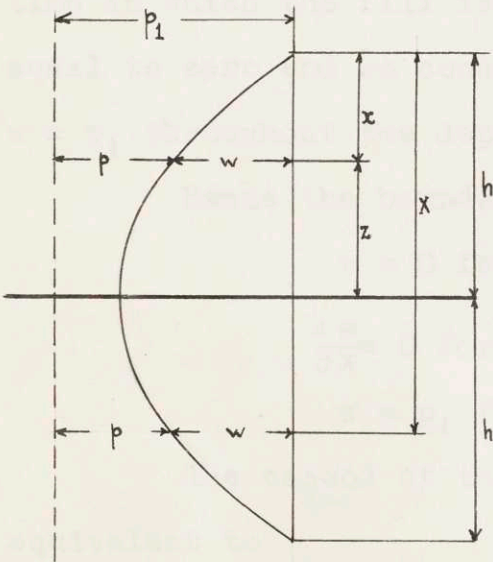


Fig. 12.

Let \underline{h} be the reduced thickness of the deposit, and let \underline{p} and \underline{w} , be the granular and the hydrodynamic pressures respectively at any section distant \underline{z} (or \underline{x}) from the bottom (or top) surface at any particular time \underline{t} .

Let $t = 0$ correspond to instant at which sedimentation has just ceased and let the fill be deposited at this same instant.

Let p_1 be the pressure per unit area exerted by the fill resting on the top of the deposit.

Since the resistance of the fill against percolation of water is neglected, the hydrodynamic pressure at the surface of the deposit will be zero and the granular pressure will be equal to p_1 (Fig.12). In order to have equilibrium we must have the following relation between p and w (10).

$$p + w = p_1 \quad (59)$$

As before, we have

$$\frac{\partial p}{\partial t} = -c \frac{\partial^2 w}{\partial z^2},$$

and from (59) we get

$$\frac{\partial w}{\partial t} = c \frac{\partial^2 w}{\partial z^2} = c \frac{\partial^2 w}{\partial x^2}, \quad (60)$$

where $x = h - z$

No water can possibly flow during the period of time in which the fill is deposited, (which is practically equal to zero and so considered) and therefore we must have $w = p_1$ throughout the deposit for $t = 0$

Hence the boundary conditions are

$$w = 0 \text{ for } x = 0 \text{ and } t > 0$$

$$\frac{\partial w}{\partial x} = 0 \text{ for } x = h \quad (61)$$

$$w = p_1 \text{ for } t = 0$$

The second of the above boundary conditions is equivalent to

$$w = 0 \text{ for } x = 2h$$

as already pointed out.

Let

$$w = e^{\alpha t + \beta x}$$

be the solution of (60).

then

$$\alpha = c\beta^2 = -c\lambda^2$$

therefore

$$w = e^{-c\lambda^2 t} \cos \lambda x \quad (62)$$

$$w = e^{-c\lambda^2 t} \sin \lambda x \quad (63)$$

are particular solutions of (60).

Now (63) is seen to satisfy both $w = 0$ for $x = 0$, and $w = 0$ for $x = 2h$, provided we set

$$\lambda = \frac{n\pi}{2h},$$

where $n = 1, 2, 3, \dots$

Hence the general solution of the differential equation (60) is given by

$$w = \sum_{n=1}^{n=\infty} a_n e^{-\frac{cn^2\pi^2}{4h^2} t} \sin \frac{n\pi x}{2h}, \quad (64)$$

where a_n represents constants multiplying every term of the series.

But $w = p_1$ for $t = 0$, and therefore

$$p_1 = \sum_{n=1}^{n=\infty} a_n \sin \frac{n\pi}{2h} x$$

$$\text{or } a_n = \frac{2}{2h} \int_0^{2h} p_1 \sin \frac{n\pi}{2h} x dx$$

$$\text{or } a_n = \frac{2p_1}{n\pi} (1 - \cos n\pi)$$

and this is zero for even values of n and equal to $\frac{4p_1}{n\pi}$

for odd values of n .

$$\text{Hence we have } w = \frac{4p_1}{\pi} \sum_{n=1,3,\dots} e^{-\frac{cn^2\pi^2}{4h^2} t} \frac{\cos \frac{n\pi}{2h} z}{n \sin \frac{n\pi}{2}} \quad (65)$$

(10) Cf. K. Terzaghi, "Erdbaumechanik" pp. 143-4.

$$\text{and } p = p_1 \left(1 - \frac{4}{\pi} \sum_{n=1,3,\dots} \frac{e^{-\frac{cn^2 \pi^2 t}{4h^2}} \cos \frac{n\pi z}{2h}}{n \sin \frac{n\pi}{2}} \right) \quad (66)$$

In regard to the boundary conditions we made use of two conditions which, at first, seem to be inconsistent, namely: $w = 0$ for $x = 0$ and $w = p_1$ for $t = 0$ throughout the deposit. Now, at the very surface of the deposit the granular pressure must always be equal to p_1 , while at any other section, p_1 is taken up partly by the capillary water and partly by the granular material (according to (59)), but at $t = 0$, when p_1 is supposed to be applied, no water can possibly flow on account of the smallness of the value of the coefficient of permeability, and since water is practically incompressible, it follows that at $t = 0$ the whole pressure p_1 must be taken up by the capillary water. Consequently there is a discontinuity in the pressure distribution at the surface of the deposit at the instant $t = 0$.

The assumption that $w = p_1$ for $t = 0$ may be readily confirmed by applying Heaviside's Operational Method⁽¹¹⁾ to the differential equation (60). Since this method

(11) Cf. Heaviside's Operators in Engineering and Physics by E.J. Berg, Journal of the Franklin Institute, Nov. 1924. Also: An Analogy between Pure Mathematics and Operational Mathematics of Heaviside by Means of the Theory of H-Functions, by J.J. Smith, Journal of the Franklin Institute, Oct., Nov., and Dec., 1925.

affords a simple way of solving the above differential equation and at the same time no assumption concerning w is involved, it will be given below.

We will now use the differential equation in terms of p instead of w . It is

$$\frac{\partial p}{\partial t} = c \frac{\partial^2 p}{\partial z^2} \quad (67)$$

The boundary conditions are

$$p = p_1 \text{ for } z = h \quad (68)$$

and
$$\frac{\partial p}{\partial z} = 0 \text{ for } z = 0 \quad (69)$$

We now introduce an operator in (67), i.e., we set

$$\frac{\partial}{\partial t} = r$$

Then

$$c \frac{d^2 p}{dz^2} = rp$$

or

$$\frac{d^2 p}{dz^2} = \frac{r}{c} p = -\lambda^2 p \quad (70)$$

where $\lambda^2 = -\frac{r}{c}$, λ being a function of r (i.e. of t) but not of z .

The solution of (70) is

$$p = C_1 \cos \lambda z + C_2 \sin \lambda z$$

where C_1 and C_2 are functions of r .

Introducing the boundary conditions (68) and (69) we have

$$C_2 = 0$$

and

$$C_1 = \frac{p_1}{\cos \lambda h}$$

$$\text{Hence } p = p_1 \frac{\cos \lambda z}{\cos \lambda h} \mathbb{1} \quad (71)$$

where $\mathbb{1}$ is the so-called "unit function"

Equation (71) may be written as follows

$$p = p_1 \frac{Y(r)}{Z(r)} \mathbb{1} \quad (72)$$

where $Y(r)$ and $Z(r)$ are functions of r .

The Solution of equation (72) is given by Heaviside in the form of a series which is called the "Expansion Theorem". It is as follows

$$p = p_1 \left[\frac{Y(0)}{Z(0)} + \sum_{r=r_1, r_2, \dots} \frac{Y(r)e^{rt}}{r \frac{dZ(r)}{dr}} \right] \quad (73)$$

r_1, r_2, \dots etc. being the roots of the equation $Z(r) = 0$, and $Y(0)$ and $Z(0)$ being the values of $Y(r)$ and $Z(r)$ when r is zero.

We can now apply the expansion theorem (73) to equation (71).

We have

$$Y(r) = \cos \lambda z \text{ and } Z(r) = \cos \lambda h$$

The roots of $Z(r) = 0 : \cos \lambda h = 0$ are

$$\lambda = \frac{n\pi}{2h} \quad (n = 1, 3, 5 \dots)$$

$$\text{But } r = -c\lambda^2$$

$$\text{Therefore } r = -\frac{cn^2\pi^2}{4h^2} \quad (74)$$

Now $\lambda = 0$ when $r = 0$, and therefore

$$\frac{Y(0)}{Z(0)} = \left(\frac{\cos \lambda x}{\cos \lambda h} \right)_{\lambda=0} = 1 \quad (75)$$

$$r \frac{dZ(r)}{dr} = c\lambda^2 \frac{dZ(r)}{d\lambda} \frac{d\lambda}{dr} = -\frac{n\pi}{4} \sin \frac{n\pi}{2} \quad (76)$$

Substituting (74), (75) and (76) together with the value of $Y(r)$ in the expansion theorem (73) we have:

$$p = p_1 \left[1 - \frac{4}{\pi} \sum_{n=1,3,\dots} \frac{e^{-\frac{en^2\pi^2 t}{4h^2}} \cos \frac{n\pi z}{2h}}{n \sin \frac{n\pi}{2}} \right]$$

which is the same as equation (66).

Before we take up any specific problem we will investigate how the state of internal stresses ~~are~~^{is} affected by the presence of evaporation at the surface of the deposit.

2. Consolidation by Evaporation

Evaporation of water at the surface of the deposit produces tension (surface tension) in the capillary water, the intensity of which depends on the temperature and degree of humidity of the atmosphere and also upon the velocity of the wind at the surface. We will not describe here the phenomenon of evaporation⁽¹²⁾, but just point out that it produces tension in the capillary water, which, in turn, affects the state of stresses in the granular material.

Let w_1 be the intensity of tension existing in the capillary water at the surface of the deposit. Since there is equilibrium, the hydrodynamic and the granular pressures

(12). Cf. K. Terzaghi, "Erdbaumechanik" pp. 137-9, 162

must be equal and opposite in sign, in every section of the deposit. It follows then that at the top surface there must be granular pressure equal to w_1 , but opposite in sign. Hence, although no granular pressure exists at the very surface of the deposit, there will always exist a pressure $p_1 (= -w_1)$ at an infinitesimal distance below the surface. We have here the same type of discontinuity in the pressure distribution at the surface as in the case of the permeable top fill.

The problem of determining the effect of evaporation on the process of consolidation of a mud deposit is largely indeterminate for the following reasons: first, w_1 is variable and does not seem to follow any definite law, its value varying from zero up to values higher than 100 kg/cm^2 ; second, as evaporation becomes intensive, the water withdraws from the surface towards the interior of the deposit (as is the case when the quantity of water percolating upwards from the interior of the deposit is less than that which is being evaporated), thus forming a more compact layer of material at the surface which is less permeable than the remainder of the deposit.

If, however, we assume a constant value for w_1 , which may be taken as the average value during a certain period of time the solution will be represented by equation (66) or

$$w = -p = -w_1 \left[1 - \frac{4}{\pi} \sum_{n=1,3,5,\dots} \frac{e^{-\frac{cn^2\pi^2}{4h^2}t} \cos \frac{n\pi}{2h}z}{n \sin \frac{n\pi}{2}} \right] \quad (77)$$

3. Settlements

The time rate of settlement of the surface of the deposit due to a granular pressure p_1 at the surface can be found from equation (56). It is

$$\frac{ds}{dt} = \frac{2kp_1}{h} \sum_{n=1,3,5,\dots} \frac{1}{n} e^{-\frac{cn^2\pi^2}{4h^2}t} \quad (78)$$

This converges very rapidly and therefore we can take

$$\frac{ds}{dt} = \frac{2kp_1}{h} e^{-\frac{c\pi^2}{4h^2}t} \quad (78a)$$

unless t is small, in which case (78) should be used.

The total settlement at any particular time t , is:

$$s = \frac{8ap_1h}{\pi^2} \sum_{n=1,3,5} \left(\frac{1 - e^{-\frac{cn^2\pi^2}{4h^2}t}}{n^3} \right) \quad (79)$$

If t is large we can use the following formula which gives sufficient accuracy

$$s = \frac{8ap_1h}{\pi^2} (1.052 - e^{-\frac{c\pi^2}{4h^2}t}) \quad (79a)$$

In both cases a was considered constant during the integrations.

4. Example. We will now investigate the conditions of stress and settlements of the mud deposit of Problem I, under the influence of a permeable fill, exerting a pressure of 1 kg/cm^2 on the top surface of the deposit.

$$\begin{aligned} \text{We have} \quad c &= 3,395, & p_1 &= 1,000 \\ a &= 0.00024, & h &= 2,500 \end{aligned}$$

The results of the computations are shown in Figures 13 to 17. Fig. 13 shows the stress distribution for the case of a top permeable fill exerting a pressure of $1,000 \text{ gr/cm}^2$ at the top surface of the deposit.

Fig. 14 shows the stress distribution due to the above pressure, and that of the own weight of the solid matter.

Fig. 15 shows the influence of evaporation for a value of w_1 equal to 10 kg/cm^2 . The value of w_1 is in general much larger and variable, but this serves the purpose of showing the general shape of the curves.

Fig. 16 shows the rates of settlement and total settlements due to the weight of the top fill.

Fig. 17 shows these same two items combined with those due to the weight of the deposit.

The shape of the curves in Fig. 14 are seen to agree with that of experimental curves obtained by Dr. C. Terzaghi and published in the Journal of the Boston Society of Civil Engineers (Vol XII, No. 10, Dec. 1925).

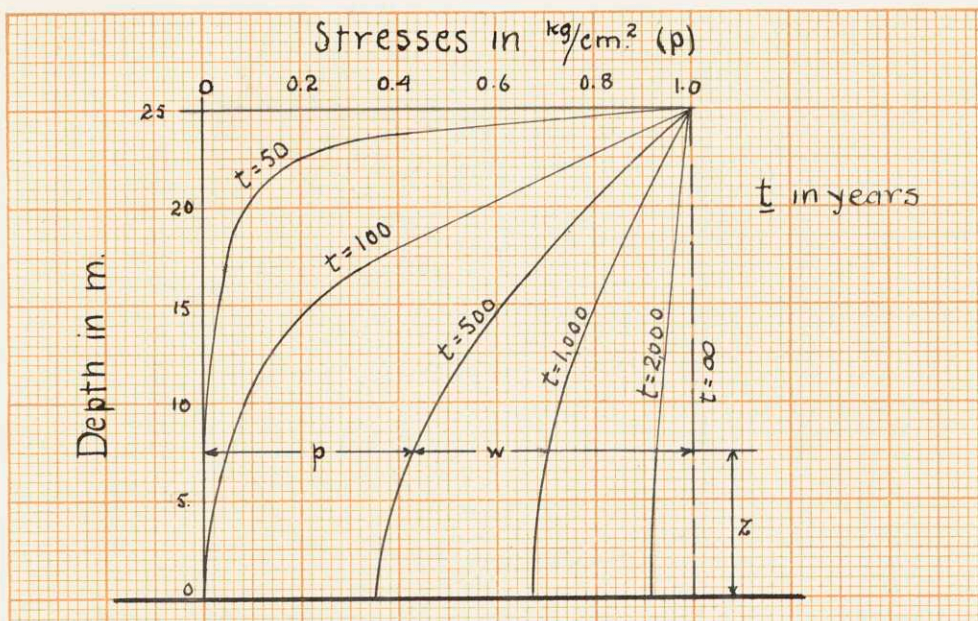


Fig. 13.

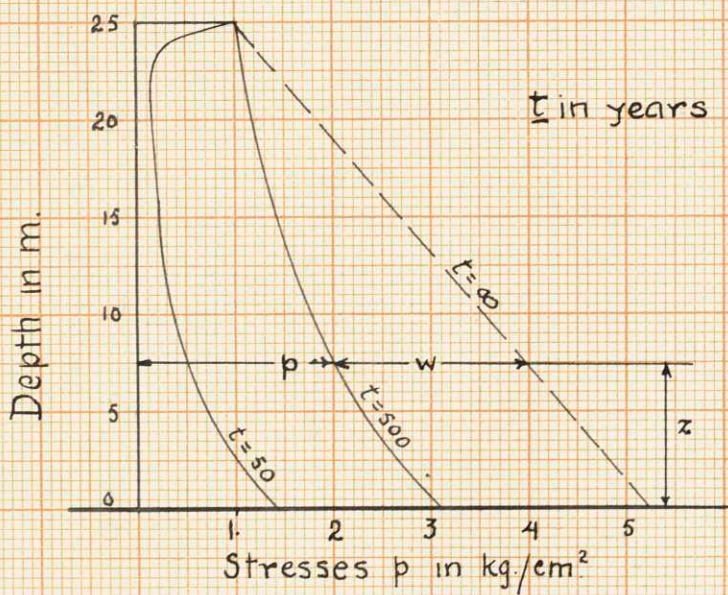


Fig. 14

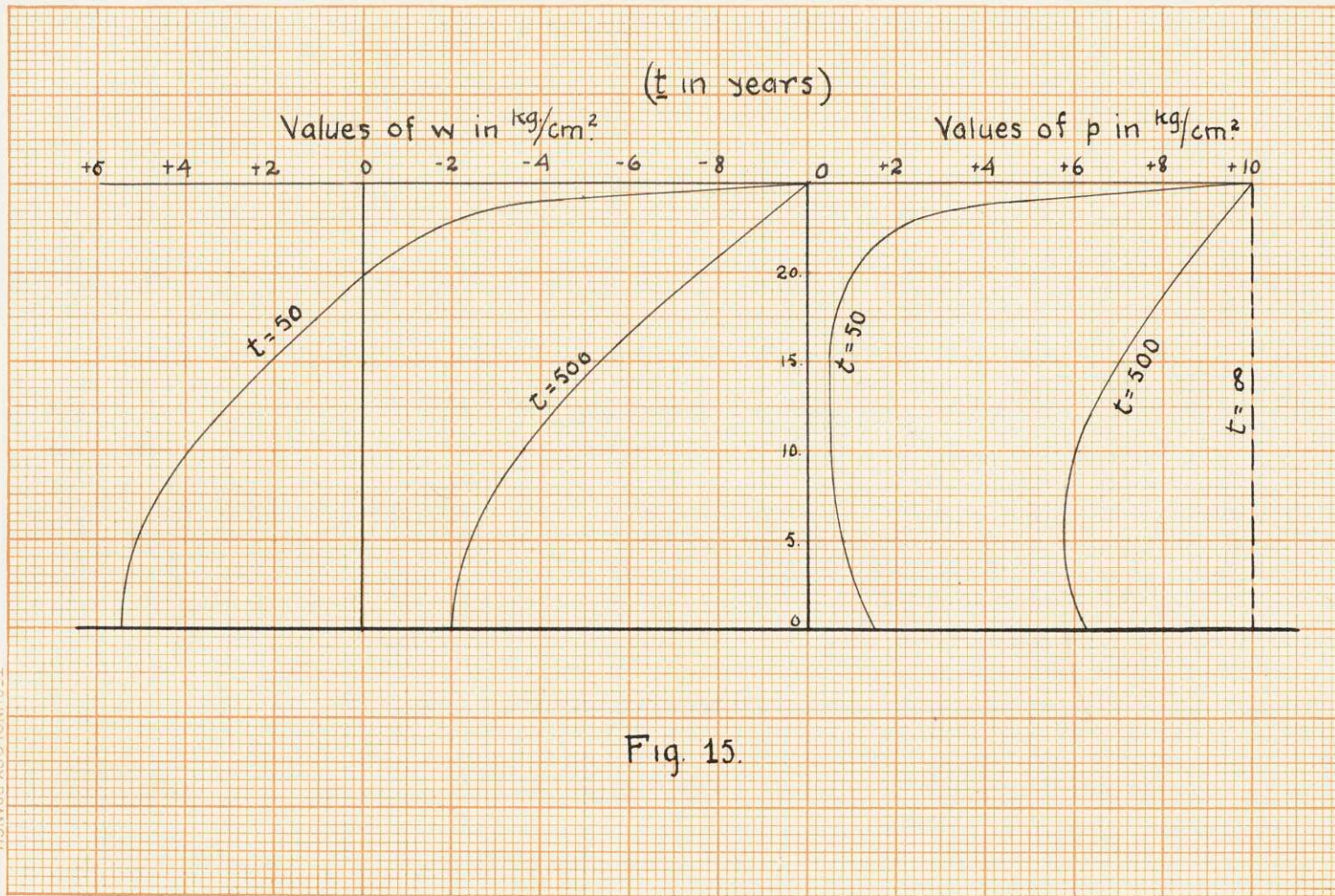


Fig. 15.

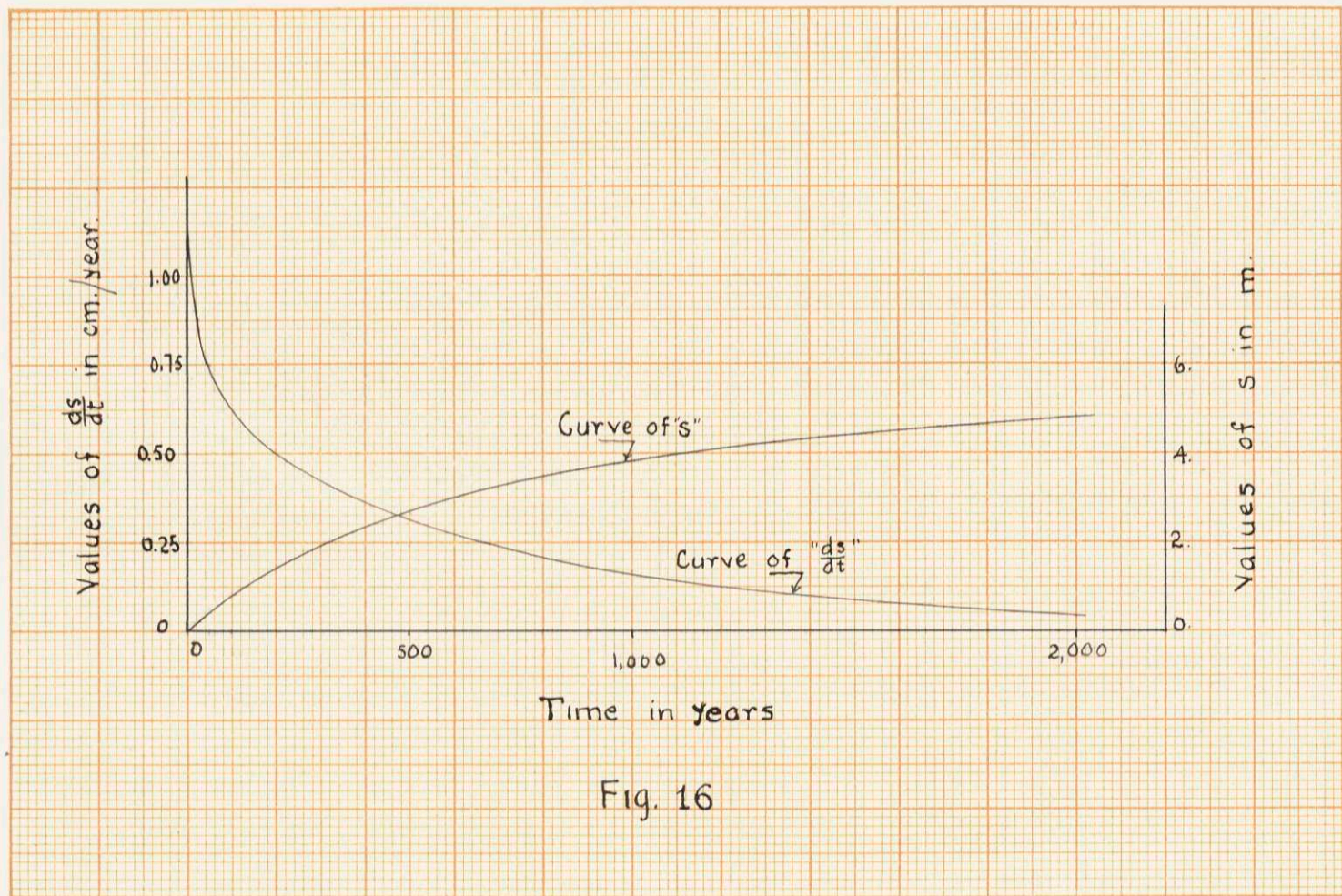


Fig. 16

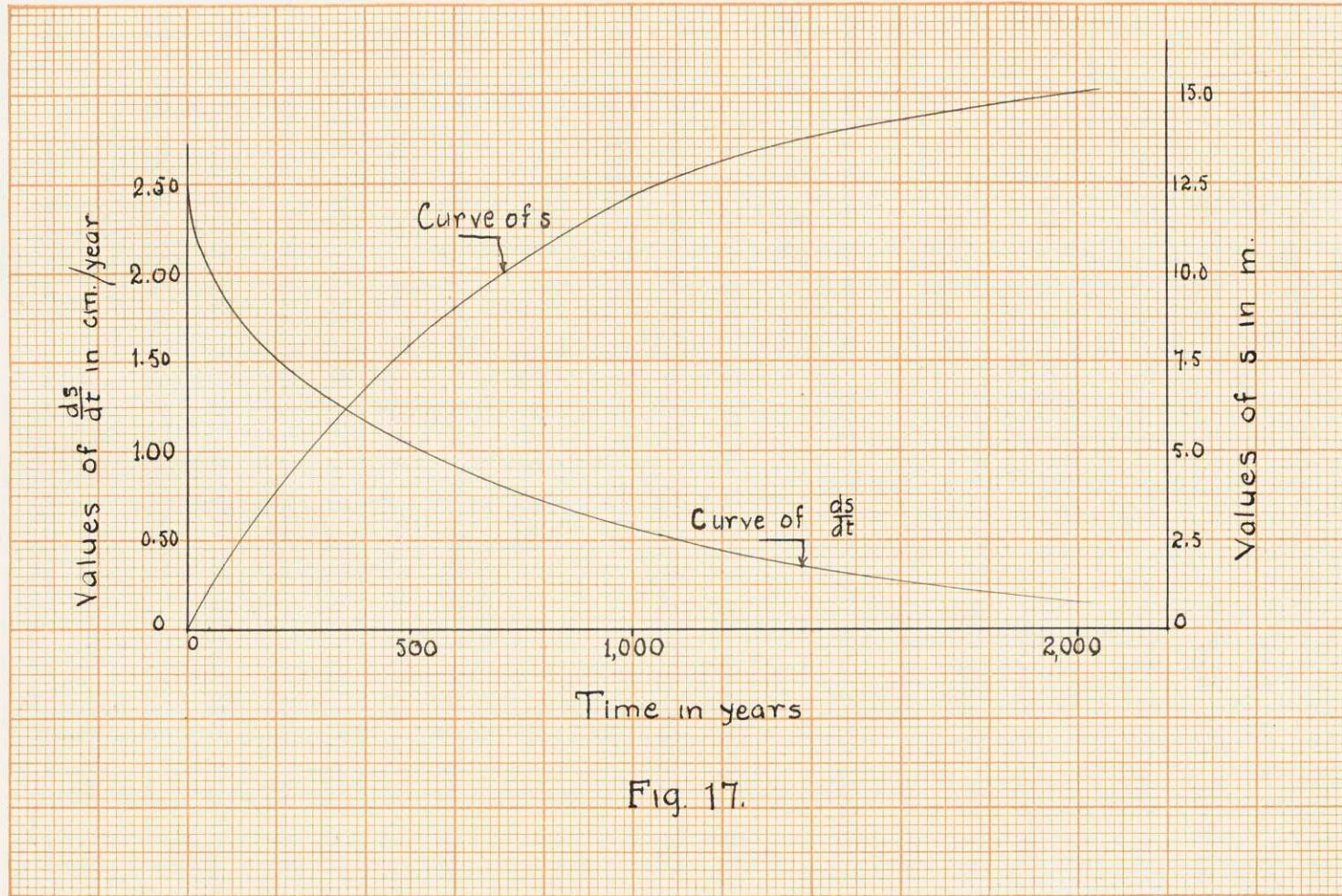


Fig. 17.

V Problem 3. Mud Deposit having within it a thin Layer of less Permeable Material.

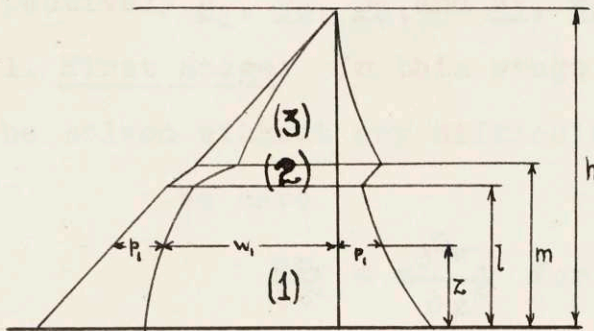


Fig. 18.

We will assume here that at a height l from the bottom of the deposit whose depth is h , there exists a layer of material of thickness $(m - l)$ having a coefficient

of permeability smaller than that of the remainder of the deposit. (See Fig. 18.)

In a problem of this sort, we can not expect to have an accurate solution by applying the simple differential equation

$$\frac{\partial p}{\partial t} = -c \frac{\partial^2 w}{\partial z^2}$$

since the conditions are too variable throughout the deposit. If we attempt to use the more complex differential equation (16) the problem will be beyond solution. Even by applying the above differential equation [equation (15)], the mathematical analysis becomes so complicated that we are forced to make some approximations.

In Fig. 18 the deposit is shown divided into three layers. Let the characteristic constants (already defined) for layers (1) and (3) be γ , q , a , and k and let those for

layer (2) be δ_1 , q_1 , a_1 , and k_1 . Let also the granular and the hydrodynamic pressures in these layers be respectively p_1 , p_2 , p_3 , and w_1 , w_2 , and w_3 .

1. First Stage: In this stage the differential equation can be solved without any difficulty.

We have

$$\frac{\partial p_1}{\partial t'} = c \frac{\partial^2 w_1}{\partial z^2} \quad \text{for } 0 \leq z \leq 1 \quad (80)$$

$$\frac{\partial p_2}{\partial t'} = c_1 \frac{\partial^2 w_2}{\partial z^2} \quad \text{for } 1 \leq z \leq m \quad (81)$$

$$\frac{\partial p_3}{\partial t'} = c \frac{\partial^2 w_3}{\partial z^2} \quad \text{for } m \leq z \leq h \quad (82)$$

The values of t' as functions of z corresponding to the above differential equations are respectively,

$$t' = \frac{\delta}{q}(1 - z) + \frac{\delta_1}{q_1}(m - 1) + \frac{\delta}{q}(h - m), \quad 0 \leq z \leq 1$$

$$t' = \frac{\delta_1}{q_1}(m - z) + \frac{\delta}{q}(h - m), \quad 1 \leq z \leq m$$

and
$$t' = \frac{\delta}{q}(h - z), \quad m \leq z \leq h$$

Hence the differential equations (80), (81), and (82) become:

$$\frac{d^2 p_1}{dz^2} + b \frac{dp_1}{dz} = 0 \quad (80a)$$

$$\frac{d^2 p_2}{dz^2} + b_1 \frac{dp_2}{dz} = 0 \quad (81a)$$

$$\frac{d^2 p_3}{dz^2} + b \frac{dp_3}{dz} = 0 \quad (82a)$$

where $b = \frac{q}{\gamma c}$

and $b_1 = \frac{q_1}{\gamma_1 c_1}$

The solutions of (80a), (81a) and (82a) are, respectively,

$$p_1 = A + Be^{-bz}, \quad (83)$$

$$p_2 = C + De^{-b_1 z}, \quad (84)$$

and

$$p_3 = E + Fe^{-bz} \quad (85)$$

where A, B, C, D, E and F are constants to be determined.

The relations between the w's and the p's are :

$$w_1 = \delta(h - m) + \gamma_1(m - l) + \gamma(1 - z) - p_1 \quad (83a)$$

$$w_2 = \delta(h - m) + \gamma_1(m - z) - p_2 \quad (84a)$$

$$w_3 = \delta(h - z) - p_3 \quad (85a)$$

therefore

$$\frac{dw_1}{dz} = -\gamma - \frac{dp_1}{dz},$$

$$\frac{dw_2}{dz} = -\gamma_1 - \frac{dp_2}{dz},$$

and

$$\frac{dw_3}{dz} = -\delta - \frac{dp_3}{dz}$$

Remembering that the quantity of water (per unit of time) which leaves layer (1) is the same as that which enters layer (2) (the same thing being true for layers (2) and (3)), we will have the following boundary conditions:

$$\text{For } z = 0 : \frac{dw_1}{dz} = 0, \quad (\text{a})$$

$$\text{" } z = h : p_3 = 0, \quad (\text{b})$$

$$\text{" } z = l : p_1 = p_2, \quad (\text{c})$$

$$\text{" } z = m : p_2 = p_3, \quad (\text{d})$$

$$\text{" } z = l : k \frac{dw_1}{dz} = k_1 \frac{dw_2}{dz} \quad (\text{e})$$

$$\text{" } z = m : k_1 \frac{dw_2}{dz} = k \frac{dw_3}{dz} \quad (\text{f})$$

Introducing these boundary conditions we get the following relations between the constants A, B, C, D, E and F:

$$-Bb = -\delta \quad (\text{a}^1)$$

$$E + Fe^{-bh} = 0 \quad (\text{b}^1)$$

$$A + Be^{-bl} = C + De^{-b_1 l} \quad (\text{c}^1)$$

$$E + Fe^{-bm} = C + De^{-b_1 m} \quad (\text{d}^1)$$

$$-k\delta + kbBe^{-bl} = -k_1\delta_1 + k_1 b_1 De^{-b_1 l} \quad (\text{e}^1)$$

$$-k_1\delta_1 + k_1 b_1 De^{-b_1 m} = -k\delta + kbFe^{-bm} \quad (\text{f}^1)$$

Solving these six equations simultaneously, we get:

$$B = \frac{\delta}{b}$$

$$D = \frac{k_1\delta_1 + k\delta(e^{-b_1 l} - 1)}{k_1 b_1 e^{-b_1 l}}$$

$$F = \frac{1}{kbe^{-bm}} \left[k e^{-bl} - \frac{k_1\delta_1 - k\delta(1 - e^{-b_1 l})}{e^{-b_1 l} (e^{-b_1 l} - e^{-b_1 m})} \right]$$

$$E = -Fe^{-bh}$$

$$A = \left(\frac{k\gamma}{k_1 b_1} - \frac{\gamma}{b} \right) e^{-bl} - F \left[e^{-bh} - e^{-bm} \left(1 - \frac{k b}{k_1 b_1} \right) \right]$$

$$C = E + Fe^{-bm} - De^{-b_1 m}$$

Hence the distribution of pressures throughout the deposit is fully determined for the first stage.

2. Second Stage. In this stage, since t is independent of z , the three differential equations will be partial instead of ordinary. Unfortunately these differential equations can not be solved and therefore we will take

$$\begin{aligned} \frac{\partial p}{\partial t} &= -r \frac{\partial^2 w}{\partial z^2} \\ \text{or} \quad \frac{\partial w}{\partial t} &= r \frac{\partial^2 w}{\partial z^2} \end{aligned} \quad (86)$$

as the general equation for the second stage, where r stands for c for layers (1) and (3), and for c_1 for layer (2). r is then variable but will be considered constant in order to make the problem solvable.

Changing the origin of coordinates to the top surface (86) becomes

$$\frac{\partial w}{\partial t} = r \frac{\partial^2 w}{\partial x^2} \quad (86a)$$

where $x = h - z$

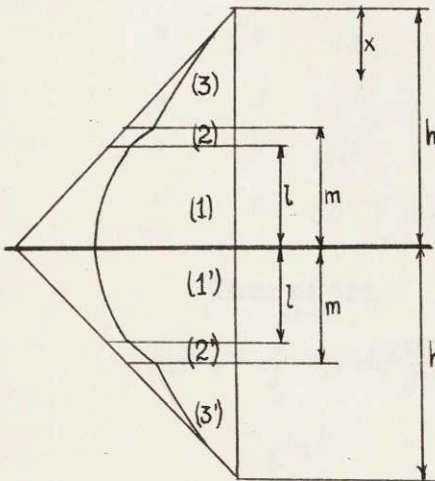
We now proceed as in the case of the mud deposit of homogeneous material.

The boundary conditions are

$$w = 0 \text{ for } x = 0,$$

$$\text{and } w = 0 \text{ for } x = 2h,$$

the last condition resulting from the fact that



$$\frac{\partial w}{\partial x} = 0.$$

The conditions are now represented by Fig. 19, where the pressure distribution is symmetric with respect to the impermeable surface.

The general solution of equation (86a) is given, as before, by:

Fig. 19

$$w = \sum_{n=1}^{n=\infty} A_n e^{-\frac{rn^2\pi^2}{4h^2} t} \sin \frac{n\pi x}{2h} \quad (87)$$

or

$$w = \sum_{n=1}^{n=\infty} A_n e^{-\frac{rn^2\pi^2}{4h^2} t} \frac{\cos \frac{n\pi z}{2h}}{\sin \frac{n\pi}{2}} \quad (87a)$$

When $t = 0$, $w = \varphi(x)$, where $\varphi(x)$ indicates the pressure distribution at the end of the first stage and is, therefore, a discontinuous function, but always finite. Therefore its integration should be performed by parts.

Since for $t = 0$ $w = \varphi(x)$, we must have

$$A_n = \frac{1}{h} \int_0^{2h} \varphi(x) \sin \frac{n\pi x}{2h} dx.$$

The expressions for $\varphi(x)$ are as follows (See Fig. 19).

$$\begin{aligned}
 \text{For layer (3)} : w_3 &= \gamma x - E - Fe^{-b(h-x)} \\
 \text{" " (2)} : w_2 &= (\gamma - \gamma_1)(h - m) + \gamma_1 x - C - De^{-b_1(h-x)} \\
 \text{" " (1)} : w_1 &= (\gamma_1 - \gamma)(m - l) + \gamma x - A - Be^{-b(h-x)} \\
 \text{" " (1')} : w_1' &= (\gamma_1 - \gamma)(m - l) - \gamma x - A - Be^{-b(x-h)} \\
 \text{" " (2')} : w_2' &= \gamma(h-m) + \gamma_1(h+m) - \gamma_1 x - C - De^{-b_1(x-h)} \\
 \text{" " (3')} : w_3' &= 2\gamma h - \gamma x - E - Fe^{-b(x-h)}
 \end{aligned}$$

Therefore

$$\begin{aligned}
 A_n h &= \int_0^{h-m} w_3 \sin \frac{n\pi x}{2h} dx + \int_{h-m}^{h-l} w_2 \sin \frac{n\pi x}{2h} dx + \int_{h-l}^h w_1 \sin \frac{n\pi x}{2h} dx + \\
 &+ \int_h^{h+l} w_1' \sin \frac{n\pi x}{2h} dx + \int_{h+l}^{h+m} w_2' \sin \frac{n\pi x}{2h} dx + \int_{h+m}^{2h} w_3' \sin \frac{n\pi x}{2h} dx,
 \end{aligned}$$

where the values of the w 's are given by the preceding equations.

Performing the above integrations and simplifying we get

$$\begin{aligned}
 A_n &= \frac{4h}{n^2\pi^2} (\gamma - \gamma_1) \left[\left(2 \sin \frac{n\pi}{2} - n\pi \cos \frac{n\pi}{2} \right) \left(\cos \frac{n\pi m}{2h} - \cos \frac{n\pi l}{2h} \right) \right. \\
 &+ \left. \frac{n\pi}{h} \sin \frac{n\pi}{2} \left(m \sin \frac{n\pi m}{2h} - l \sin \frac{n\pi l}{2h} \right) \right] + \frac{8\gamma h}{n^2\pi^2} \sin \frac{n\pi}{2} + \\
 &\frac{2}{n\pi} \left\{ 2 \sin \frac{n\pi}{2} \left[(E - C) \sin \frac{n\pi}{2h} \cdot m + (C - A) \sin \frac{n\pi l}{2h} \right] + \right. \\
 &+ E(\cos n\pi - 1) - (\gamma - \gamma_1) \left[2 \sin \frac{n\pi}{2} \left(m \sin \frac{n\pi m}{2h} - l \sin \frac{n\pi l}{2h} \right) \right. \\
 &\left. \left. + 2h \cos \frac{n\pi}{2} \left(\cos \frac{n\pi l}{2h} - \cos \frac{n\pi m}{2h} \right) \right] \right\} -
 \end{aligned}$$

$$\begin{aligned}
& - \frac{4h}{4h^2 b^2 + n^2 \pi^2} \left\{ 2 \sin \frac{n\pi}{2} \left[Fe^{-bm} \left(b \cos \frac{n\pi}{2h} m - \frac{n\pi}{2h} \sin \frac{n\pi}{2h} m \right) - \right. \right. \\
& \left. \left. Be^{-bl} \left(b \cos \frac{n\pi l}{2h} - \frac{n\pi}{2h} \sin \frac{n\pi l}{2h} \right) \right] + Fe^{-bh} \frac{n\pi}{2h} (1 - \cos n\pi) \right. \\
& \left. + 2 Bb \sin \frac{n\pi}{2} \right\} - \frac{4h}{4h^2 b_1^2 + n^2 \pi^2} \left\{ 2D \sin \frac{n\pi}{2} \left[e^{-b_1 l} \left(\right. \right. \right. \\
& \left. \left. \left(b_1 \cos \frac{n\pi l}{2h} - \frac{n\pi}{2h} \sin \frac{n\pi l}{2h} \right) - e^{-b_1 m} \left(b_1 \cos \frac{n\pi}{2h} m - \right. \right. \right. \\
& \left. \left. \left. - \frac{n\pi}{2h} \sin \frac{n\pi}{2h} m \right) \right] \right\} \quad (88)
\end{aligned}$$

Now once the values of the trigonometric functions are tabulated, expression (88) is not very hard to compute, but it seems to be too long to be of any practical value.

In case the thickness of layer (2) is not large, we can set $l = m$ and get a much simpler equation for A_n . This will be shown later, by an example, to involve a very small error. Also A_n is seen to be small for even values of n , the prevailing term in (88) being $\frac{8\gamma h}{n^2 \pi^2} \sin \frac{n\pi}{2}$

Setting $l = m = l'$ (where l' is now the average between l and m) in (88) we get:

$$\begin{aligned}
A_n = & \frac{8\gamma h}{n^2 \pi^2} \sin \frac{n\pi}{2} + \frac{2}{n\pi} \left[2 \sin \frac{n\pi}{2} (E - A) \sin \frac{n\pi}{2h} l' + \right. \\
& \left. E(\cos n\pi - 1) \right] - \frac{4h}{4h^2 b^2 + n^2 \pi^2} \left[2 (F - B) \sin \frac{n\pi}{2} e^{-bl'} \left(\right. \right. \\
& \left. \left. \left(b \cos \frac{n\pi}{2h} l' - \frac{n\pi}{2h} \sin \frac{n\pi}{2h} l' \right) + F \frac{n\pi}{2h} e^{-bh} (1 - \cos n\pi) + \right. \right. \\
& \left. \left. + 2 Bb \sin \frac{n\pi}{2} \right] \quad (89)
\end{aligned}$$

A still more approximate expression is

$$A_n = \frac{8\gamma h}{n^2 \pi^2} \sin \frac{n\pi}{2} - \frac{8hBb}{4h^2 b^2 + n^2 \pi^2} \sin \frac{n\pi}{2} \quad (90)$$

It should be remembered that the assumption or approximate $l = m$ was made only in order to obtain a simpler expression for A_n and should not affect the constants A, B, C etc.

The fact that the state of stress for the first stage of consolidation can be determined, furnishes already valuable information. For the second stage of consolidation, we can obtain only an approximate solution, and therefore we will not enter into many details with regard to the mathematical analysis.

We will now determine the value of \underline{r} which will represent, approximately, the state of stress in layer (1). To do this, we know that the quantity of water which leaves layer (1) per unit time must be the same as that which enters layer (2) per unit of time. If a solution could be obtained for this stage, we would have three stress equations - one for each layer, and instead of \underline{r} , we would have \underline{c} for layers (1) and (3), and \underline{c}_1 for layer (2). Assuming that equation (87) represents the solution of our problem is equivalent to assuming that

$$k \left(\frac{\partial w}{\partial z} \right)_{\substack{z=1, \\ r=c}} = K_1 \left(\frac{\partial w}{\partial z} \right)_{\substack{z=1, \\ r=c_1}}$$

Although this is not true, it affords a means of estimating the value of \underline{r} with the following modifications. Compute values of $k\left(\frac{\partial w}{\partial z}\right)_{z=1, r=c}$ for several values of \underline{t} , within a certain period of time, and then compute the corresponding values of $k_1\left(\frac{\partial w}{\partial z}\right)_{z=1, r=c_1}$ for the same values of \underline{t} . Now take the average of each pair of values for the same value of \underline{t} which then represents the quantity of water percolating through the section $z = 1$ per unit of time and per unit of area. After this is done we determine the value of \underline{r} , say \underline{r}_1 , which will make the same average values between

$$k\left(\frac{\partial w}{\partial z}\right)_{z=1, r=r_1} \text{ and } k_1\left(\frac{\partial w}{\partial z}\right)_{z=1, r=r_1}$$

approximately the same as the average values already obtained.

Another method which may be used to determine \underline{r} is the following. The rate at which the granular stress \underline{p} is increasing with the time $\left(\frac{\partial p}{\partial t}\right)$ at the end of the first stage of consolidation for sections far away from the top surface of the deposit, should be about the same as its rate of increase^{for} at the beginning of the second stage. In other words,

$$-c\left(\frac{d^2 w}{dz^2}\right)_{z=z_1} = -r\left(\frac{\partial^2 w}{\partial z^2}\right)_{z=z_1, t=0}$$

The right hand member of this equation can be computed only by taking $t > 0$, and finding its limit as \underline{t} is made to approach zero, otherwise a very large number of terms would be required in the series (87). The method of procedure is:

compute $\frac{d^2 w}{dz^2}$ for, say, $z = 0$ and $z = 1$ (equation (83)).

Then compute $\frac{\partial^2 w}{\partial z^2}$ (equation (87)) for the same values of z and for several values of \underline{t} , say, 50, 100, and 500 years by assigning to \underline{r} a trial value. With these values, plot curves between $(\frac{\partial^2 w}{\partial z^2})_{z=z_1}$ and \underline{t} and produce them to meet the axis $t = 0$. \underline{r} should then be such as to satisfy the above equation.

In all cases \underline{r} must be $c_1 < r < c$.

We are not interested in the stress distribution in layer (2) which may be considered as a plane of separation between layers (1) and (3).

The stress distribution in layer (3) can not be determined unless we make some approximations. In this case, however, we know the limiting values between which the pressures must lie. Let the granular and hydrodynamic stress for this layer be \underline{p}_3 and \underline{w}_3 , respectively. The two limiting values for w_3 are given by equation (87) for $r = r_1$ and $r = c$ for $0 \leq x \leq h-m$. They are

$$w_3^1 = \sum_{n=1}^{n=\infty} A_n e^{-\frac{cn^2 \pi^2}{4h^2} t} \sin \frac{n\pi x}{2h}, \quad (91)$$

and

$$w_3^{c1} = \sum_{n=1}^{n=\infty} A_n e^{-\frac{r_1 n^2 \pi^2}{4h^2} t} \sin \frac{n\pi x}{2h}. \quad (91a)$$

Now the change in water-content of layer (2) is very small, and can be neglected. Therefore we can assume that the quantity of water which leaves layer (1) per unit time is the same as that which enters layer (3) per unit

of time. This leads to the equation:

$$\left(\frac{\partial w_1}{\partial z}\right)_{z=1} = - \left(\frac{\partial w_3}{\partial x}\right)_{x=h-m} \quad (92)$$

where $\frac{\partial w_3}{\partial x}$ has an intermediate value between $\frac{\partial}{\partial x} (w_3^I \text{ and } w_3^{II})$. This equation serves either as a check on the results or as a means of determining to which of the equations (91) and (91a) the pressure w_3 is closer.

3. Settlements and Rates of Settlements.

These can be determined by directly applying equations (56) and (57). Let w_3 be the average between w_3^I and w_3^{II} and let l' be the distance of the center of layer (2) from the bottom of the deposit. Then, taking a constant, we have

$$\begin{aligned} \frac{ds}{dt} = a \int_0^{l'} \frac{\partial p_1}{\partial t} dz + a \int_0^{l'} \frac{\partial p_2}{\partial t} dx = \\ -a \int_0^{l'} \frac{\partial w_1}{\partial t} dz - \frac{a}{2} \int_0^{l'} \left(\frac{\partial w_3^I}{\partial t} + \frac{\partial w_3^{II}}{\partial t} \right) dx \end{aligned}$$

Integrating, we will have for the rate of settlement,

$$\frac{ds}{dt} = \frac{a\pi}{4h} \sum_{n=1}^{\infty} \left\{ r e^{-\frac{rn^2\pi^2}{4h^2}t} (1+2 \sin \frac{n\pi l'}{2h} - \cos \frac{n\pi l'}{2h}) + c e^{-\frac{cn^2\pi^2}{4h^2}t} (1-\cos \frac{n\pi l'}{2h}) \right\} A_n^n. \quad (93)$$

The total settlement of the top surface of the deposit at any time t is:

$$\begin{aligned} s = \frac{ah}{\pi} \sum_{n=1}^{\infty} \frac{A_n}{n} \left\{ (1+2 \sin \frac{n\pi}{2} \sin \frac{n\pi l'}{2h} - \cos \frac{n\pi l'}{2h}) (1-e^{-\frac{rn^2\pi^2}{4h^2}t}) + \right. \\ \left. + (1-\cos \frac{n\pi l'}{2h}) (1-e^{-\frac{cn^2\pi^2}{4h^2}t}) \right\} \quad (94) \end{aligned}$$

In computing the values of $\frac{ds}{dt}$ and s from (93) and (94) for very small values of \underline{t} , two or more terms should be considered. For large values of \underline{t} , only one term in the above series will give results accurate enough for any practical purpose.

4. Example. We will now investigate the state of stresses in a mud deposit with a total reduced thickness of 25 m. and having within it a layer of less permeable material with a reduced thickness of 50 cm. Such a large thickness for this layer is chosen in order to give an idea of the discrepancies in the values of A_n as computed from equations (89) and (90).

The characteristic constants of the materials are (1)

Layers (1) and (3): $\gamma = 1.7$, $a = 0.00024$, $k = 0.815$

Layer (2) $\gamma_1 = 1.9$, $a_1 = 0.00050$, $k_1 = 0.20$

Take $q = q_1 = 8.5$

Therefore $c = \frac{k}{a} = 3,395$, $b = \frac{q}{\gamma c} = 0.001472$,

and $c_1 = \frac{k_1}{a_1} = 400$, $b = \frac{q_1}{\gamma_1 c_1} = 0.01118$.

The reduced dimensions are

$l = 1,200$ cm., $m = 1,250$ cm. and $h = 2,500$ cm.

(1) Cf. K. Terzaghi, "Erdbaumechanik" Fig. 30, p.171. Also Principles of Soil Mechanics, Engineering News-Record, Nov. 26, 1925.

It would take too much space if we were to reproduce here all the computations involved in the determination of the state of stress in each layer. Therefore we will illustrate only the steps in which some doubt might arise in the interpretation of the equations.

(a). First Stage. There is nothing in particular to be said about the determination of \underline{p} and \underline{w} for this stage. We first compute the values of the constants A, B, C, D, E and F. Then the granular pressures are determined from equations (83), (84) and (85) and the hydrodynamic pressures from equations (83a), (84a) and (85a). The results of the computations are graphically represented by Fig. 20.

(b). Second Stage. The values of A_n , computed from equations (88)⁽⁸⁹⁾ and (90) for $n=1$, $n=2$ and $n=3$ are

Formula (88): $A_1 = 2,820$, $A_2 =$ practically zero

" (89): $A_1 = 2,825$, $A_2 = 0$ $A_3 = - 330$

" (90): $A_1 = 2,930$, $A_2 = 0$, $A_3 = - 150$.

This shows that, for $n = 1$, either formula (89) or (90) may be used to compute A_n , while for large values of n , formula (89) should be used.

Determination of \underline{r} : The average values between $k\left(\frac{\partial w}{\partial z}\right)_{z=1, r=c}$ and $k_1\left(\frac{\partial w}{\partial z}\right)_{z=1, r=c_1}$, for $t_1 = 100, 500$ and $1,000$ years are, respectively,

0.91, 0.68 and 0.46 cm.³/year.

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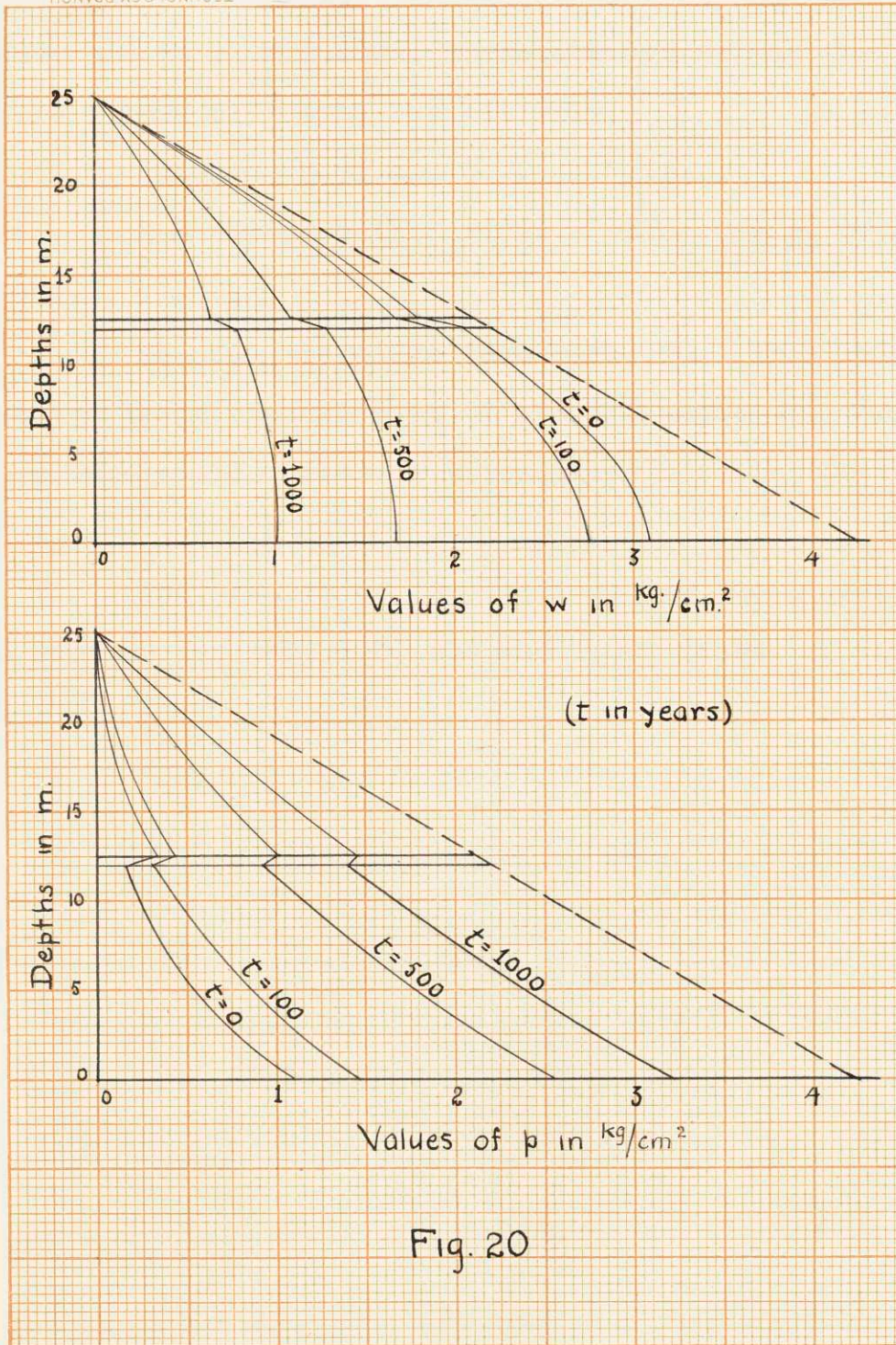


Fig. 20

The value of \underline{r} , (\underline{r}_1), which gives average values of $k\left(\frac{\partial w}{\partial z}\right)_{z=1, r=r_1}$ and $k_1\left(\frac{\partial w}{\partial z}\right)_{z=1, r=r_1}$ closer to the above ones is $r_1 = 2,600$. These average values for the above values of \underline{t} are, respectively,

$$0.91+ , 0.71 \text{ and } 0.42 \text{ cm.}^3/\text{year.}$$

The second method gives a value for \underline{r}_1 of about 2000+.

We will therefore take $r_1=2,300$. This value is not too large because part of the influence exerted by the more impermeable layer is already included in the values of A_n .

The results of computations are shown graphically in Fig. 20 for $t=0, 100, 500$ and $1,000$ years.

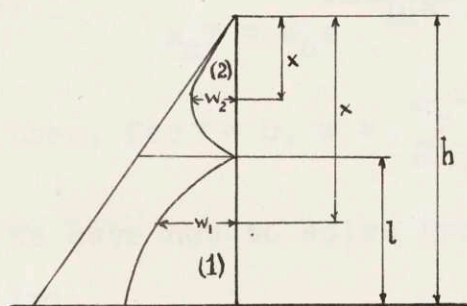
VI. PROBLEM 4. Consolidation of Mud Deposits by Drainage.

1. General. We have seen how slow is the process of consolidation of mud deposits (or of fine-grained materials) when under the influence of the own weight of the material. This is due to the extremely low value of the coefficient of permeability of fine-grained soils. It is evident that in order to effectively drain a mud deposit, we must have at least one layer of coarser material (like sand) within the deposit. If, in the mud deposit discussed in Problem III, the interposed layer had been composed of a coarser material like sand, the hydrodynamic pressure, at the bottom and top of the layer, would be the same because of the extremely high value of the coefficient of permeability of sand compared with that of mud or clay. Hence, when a layer of very permeable material (which we may call sand) comes between the layers of mud, this layer can be entirely disregarded in the computation of the stresses, using as reduced dimension, that of the deposit without the sand layer.

2. Determination of Stresses. Consider now a mud deposit of total reduced thickness h , interposed by a sand layer whose center lies at a distance (reduced) l from the bottom surface which we assume horizontal and impermeable.

Let us now suppose that a pipe is driven into the deposit to reach the layer of sand and that the water is discharged at the top surface of the deposit. Under these conditions, the pressure in the water at the section $z=1$ is hydrostatic, i.e., the hydrodynamic pressure is zero. Let $\tau = 0$ correspond to the instant at which drainage begins. This instant is a short time after the pipe has been driven, to account for the rapid change in the hydrodynamic pressure in the sand layer itself.

The boundary conditions are (Fig. 21):



$$\begin{aligned} \frac{\partial w}{\partial z} &= 0 \quad \text{for } z = 0, \\ w &= 0 \quad \text{for } z = 1, \\ w &= 0 \quad \text{for } z = h, \quad \text{and} \\ w &= f(z) \quad \text{for } \tau = 0. \end{aligned}$$

Fig. 21.

We will need to consider only the case in which the deposit is in its second stage of consolidation. The effect of drainage at a particular location of the deposit is effective over a considerable distance from it on account of the high value of the coefficient of permeability of the sand, unless the sand layer is discontinuous.

For $\tau = 0$, and for any value of \underline{t} , the pressure distribution is given by equation (38) or

$$w = \sum_{n=1,3,\dots} a_n T \sin \frac{n\pi}{2h} x \quad (95)$$

$$\text{where } T = e^{-\frac{c n^2 \pi^2}{4h^2} t} \quad (96)$$

and a_n is given by equation (39).

τ is not a function of T , because \underline{t} refers to the time before drainage has started.

Let p_1 , w_1 , and $p_2^{w_2}$ be the granular and hydrodynamic pressures at the portion of the deposit below and above the sand layer, respectively (See Fig. 21).

Let also

$$a_n T = a_n e^{-\frac{c n^2 \pi^2}{4h^2} t} = K_n \quad (97)$$

$$\text{then, for } \tau = 0, w = \sum_{n=1,3,\dots} K_n \sin \frac{n\pi}{2h} x \quad (95a)$$

We have now to solve two differential equations, namely:

$$\frac{\partial w_1}{\partial \tau} = c \frac{\partial^2 w_1}{\partial x^2} \quad (98)$$

and

$$\frac{\partial w_2}{\partial \tau} = c \frac{\partial^2 w_2}{\partial x^2} \quad (99)$$

(a). Solution of equation (99): This equation is subjected to the following boundary conditions:

$$w_2 = 0 \text{ for } x = 0 \quad (100)$$

$$w_2 = 0 \text{ for } x = h - 1 \quad (101)$$

and $w_2 = w = f(x)$, for $\tau = 0$ and

$$0 \leq x \leq h - 1 \quad (102)$$

The particular solutions of (99) are, as before,

$$w_2 = e^{-c\lambda^2\tau} \sin \lambda x \quad (103)$$

$$\text{and } w_2 = e^{-c\lambda^2\tau} \cos \lambda x \quad (104)$$

Now the particular solution (103) will satisfy both (100) and (101), for all values of τ , provided that we take

$$\lambda = \frac{m\pi}{h-1} \quad m = 1, 2, 3, \dots$$

Hence the general solution of (99) is

$$w_2 = \sum_{m=1}^{m=\infty} R_m e^{-\frac{c m^2 \pi^2}{(h-1)^2} \tau} \sin \frac{m\pi}{h-1} x \quad (105)$$

To determine R_m we have that, for $\tau = 0$,

$$w_2 = \sum_{n=1,3,\dots} K_n \sin \frac{n\pi}{2h} x = f(x),$$

$$\text{or } \sum_{m=1}^{m=\infty} R_m \sin \frac{m\pi}{h-1} x = \sum_{n=1,3,\dots} K_n \sin \frac{n\pi}{2h} x \quad (106)$$

Therefore

$$R_m = \frac{2}{h-1} \int_0^{h-1} \sum_{n=1,3,\dots} K_n \sin \frac{n\pi}{2h} x \sin \frac{m\pi}{h-1} x dx \quad (107)$$

Integrating, we get

$$R_m = \frac{2\pi}{(h-1)^2} \sum_{n=1,3,\dots} K_n \frac{m \sin \frac{n\pi}{2} \cos m\pi \cos \frac{n\pi}{2h}}{\left(\frac{n\pi}{2h}\right)^2 - \left(\frac{m\pi}{h-1}\right)^2} \quad (108)$$

Therefore

$$w_2 = \frac{2\pi}{(h-1)^2} \sum_{m=1}^{m=\infty} \sum_{n=1,3,\dots} \left[K_n \frac{m \sin \frac{n\pi}{2} \cos m\pi \cos \frac{n\pi}{2h}}{\left(\frac{n\pi}{2h}\right)^2 - \left(\frac{m\pi}{h-1}\right)^2} \right] \times \left[e^{-\frac{c m^2 \pi^2}{(h-1)^2} \tau} \sin \frac{m\pi}{h-1} x \right] \quad (109)$$

$$\text{and } p_2 = \gamma x - w_2 \quad (109a)$$

(b). Solution of equation (98): The solution has to hold for $h-1 \leq x \leq h$. Let (Fig. 22)

$$y = x - h + 1 \quad \text{or} \quad x = y + h - 1$$

$$\text{then} \quad \frac{\partial w_1}{\partial \tau} = c \frac{\partial^2 w}{\partial y^2} \quad (98a)$$

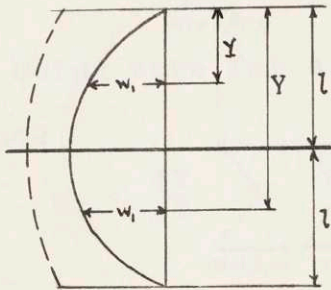


Fig. 22.

The boundary conditions are:

$$w_1 = 0 \quad \text{for} \quad y = 0 \quad (110)$$

$$\frac{\partial w_1}{\partial y} = 0 \quad \text{for} \quad y=l \quad \text{or} \quad w_1=0 \quad \text{for} \quad y=2l \quad (111)$$

$$w_1 = \varphi(y), \quad \text{for} \quad y=0 \quad \text{and} \quad 0 \leq y \leq 1.$$

This latter condition reduces to

$$\begin{aligned} \varphi_1(y) &= \sum_{n=1,3,\dots} K_n \sin \frac{n\pi}{2h}(y+h-1) \quad \text{for} \quad 0 \leq y \leq 1, \\ \text{and} \quad \varphi_2(y) &= \sum_{n=1,3,\dots} K_n \sin \frac{n\pi}{2h}(h+1-y) \quad \text{for} \quad 1 \leq y \leq 2l. \end{aligned} \quad (112)$$

Now the particular solution (103):

$$w_1 = e^{-c\lambda^2\tau} \sin \lambda y$$

will satisfy both (110) and (111), if we set

$$\lambda = \frac{m\pi}{2l} \quad m = 1, 2, 3, \dots$$

Hence the general solution of (98a) is:

$$w_1 = \sum_{m=1}^{\infty} S_m e^{-\frac{cm^2\pi^2}{4l^2}\tau} \sin \frac{m\pi}{2l} y \quad (113)$$

When $\tau = 0$, $w_1 = \varphi(y)$ and therefore

$$S_m = \frac{1}{l} \int_0^l \left\{ \sum_{n=1,3,\dots}^{2l} K_n \sin \frac{n\pi}{2h}(y+h-1) \right\} \sin \frac{m\pi}{2l} y \, dy + \\ + \frac{1}{l} \int_l^{2l} \sum_{n=1,3,\dots}^{2l} K_n \sin \frac{n\pi}{2h}(h+1-y) \sin \frac{m\pi}{2l} y \, dy.$$

Integrating we shall have:

$$S_m = \frac{4\pi}{l^2} \sum_{n=1,3,\dots} K_n \frac{m \sin \frac{n\pi}{2} \cos \frac{n\pi}{2h} l}{\left(\frac{m\pi}{l}\right)^2 - \left(\frac{n\pi}{h}\right)^2} \quad (114)$$

S_m being zero for even values of m .

Hence

$$w_1 = \frac{4\pi}{l^2} \sum_{m=1,3,\dots} \sum_{n=1,3,\dots} K_n \frac{m \sin \frac{n\pi}{2} \cos \frac{n\pi}{2h} l}{\left(\frac{m\pi}{l}\right)^2 - \left(\frac{n\pi}{h}\right)^2} e^{-\frac{cm^2\pi^2}{4l^2} \tau} \times \\ \times \sin \frac{m\pi}{2l} y \quad (115)$$

It is more convenient to have our equations referred to a single origin of coordinates. Take this as the bottom surface and let z be the distance of any section from this surface.

We have

$$x = h-z \quad \text{for } l \leq z \leq h,$$

$$y = l - z \quad \text{for } 0 \leq z \leq l.$$

And therefore

$$w_2 = \frac{2\pi}{(h-1)^2} \sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots} K_n \frac{m \sin \frac{n\pi}{2} \cos m\pi \cos \frac{n\pi l}{2h}}{\left(\frac{n\pi}{2h}\right)^2 - \left(\frac{m\pi}{h-1}\right)^2} \times \\ \times e^{-\frac{cm^2\pi^2}{(h-1)^2} \tau} \sin \frac{m\pi}{h-1} (h-z) \quad (116)$$

for $l \leq z \leq h$, and

$$w_1 = \frac{4}{\pi l^2} \sum_{m=1,3,\dots} \sum_{n=1,3,\dots} K_n \frac{m \sin \frac{n\pi}{2} \cos \frac{n\pi}{2h} l}{\left(\frac{m}{l}\right)^2 - \left(\frac{n}{h}\right)^2} e^{-\frac{cm^2\pi^2}{4l^2}\tau} \times \sin \frac{m\pi}{2l}(1-z) \quad (117)$$

for $0 \leq z \leq l$.

In both cases: $p_{1,2} = \gamma(h-z) - w_{1,2}$.

3. Sand Layer at the Bottom of the Deposit.

In case the layer of sand is located immediately above the bottom impermeable surface, the pressure distribution will be the same as that given by equation (116) by setting $l = 0$. It is:

$$w = \frac{8}{\pi} \sum_{m=1}^{m=\infty} \sum_{n=1,3,\dots} K_n \frac{m \sin \frac{n\pi}{2} \cos m\pi}{n^2 - 4m^2} e^{-\frac{cm^2\pi^2}{h^2}\tau} \times \sin \frac{m\pi}{h}(h-z) \quad (118)$$

and $p = \gamma(h-z) - w$, the solution now holding for $0 \leq z \leq h$.

4. Settlements and Rates of Settlement.

The effect of drainage on the settlements and rates of settlement of the top surface of the deposit can be found by applying equations (56) and (57) to the above equations.

We have

$$\frac{ds}{d\tau} = - \int_0^l a \frac{\partial w_1}{\partial \tau} dz - \int_l^h a \frac{\partial w_2}{\partial \tau} dz \quad (119)$$

Substituting the values of $\frac{\partial W_1}{\partial \tau}$ and $\frac{\partial W_2}{\partial \tau}$ from (116) and (117) and taking a constant during integrations, we get

$$\begin{aligned} \frac{ds}{d\tau} = & \frac{2k}{l^3} \sum_{m=1,3,\dots} \sum_{n=1,3,\dots} K_n \frac{m^2 \sin \frac{n\pi}{2} \cos \frac{n\pi}{2h} l}{\left(\frac{m}{l}\right)^2 - \left(\frac{n}{h}\right)^2} e^{-\frac{cm^2\pi^2}{4l^2}\tau} + \\ & + \frac{4k}{(h-1)^3} \sum_{m=1,3,\dots} \sum_{n=1,3,\dots} K_n \frac{m^2 \sin \frac{n\pi}{2} \cos \frac{n\pi}{2h} l}{\left(\frac{m}{h-1}\right)^2 - \left(\frac{n}{2h}\right)^2} e^{-\frac{cm^2\pi^2}{(h-1)^2}\tau} \quad (120) \end{aligned}$$

as the rate of settlement of the top surface of the deposit. The total settlement of the top surface at any time τ , due to drainage alone, is:

$$\begin{aligned} s = \int_0^\tau \frac{ds}{d\tau} d\tau = & \frac{8a}{\pi^2 l} \sum_{m=1,3,\dots} \sum_{n=1,3,\dots} K_n \frac{\sin \frac{n\pi}{2} \cos \frac{n\pi}{2h} l}{\left(\frac{m}{l}\right)^2 - \left(\frac{n}{h}\right)^2} (1 - e^{-\frac{cm^2\pi^2}{4l^2}\tau}) + \\ & + \frac{4a}{\pi^2 (h-1)} \sum_{m=1,3,\dots} \sum_{n=1,3,\dots} K_n \frac{\sin \frac{n\pi}{2} \cos \frac{n\pi}{2h} l}{\left(\frac{m}{h-1}\right)^2 - \left(\frac{n}{2h}\right)^2} (1 - e^{-\frac{cm^2\pi^2}{(h-1)^2}\tau}) \quad (121) \end{aligned}$$

Equations (120) and (121) refer to the case in which the layer of sand is located at a distance l from the bottom surface.

For the case of the sand layer located at the bottom surface, we have from equation (120) by setting $l = 0$ in the second term of the right hand member,

$$\frac{ds}{d\tau} = \frac{16k}{h} \sum_{m=1,3,\dots} \sum_{n=1,3,\dots} K_n \frac{m^2 \sin \frac{n\pi}{2}}{4m^2 - n^2} e^{-\frac{cm^2\pi^2}{h^2}\tau} \quad (122)$$

Placing $l = 0$ in the second term of the right hand member

of equation (121) we get

$$s = \frac{16 a h}{\pi^2} \sum_{m=1,3,\dots} \sum_{n=1,3,\dots}^K \frac{\sin \frac{n\pi}{2}}{4m^2 - n^2} \left(1 - e^{-\frac{cm^2 \pi^2}{h^2} \tau} \right) \quad (123)$$

These two last equations give the rates of settlement and total settlement of the top surface of the deposit at any time, τ .

It should be noticed that the total settlement given by equation (121) and (123) is that due to drainage alone. Therefore, if the total settlement from the beginning of the second stage, to a time $\tau = \tau_1$, is desired, we will have to add to the above equations, a constant term giving the total settlement at a time $t = t_1$ corresponding to $\tau = 0$.

(5). Example. We will now determine the stress distribution and settlements of the mud deposit discussed in Problem I, for $t = 500$ and $t = 1,000$ years. Two cases will be considered: that in which the sand layer is located at the center of the deposit, and that in which it is located at the bottom.

The constants are:

$$c = 3,395, \quad a = 0.00024, \quad h = 2,500 \text{ cm.} \quad l = 1,250 \text{ cm.}$$

$$\text{and } l = 0.$$

(a). Sand Layer at Middle of Deposit. The results of computations are as follows:

Initial State: t = 500 years				Initial state: t=1,000 years		
$\frac{z}{h}$	Values of w for (gr/cm^2)			Values of w for (gr/cm^2)		
	$\tau = 50$	$\tau = 100$	$\tau = 500$	$\tau = 50$	$\tau = 100$	$\tau = 500$
0	1,370	1,060	97	700	540	50
1/4	950	750	68	485	380	35
3/8	514	405	37	260	210	27
5/8	193	64	-	96	33	-
3/4	266	91	-	136	47	-

The rates of settlement and total settlements are:

	Initial state: $t = 500$ years				Initial state $t = 1,000$ years.			
	$\tau=10$	$\tau=50$	$\tau=100$	$\tau=500$	$\tau=10$	$\tau=50$	$\tau=100$	$\tau=500$
$\frac{ds}{d\tau}$ in cm./yr.	5.83	2.65	1.43	0.12	2.98	1.35	0.73	0.07
s in m.	0.69	2.14	3.09	5.50	0.35	1.09	1.57	2.81

(b). Sand Layer at the Bottom of the Deposit:

The hydrodynamic pressures are:

Initial state: t=500 years				Initial state: t = 1,000 years		
$\frac{z}{h}$	Values of w for (gr/cm^2)			Values of w for (gr/cm^2)		
	$\tau = 50$	$\tau = 100$	$\tau = 500$	$\tau = 50$	$\tau = 100$	$\tau = 500$
$\frac{1}{8}$	501	330	34	256	168	17
$\frac{1}{4}$	873	590	62	445	296	32
$\frac{1}{2}$	986	750	88	503	382	45
$\frac{3}{4}$	523	470	34	262	240	17

The rates of settlement and total settlements are:

	Initial state: t=500 yrs.				Initial state: t=1000 yrs.			
	$\tau=10$	$\tau=50$	$\tau=100$	$\tau=500$	$\tau=10$	$\tau=50$	$\tau=100$	$\tau=500$
$\frac{ds}{dt}$ in cm./year	4.26	2.02	1.53	0.14	2.18	1.03	0.79	0.07
s in m.	0.52	1.69	2.59	5.21	0.27	0.86	1.32	2.66

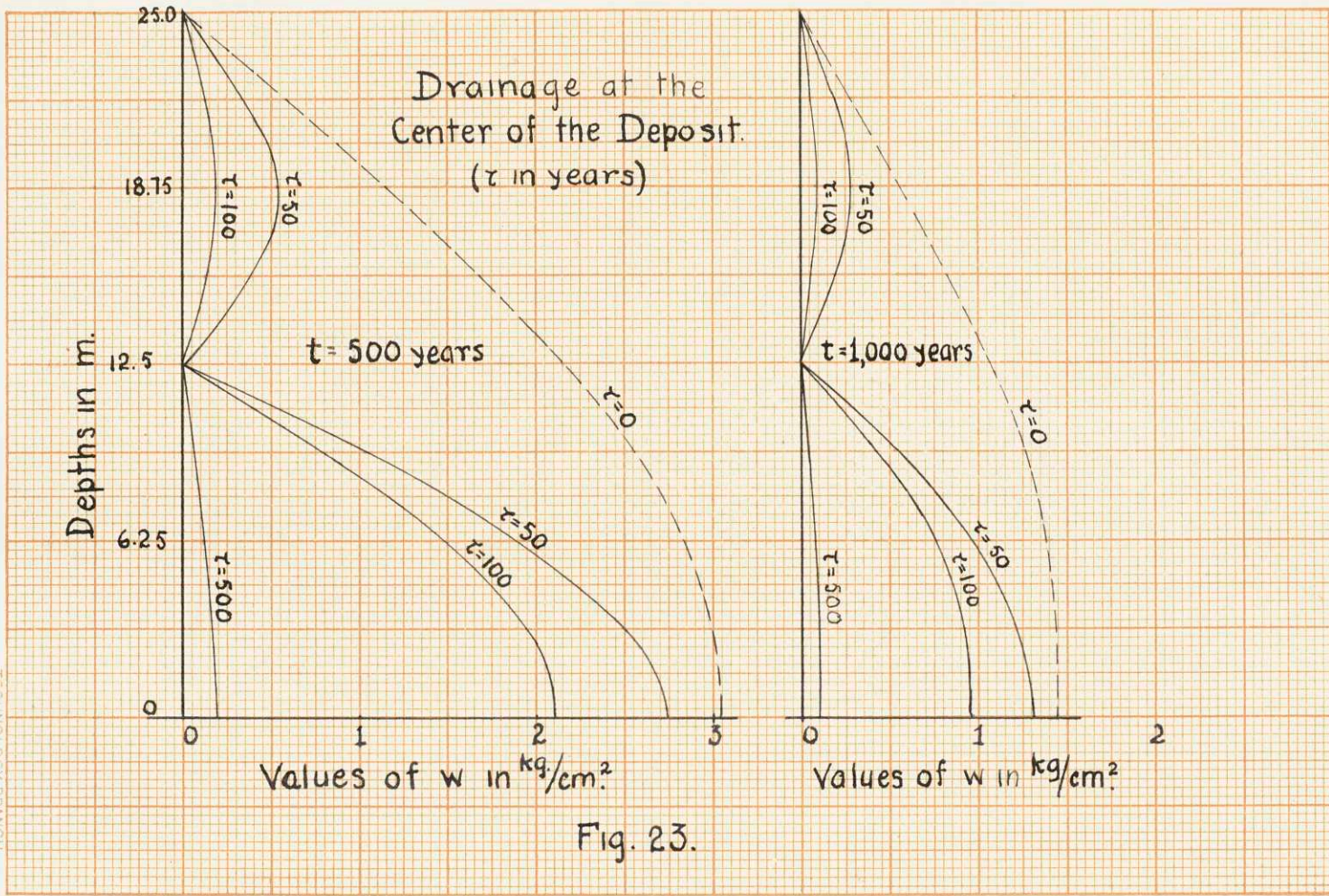
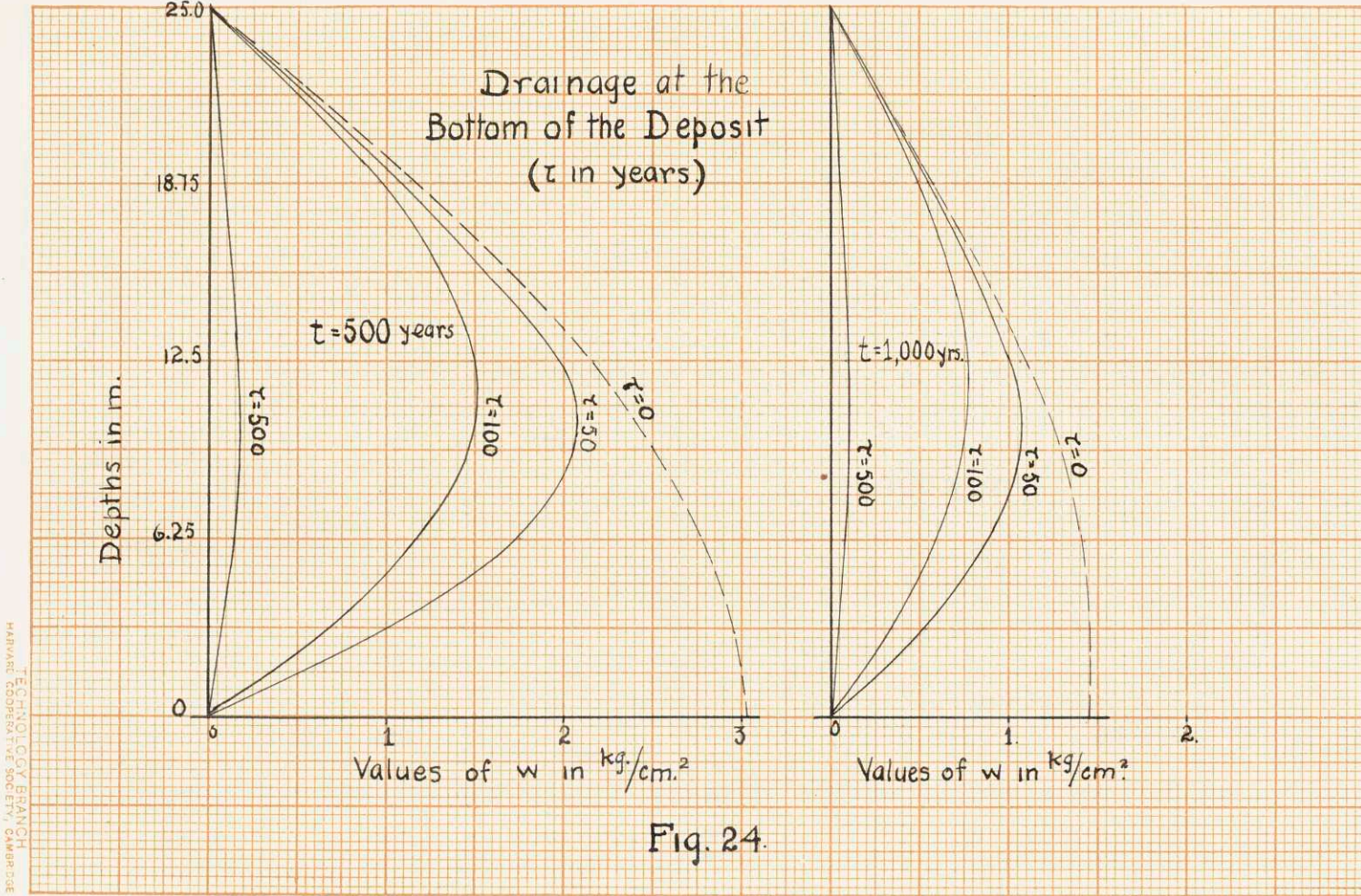


Fig. 23.



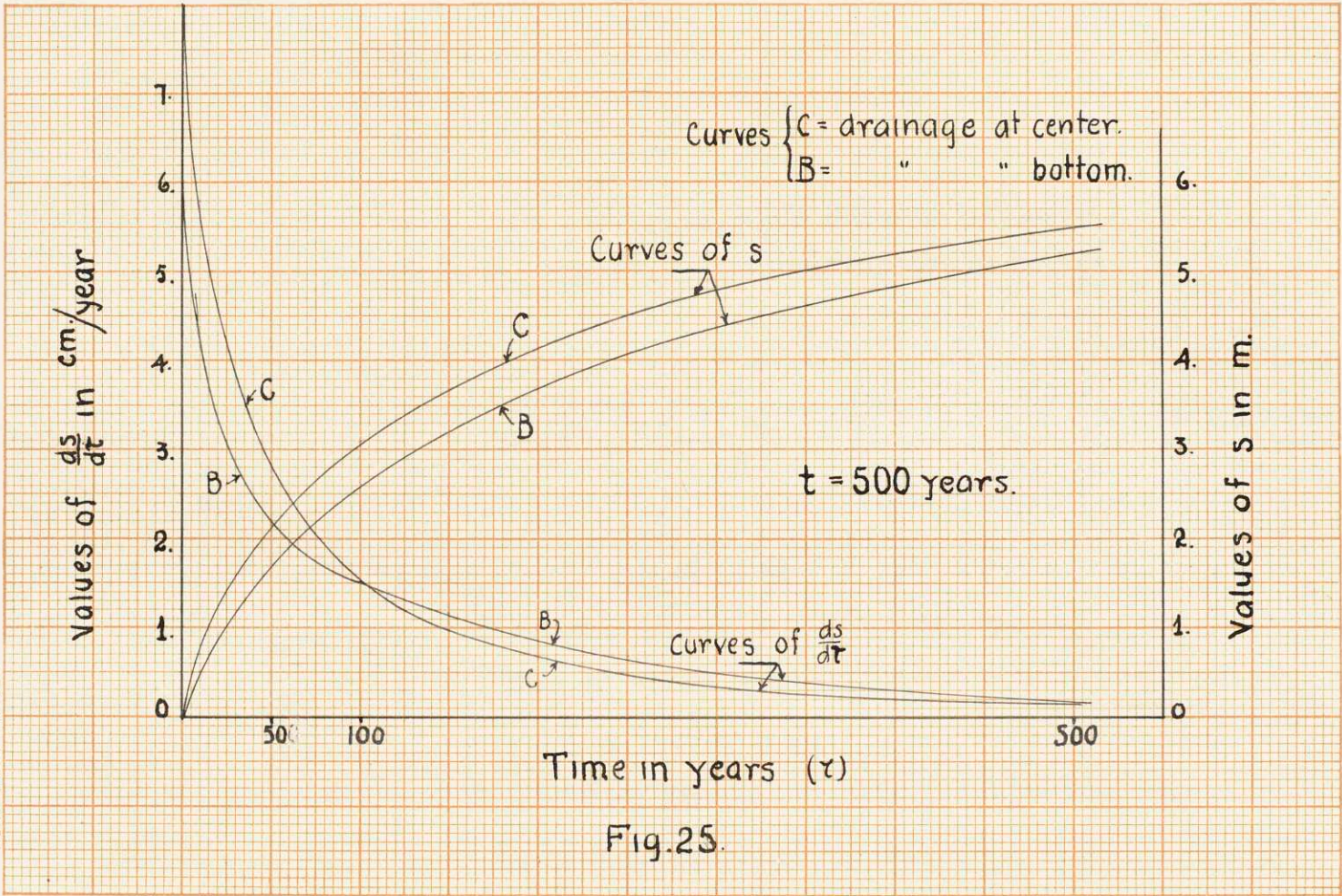


Fig. 25.

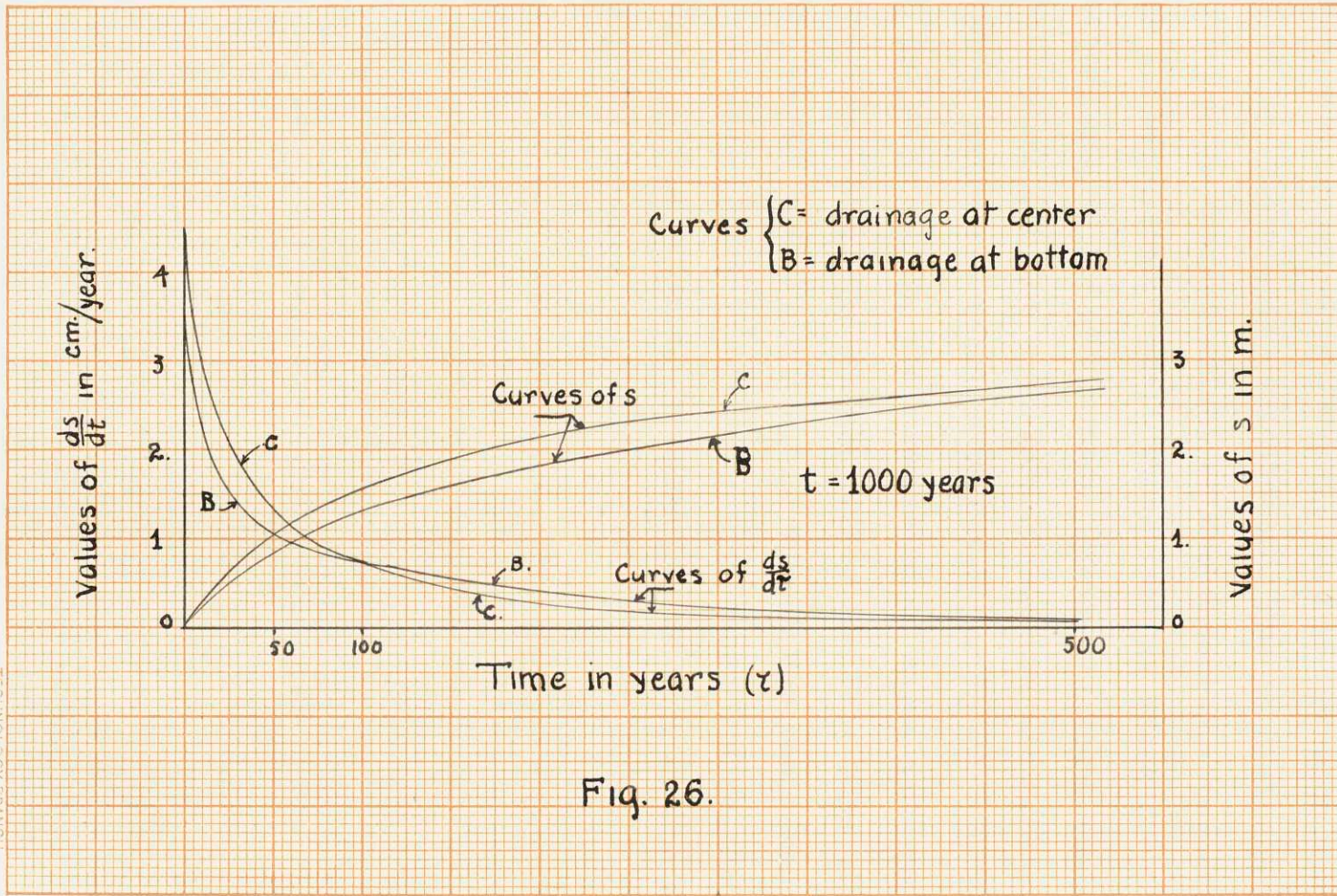


Fig. 26.

These results are plotted in Figs. (23), (24), (25), and (26). From them we learn of how much more rapid is the process of consolidation due to drainage than that due to the own weight of the material. Consider, for instance, the deposit whose initial state ($\tau = 0$) corresponds to $t = 500$ years. After 500 years the total settlements for the two cases considered are 5.50 m. and 5.21 m. while the total settlement due to the own weight of the material for the same period of time, ($t = 500$ to $t = 1,000$ years) is 3.03 m. (Problem I). This difference is not very large, but if we compare the rates of settlement due to drainage with those due to the own weight of the material (Problem I), we will notice a very large difference.

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Graduated from Mackenzie College (under charter of the New York State University), S. Paulo, Brazil, in Civil Engineering, December 1922.

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