GLOBAL SOLVABILITY OF INVARIANT DIFFERENTIAL OPERATORS

by

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ABSTRACT

Global solvability of every non-zero semi-bi-invariant differential operator on simply-connected solvable Lie groups and the Laplacian on symmetric spaces G/H (where G is a non-compact ,connected semisimple Lie group with finite center and H is an open subgroup of the fixed point group of an involution of G) is proved. Also, the convexity of simply-connected split solvable Lie group with respect to all non-zero left invariant differential operators is shown. This gives a new proof to Helgason's global solvability theorem of invariant differential operators on symmetric spaces of non-compact type.

Thesis Supervisor: Sigurdur Helgason

Title: Professor of Mathematics

"Geneviève, lis-nous des vers..."

Tu lisais, et, pour nous, c'étaient des enseignments sur le monde, sur la vie, qui nous venaient non du poète, mais de ta sagesse. Et les détresses des amants et les pleurs des reines devenaient de grandes choses tranquilles. On mourait d'amour avec tant de calme dans ta voix...

"Geneviève, est-ce vrai que l'on meurt d'amour?"

Tu suspendais les vers, tu réfléchissais gravement. Tu cherchais sans doute la réponse chez les fougères, les grillons, les abeilles et tu répondais "oui" puisque les abeilles en meurent. C'était nécessaire et paisible.

Antoine Saint-Exupery "Courrier Sud"

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Mogens Flensted-Jensen taught me basic facts about pseudo-Riemannian symmetric spaces.

Michel Duflo and David Wigner informed me that they recently proved the global solvability of bi-invariant operators using a different method. I very much appreciate their comments.

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CHAPTER 0

Introduction

The solvability problem of invariant differential operators on homogeneous manifolds has been studied by several mathematicians in recent years. Many theorems on differential operators with constant coefficients have been generalized. Among the most notable recent advances is the local solvability of all non-zero bi-invariant differential operators on all Lie groups proved by Duflo [4]. Local solvability of left-invariant operators is false in general as was shown by Cerèzo-Rouvière [3]. In this thesis, we consider the global sovability problem rather than the local one, and we will work in the category of smooth functions. We call a differential operator P defined on a smooth manifold M globally solvable on M if for any smooth function f on M, we can find a smooth function u on M so that $Pu = f$ holds on M. It is known that if P is linear and has smooth coefficients, the semi-global solvability of P (the solvability on each compact set of M) and P-convexity of M (See Definition 1.2) imply the global solvability of P on M. Our main results in this thesis are:

(1) The global solvability of all non-zero semi-bi-invariant differential operators (Definition 2.1.3)

on simply connected solvable Lie groups (Corollary 2.3.4).

(2) The P-convexity of a simply connected split solvable Lie group (Definition 2.1.2) for each non-zero left-invariant differential operator P (Theorem 2.3.5).

(3) The global solvability of the Laplacian on non-compact semisimple symmetric spaces G/H where G is a non-compact semisimple Lie group with finite center (connected) and H is an open subgroup of the fixed point group of an involution of G (Theorem 3.4).

Although (2) does not imply any solvability result by itself, it can be applied to symmetric spaces of non-compact type and gives a new proof to the P-convexity part of Helgason's global solvability theorem of non-zero invariant differential operators (Helgason [10]).
(1) is a generalization of the global solvability of non-zero bi-invariant operators on simply connected nilpotent Lie groups proved by Wigner [19]. (3) is a generalization of Raugh-Wigner's result [13] that the Casimir operator on a non-compact semisimple Lie group with finite center is globally solvable. In fact, the proof of the semi-global solvability in (3) is analogous to that in [13] in the sense that by investigating bicharacteristic curves, we use a theorem in

Duistermaat-Hörmander [10]. However, to prove the P-convexity part in (3), we will use a theorem in Flensted-Jensen [7] and Helgason's theorem on the radial part of the Laplacian.

Chapter I is devoted to general preliminaries.

In Chapter II we consider invariant operators on simply connected solvable groups. By reproducing Rouvière [15], we obtain the semi-global solvability of all non-zero semi-bi-invariant operators in §2. (If one wants a shorter proof, one could say that the semi-global solvability is immediate from Rouvière's work just by noting the commutativity of semi-bi-invariant operators.) Note that for exponential solvable groups, Duflo-Rais [5] proved the same result. §3 is devoted to the P-convexity results and global solvability.

In Chapter III we study the Laplacian on a class of pseudo-Riemannian symmetric spaces called "semisimple". In §1, after some preliminaries, we give a complete proof for the following fact: All bicharacteristic curves of the Laplacian on pseudo-Riemannian spaces are geodesics.

Of course, this is well known but since it is hard to find an explicit proof in the literature, we find it worth including in the thesis. In §2, we prove that on our symmetric spaces, no null bicharacteristic curve of the Laplacian stays inside a compact set. In § 3 we prove the P-convexity part and also show the injectivity of the Laplacian on the space of smooth functions with compact support. § 4 gives the final conclusion.

Chapter I

General Preliminaries

We fix our basic notation. Let R, C, Z, Z^+ denote respectively the set of real numbers, complex numbers, integers, positive integers. If A and B are sets, $A\setminus B$ shall denote the complement of B in A. Let M be a smooth manifold countable at infinity. T^*M shall denote the cotangent bundle of M and $\pi: T^*M \to M$ the projection. Let $C^\infty(M)$, $C_0^\infty(M)$, $\mathcal{D}'(M)$, $\mathcal{E}'(M)$ denote respectively the space of smooth functions, smooth functions with compact support, distributions, distributions with compact support on M. If u is either a function or a distribution on M, supp u denotes the support of u.

Let $(x_1,\ldots,x_n)$ be local coordinates of M. Then the <u>induced coordinates</u> $(x_1,\ldots,x_n,\xi_1,\ldots,\xi_n)$ of T^*M is defined in such a way that $(x_1,\ldots,x_n,\xi_1,\ldots,\xi_n)$ represents the cotangent vector $\xi_1 dx_1 + \ldots + \xi_n dx_n$ at $x = (x_1,\ldots,x_n)$.

For $f \in C^\infty(M)$, let df denote the differential of f. $df(x_0)$ shall denote the cotangent vector at x_0 given as the value of df at x_0. In terms of the local coordinates,

$$df(x_0) = \frac{\partial f}{\partial x_1}(x_0)(dx_1)_{x_0} + \ldots + \frac{\partial f}{\partial x_n}(x_0)(dx_n)_{x_0}$$

where the $(dx_i)_{x_0}$ are the cotangent vector dx_i at x_0.

Throughout this thesis, we use the standard multi-index notation, e.g. $\alpha = (\alpha_1,\ldots,\alpha_n)$,

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \quad \xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \ldots + \alpha_n$$

etc.

By $[\ ,\ ]$ we denote the commutator of differential operators. Let D be a linear differential operator on M. (In this thesis, we treat only linear differential operators with smooth coefficients).

By deg D, we denote the degree of D.

<u>Definition 1.1</u>

The principal symbol $\sigma(D)$ of a differential operator D on M is a map $T^*M \to C$ given by

$$\sigma(D)(df(x_0)) = \frac{1}{m!}D_x(f(x)-f(x_0))^m|_{x=x_0}$$

where $m = \deg D$ and D_x denotes that D is acting on the x variable. This is well-defined.

<u>Remark</u>

We can show the well-definedness of $\sigma(D)$ as follows. Take local coordinates $(x_1,\ldots,x_n)$ of M so that

$$D = \sum_{|\alpha|\leq m} a_\alpha(x)\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

Then by induction on m, we can show that

$$(*) \quad \frac{1}{m!} D_x (f(x) - f(x_0))^m \Big|_{x=x_0}$$

$$= \sum_{|\alpha|=m} a_\alpha(x) \left(\frac{\partial f}{\partial x_1}(x_0)\right)^{\alpha_1} \cdots \left(\frac{\partial f}{\partial x_n}(x_0)\right)^{\alpha_n}.$$

Hence in the induced coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ we have

$$(**) \quad \sigma(D)(x_1, \dots, x_n, \xi_1, \dots, \xi_n) = \sum_{|\alpha|=m} a_\alpha(x) \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$$

and this shows the well-definedness of $\sigma(D)$. Also (**) shows that $\sigma(D_1 D_2) = \sigma(D_1)\sigma(D_2)$ for two differential operators with C^∞-coefficients D_1, D_2.

In the following definitions, M is a smooth manifold countable at infinity, D is a linear differential operator with smooth coefficients on M.

<u>Definition 1.2</u>

M is called <u>D-convex</u> if for any compact set K of M, there exists a compact set K' of M such that

$$u \in \mathcal{E}'(M), \ \operatorname{supp} {}^t\!Du \subset K \Rightarrow \operatorname{supp} u \subset K'.$$

Here ${}^tD$ denotes the transpose of D. Namely let

$\langle\ ,\ \rangle$ denote the pairing of distributions and smooth functions with compact support on M. Then ${}^tD$ is defined by $\langle {}^tDu, v\rangle = \langle u, Dv\rangle$ for $u \in \mathcal{D}'(M)$, $v \in C_0^\infty(M)$.

Definition 1.3

Assume that M is given a fixed nowhere vanishing smooth measure so that $C_0^\infty(M)$ is identified with a subspace of $\mathcal{D}'(M)$.

Then a closed set $F \subset M$ is called D-full if

$$u \in \mathcal{E}'(M),\ \operatorname{supp} Du \subset F \Rightarrow \operatorname{supp} u \subset F$$

Remark In Chapter II where M = a simply connected solvable Lie group, we will use the right invariant measure. In Chapter III where $M = G/H$ = a non compact semisimple symmetric space, we will use the G-invariant Riemannian measure. In order to show the D-convexity of M, we shall show that any compact set is contained in a compact ${}^tD$-full set.

Definition 1.4

D is called semi-globally solvable on M if for any $f \in C^\infty(M)$ and any compact set K of M, we can find $u \in C^\infty(M)$ so that $Du = f$ holds on K.

Definition 1.5

D is called globally solvable on M if for any $f \in C^\infty(M)$ there exists $u \in C^\infty(M)$ such that $Du = f$ holds on M.

We have the following sufficient condition for the global solvability.

Theorem 1.6 (Trèves [16] Theorem 3.3)

Suppose D and M are as above. Then D is globally solvable on M if

(1) D is semi-globally solvable on M and

(2) M is D-convex.

We also want to remark the following fact.

Theorem 1.7 (Hörmander [11] Theorem 3.5.1).

Let P be a non-zero linear differential operator with constant coefficeitns on R^n. Then every convex closed set is P-full.

The following uniqueness theorem of Holmgren plays a significant role in our work.

The uniqueness theorem of Holmgren (Hörmander [11], Theorem 5.3.1).

Let Ω be an open subset of R^n, D a differential operator with analytic coefficients in Ω. Let ϕ be a real valued smooth function on Ω and let $x_0 \in \Omega$ be such that $\sigma(D)(d\phi(x_0)) \neq 0$ (i.e. the level surface of ϕ is non-characteristic to D at x_0). Then there exists a neighborhood $\Omega' \subset \Omega$ of x_0 such that every $u \in \mathcal{D}'(\Omega)$ satisfying $Du \equiv 0$ on Ω and vanishing on $\phi(x) > \phi(x_0)$, $x \in \Omega$ must also vanish on Ω'.

The following version of Holmgren's theorem will be used in the sequel. It is stated in a bit artificial way. But instead, we will be able to avoid repetitions of similar arguments in the later chapters.

Proposition 1.8

Let M be a real analytic manifold and D a linear differential operator with analytic coefficients on M. Let F be a closed set of M and assume that F is D-full. Let ϕ be a real valued smooth function on M, N a positive constant so that

$$\sigma(D)(d\phi(x)) \neq 0 \quad \text{for} \quad x \in F\,,\ |\phi(x)| \geqq N.$$

Then for any $L \geqq N$, the closed set $\{x \in M \mid |\phi(x)| \leqq L\} \cap F$ is D-full.

<proof> Take $L \geqq N$.

Let $u \in \mathcal{E}'(M)$ be such that

supp $Du \subset \{x \in M \mid |\phi(x)| \leqq L\} \cap F$.

Our objective is to show that

$$\text{supp } u \subset \{x \in M \mid |\phi(x)| \leqq L\} \cap F\,.$$

By the D-fullness of F we have

$$\text{supp } u \subset F.$$

Assume that supp $u \not\subset \{x \in M \mid |\phi(x)| \leqq L\}$.

We want to derive a contradiction.

We have $\sup_{x \in \text{supp } u} |\phi(x)| > L.$

Without loss of generality we may assume that there is a point x_0 such that

$$\phi(x_0) = \sup_{x \in \text{supp } u} |\phi(x)|, \quad x_0 \in \text{supp } u.$$

Note $\phi(x_0) > L$ and $x_0 \in F$. Since by our assumption $\sup_{x \in \text{supp } Du} \phi(x) \leqq L$, it is clear that there is a neighborhood Ω of x_0 in M such that $Du \equiv 0$ on Ω and $u \equiv 0$ on $\phi(x) > \phi(x_0)$, $x \in \Omega$. Note that since $\phi(x_0) > L \geqq N$, $\sigma(D)(d\phi(x_0)) \neq 0$. By Holmgren's theorem, we have a neighborhood of x_0 where u vanishes. But $x_0 \in \text{supp } u$. This is a contradiction. Therefore $\text{supp } u \subset \{x \in M \,|\, |\phi(x)| \leqq L\} \cap F$.

<u>q.e.d.</u>

Chapter II

Solvable Groups

§1. Preliminaries for solvable groups.

Let $\mathfrak{g}$ be a Lie algebra over R. By $\mathfrak{g}_C$ we denote the complexification of $\mathfrak{g}$. Suppose that $\mathfrak{g}$ is solvable. By $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ we denote the vector subspace of $\mathfrak{g}$ spanned by the elements of the form $[X,Y]$, $X \in \mathfrak{g}$, $Y \in \mathfrak{g}$. Then $\mathfrak{g}^1$ is an ideal of $\mathfrak{g}$. We define $\mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}^1]$, ... $\mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}^i]$ ..., in the similar manner. Each $\mathfrak{g}^i$ is an ideal of $\mathfrak{g}$ called the i-th derived ideal. By the solvability assumption on $\mathfrak{g}$, we have $\mathfrak{g} \supsetneqq \mathfrak{g}^1 \supsetneqq \cdots \supsetneqq \mathfrak{g}^\ell = \{0\}$ for some integer ℓ. So we can take an ordered basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ in such a way that if $i \leqq j$ and $X_i \in \mathfrak{g}^k$, then $X_j \in \mathfrak{g}^k$. For each i, $\{X_i, X_{i+1}, \ldots, X_n\}$ spans a subalgebra of $\mathfrak{g}$ and the span of $\{X_{i+1}, X_{i+2}, \ldots, X_n\}$ is an ideal of the span of $\{X_i, X_{i+1}, \ldots, X_n\}$. In fact suppose $X_i \in \mathfrak{g}^k \setminus \mathfrak{g}^{k+1}$. Then the span of $\{X_{i+1}, X_{i+2}, \ldots, X_n\}$ contains $\mathfrak{g}^{k+1}$. For any $j_1, j_2 \geqq i$, $[X_{j_1}, X_{j_2}] \subset [\mathfrak{g}^k, \mathfrak{g}^k] \subset \mathfrak{g}^{k+1} \subset$ the span of $\{X_{i+1}, \ldots, X_n\}$. Hence the span of $\{X_{i+1}, \ldots, X_n\}$ is an ideal of the span of $\{X_i, \ldots, X_n\}$.

In general, suppose $Y_1, \ldots, Y_n$ is a basis of $\mathfrak{g}$ such that for each i, the span of $\{Y_i, Y_{i+1}, \ldots, Y_n\}$

is a subalgebra of $\mathfrak{g}$ and the span of $\{Y_{i+1}, Y_{i+2}, \ldots, Y_n\}$ is an ideal of the span of $\{Y_i, Y_{i+1}, \ldots, Y_n\}$. Then there is a diffeomorphism from the simply connected solvable Lie group G with the Lie algebra $\mathfrak{g}$ onto R^n given by

$$\exp t_1Y_1 \ldots \exp t_nY_n \longrightarrow (t_1, \ldots, t_n)$$

(See Varadarajan [17] Theorem 3.18.11).

In the sequel, we shall frequently make an identification between G and R^n after fixing such a basis. Note that under this identification, the left-invariant differential operator Y_n on G is identified with $\frac{d}{dt_n}$.

Also remark the following.

(1) If G is a simply connected solvable Lie group, then every analytic subgroup of G is closed and simply connected. ([17]. Theorem 3.18.12).

(2) Let G be as in (1). If N is a normal analytic subgroup of G, then G/N is a simply connected solvable Lie group. ([17] Theorem 3.18.2).

These two remarks will enable us to work on our problem using induction on dim G.

Lemma 2.1.1

Let $\mathfrak{g}$ be a solvable Lie algebra over R of dimension n and let $X_n \in \mathfrak{g}$ be a non-zero element which spans an ideal of $\mathfrak{g}$. Then we can take elements $X_1, \ldots, X_{n-1} \in \mathfrak{g}$ so that $X_1, \ldots, X_n$ form a basis for $\mathfrak{g}$, and for each i, the span of $\{X_i, X_{i+1}, \ldots, X_n\}$ is a subalgebra of $\mathfrak{g}$ and the span of $\{X_{i+1}, X_{i+2}, \ldots, X_n\}$ is an ideal of the span of $\{X_i, X_{i+1}, \ldots, X_n\}$. In particular, the map $\exp t_1X_1 \cdot \ldots \cdot \exp t_nX_n \longrightarrow (t_1, \ldots, t_n)$ is a diffeomorphism of G onto R^n where G is the simply connected Lie group for $\mathfrak{g}$.

<proof> We use induction on dim $\mathfrak{g}$.

If dim $\mathfrak{g} = 1$, the statement is obvious. Let dim $\mathfrak{g} > 1$ and assume that the statement is true for all solvable Lie algebras of dimension less than dim $\mathfrak{g}$. Let $\mathfrak{n}$ denote the ideal spanned by X_n. Then applying the induction hypothesis to $\mathfrak{g}/\mathfrak{n}$, we have a basis $Y_1, \ldots, Y_{n-1}$ for $\mathfrak{g}/\mathfrak{n}$ such that for each i, the span of $\{Y_i, Y_{i+1}, \ldots, Y_{n-1}\}$ is a subalgebra of $\mathfrak{g}/\mathfrak{n}$ and the span of $\{Y_{i+1}, Y_{i+2}, \ldots, Y_{n-1}\}$ is an ideal of the span of $\{Y_i, Y_{i+1}, \ldots, Y_{n-1}\}$. Take $X_1, \ldots, X_{n-1} \in \mathfrak{g}$ so that the equivalence classes represented by the X_i are the Y_i. $(i = 1, \ldots, n-1)$. Now, it is obvious that $X_1, \ldots, X_{n-1}, X_n$ satisfy the desired condition.

Definition 2.1.2

Let $\mathfrak{g}$ be a solvable Lie algebra over R of dimension n. $\mathfrak{g}$ is called split if there is a chain of ideals $\mathfrak{g}_i$ $i = 0, \ldots, n$, of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{g}_0 \supsetneqq \mathfrak{g}_1 \supsetneqq \mathfrak{g}_2 \supsetneqq \cdots \supsetneqq \mathfrak{g}_{n-1} \supsetneqq \mathfrak{g}_n = \{0\}$$

(hence $\dim (\mathfrak{g}_i / \mathfrak{g}_{i+1}) = 1$ for each i).

A solvable Lie group is called split if its Lie algebra is split.

<u>Remark</u>

1) Nilpotent Lie aglebras are split. ([17] Cor. 3.5.6).

2) Let $\mathfrak{g}$ be a real semi-simple Lie algebra with an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$.

The solvable Lie algebra $\mathfrak{a} + \mathfrak{n}$ is split. In fact let $\alpha_1, \ldots, \alpha_\ell$ be the restricted positive roots so that $\mathfrak{n} = \sum_{i=1}^{\ell} \mathfrak{g}_{\alpha_i}$ where $\mathfrak{g}_{\alpha_i}$ is the root space corresponding to α_i. We may assume that if $i < j$, then $\alpha_j \not\prec \alpha_i$. (We write $\alpha \prec \beta$ if $(\beta - \alpha)(\mathfrak{a}^+) > 0$, where $\mathfrak{a}^+$ is the positive Weyl chamber of $\mathfrak{a}$). Take a basis $H_1, \ldots, H_p$ of $\mathfrak{a}$ and a basis

$$X_{1,1}, \ldots, X_{1,n(\alpha_1)}, \quad \ldots\ldots\ldots\ldots\ldots \quad , X_{\ell,1}, \ldots, X_{\ell,n(\alpha_\ell)}$$

of $\mathfrak{n}$ so that for each $1 \leqq j \leqq \ell$, $X_{j,1}, \ldots, X_{j,n(\alpha_j)}$ is a basis of $\mathfrak{g}_{\alpha_j}$. ($n(\alpha_j) = \dim \mathfrak{g}_{\alpha_j}$, $p = \dim \mathfrak{a}$)

Renumber the above ordered basis of $\mathfrak{a} + \mathfrak{n}$,

$H_1, \ldots, H_p, X_{1,1}, \ldots, X_{1,n(\alpha_1)}, X_{2,1}, \ldots, X_{2,n(\alpha_2)}, \ldots$

$\ldots, X_{\ell,1}, \ldots, X_{\ell,n(\alpha_\ell)}$ as $Y_1, \ldots, Y_q$ where

$q = p + \sum_{i=1}^{\ell} n(\alpha_i)$. Then it is clear that for each

$1 \leqq i \leqq q$, the span of $\{Y_i, \ldots, Y_q\}$ is an ideal of

$\mathfrak{a} + \mathfrak{n}$. (To see this, one has only to recall

$[\mathfrak{a}, \mathfrak{g}_\alpha] \subset \mathfrak{g}_\alpha$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, and $\alpha \prec \alpha+\beta$ for

$\alpha \succ 0$, $\beta \succ 0$).

3) A subalgebra of a split solvable Lie algebra over R is split.

4) A factor algebra of a split solvable Lie algebra is split solvable.

We are now going to define semi-bi-invariant operators. Let $\mathfrak{g}$ be any Lie algebra over R and G a Lie group with Lie algebra $\mathfrak{g}$. Let $U(\mathfrak{g})$ denote the universal envelopping algebra of $\mathfrak{g}$ over R (not complexified yet!) and let $Z(\mathfrak{g})$ denote its center. Let $U(\mathfrak{g})_C$, $Z(\mathfrak{g})_C$ denote the complexifications of $U(\mathfrak{g})$, $Z(\mathfrak{g})$ respectively. $U(\mathfrak{g})_C$, $Z(\mathfrak{g})_C$ are respectively regarded as the algebra of complex coefficients left invariant differential operators on G, bi-invariant differential operators on G. $U(\mathfrak{g})_C$ and $U(\mathfrak{g}_C)$ are isomorphic and we identify them occasionally. Let $\mathfrak{g}^*$, $\mathfrak{g}^*_C$ denote the real dual of $\mathfrak{g}$, the complex dual of $\mathfrak{g}$. Let $^-$ denote the complex conjugation. For example, $\overline{P + iQ} = P - iQ$ for $P \in U(\mathfrak{g})$, $Q \in U(\mathfrak{g})$. Note that if $\lambda \in \mathfrak{g}_C^*$, $\overline{\lambda} \in \mathfrak{g}_C^*$ is given by $\overline{\lambda}(X) = \lambda(\overline{X})$.

Definition 2.1.3

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A left invariant differential operator $P \in U(\mathfrak{g})_C$ on G is called semi-bi-invariant if there exists $\lambda \in \mathfrak{g}_C^*$ such that

$$[X,P] = \lambda(X)P \quad \text{for} \quad X \in \mathfrak{g}.$$

We put $U(\mathfrak{g})_C^\lambda = \{Q \in U(\mathfrak{g})_C \mid [X,Q] = \lambda(X)Q \quad \text{for} \quad X \in \mathfrak{g}_C\}$. The set of all semi-bi-invariant operators is $\bigcup_{\lambda \in \mathfrak{g}_C^*} U(\mathfrak{g})_C^\lambda$.

Remark

1) $U(\mathfrak{g})_C^0 = Z(\mathfrak{g})_C$

2) Suppose $U(\mathfrak{g})_C^\lambda \neq 0$ for $\lambda \in \mathfrak{g}_C^*$.

Then $\ker \lambda$ is a complex ideal of $\mathfrak{g}_C$.

In fact take $0 \neq Q \in U(\mathfrak{g}_C)^\lambda$. Then $[X,Q] = \lambda(X)Q$ for all $X \in \mathfrak{g}_C^*$. If $X, Y \in \mathfrak{g}_C^*$ then by the Jacobi Identity,

$$[[X,Y],Q] = -[[Y,Q],X] - [[Q,X],Y]$$
$$= -\lambda(Y)[Q,X] + \lambda(X)[Q,Y] = 0.$$

So $\lambda([X,Y]) = 0$. Hence $\ker \lambda \supset [\mathfrak{g},\mathfrak{g}]$.

So $\ker \lambda$ is an ideal of $\mathfrak{g}_C$.

q.e.d.

The following lemma due to Borho is of great importance to us.

Bohro's lemma (Borho [2] page 58)

Let $\mathfrak{g}$ be a solvable Lie algebra over R. If there exists $\lambda \neq 0$ in $\mathfrak{g}_C^*$ such that $U(\mathfrak{g})_C^\lambda \neq 0$, then all semi-bi-invariant operators are contained in $U(\ker \lambda)$. i.e.

$$\bigcup_{\lambda \in \mathfrak{g}_C^*} U(\mathfrak{g}_C)^\lambda \subset U(\ker \lambda).$$

(Recall by the Remark (2) above that $\ker \lambda$ is a complex ideal of $\mathfrak{g}_C$).

We use the following consequences of Bo o's lemma.

Lemma 2.1.4

Let $\mathfrak{g}$ be a solvable Lie algebra over R.

(1) If there is a semi-bi-invariant operator in $U(\mathfrak{g})_C$ which is not bi-invariant, then there exists an ideal $\mathfrak{h}$ of $\mathfrak{g}$ of codimension one such that every semi-bi-invariant operator in $U(\mathfrak{g})_C$ is contained in $U(\mathfrak{h})_C$.

(2) If the center of $\mathfrak{g}$ is zero, there exists an ideal $\mathfrak{h}$ of $\mathfrak{g}$ of codimension one such that every semi-bi-invariant operator in $U(\mathfrak{g})_C$ is contained in $U(\mathfrak{h})_C$.

<proof>

Let $0 \neq P \in U(\mathfrak{g})_C^\lambda$ for some $\lambda \in \mathfrak{g}_C^*$. If λ is pure imaginary, i.e. $\lambda \in i\mathfrak{g}^* \subset \mathfrak{g}_C^*$ then $\ker \lambda = (\mathfrak{h})_C$ for some ideal $\mathfrak{h}$ of $\mathfrak{g}$ of codimension one.

And by Borho's lemma, we have (1). Assume that λ is not pure imaginary. Observe $\overline{P} \in U(\mathfrak{g})_C^{\overline{\lambda}}$. In fact, for $X \in \mathfrak{g}_C^*$, $[X,P] = \lambda(X)P$, so taking the complex conjugation, $\overline{[X,P]} = \overline{\lambda(X)}\ \overline{P}$.

Hence $[\overline{X},\overline{P}] = \overline{\lambda(X)}\ \overline{P}$ for $X \in \mathfrak{g}_C^*$

i.e. $[Y,\overline{P}] = \overline{\lambda(\overline{Y})}\ \overline{P}$ for $Y \in \mathfrak{g}_C^*$.

Therefore we have $\overline{P} \in U(\mathfrak{g})_C^{\overline{\lambda}}$. It is easy to see $U(\mathfrak{g})_C^{\lambda} \cdot U(\mathfrak{g})_C^{\overline{\lambda}} \subset U(\mathfrak{g})_C^{\lambda+\overline{\lambda}}$. So we have $P \cdot \overline{P} \in U(\mathfrak{g})_C^{2\,\mathrm{Re}\,\lambda}$ and $P \cdot \overline{P} \neq 0$. Note $2\,\mathrm{Re}\,\lambda \neq 0$. Obviously $\ker(2\,\mathrm{Re}\,\lambda) = (\mathfrak{h})_C$ for an ideal $\mathfrak{h}$ of $\mathfrak{g}$ of codimension one. And by Borho's lemma, all semi-bi-invariant operators are contained in $U(\mathfrak{h})_C$. So (1) is proved.

(2) is a special case of (1). In fact, by Lie's theorem there is a complex one-dimensional ideal $\mathfrak{g}_1$ of $\mathfrak{g}_C$. The assumption that the center of $\mathfrak{g}$ is zero amounts to that there exists $0 \neq \lambda \in \mathfrak{g}_C^*$ such that $0 \neq \mathfrak{g}_1 \subset U(\mathfrak{g})_C^{\lambda}$. So by (1), we are done.

q.e.d.

Definition 2.1.5

Let $\mathfrak{g}$ be a Lie algebra over R. Choose an ordered basis $X_1, \ldots, X_n$ for $\mathfrak{g}$. Then each element of $U(\mathfrak{g})_C$ is uniquely expressed as

$$\sum_{\alpha} c_{\alpha} X^{\alpha} = \sum_{\alpha} c_{\alpha} X_1^{\alpha_1} \cdots X_n^{\alpha_n}$$

$$c_{\alpha} \in C, \quad \alpha = (\alpha_1, \ldots, \alpha_n),$$

The above expression is called the <u>canonical expression in terms of the ordered basis $X_1, \ldots, X_n$</u>.

$|\alpha| = \alpha_1 + \ldots + \alpha_n$ is called the <u>degree</u> of the term $c_{\alpha} X_1^{\alpha_1} \cdots X_n^{\alpha_n}$.

<u>Lemma 2.1.6</u>

Let G be a Lie group with Lie aglebra $\mathfrak{g}$. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$. Let $P \in U(\mathfrak{g})_C$ be expressed as $P = \sum_{|\alpha| \leq m} c_{\alpha} X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ where $m = \deg P$. Then for $f \in C^{\infty}(G)$, $x_0 \in G$,

$$\sigma(P)(df(x_0)) = \sum_{|\alpha|=m} (X_1 f(x_0))^{\alpha_1} \cdots (X_n f(x_0))^{\alpha_n}$$

<proof>

From the definition of principal symbol, only the highest degree term of P in the above expression influences $\sigma(P)$.

$$\sigma(X_i)(df(x_0)) = X_i(f - f(x_0))|_{x=x_0} = (X_i f)(x_0)$$

Recalling that $\sigma(D_1 D_2) = \sigma(D_1)\sigma(D_2)$ for any two differential operators we get the above result. <u>q.e.d.</u>

Definition 2.1.7

Let $\mathfrak{g}$ be a solvable Lie algebra over R. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ such that $X_1, \ldots, X_\ell \notin [\mathfrak{g}, \mathfrak{g}]$ and $X_{\ell+1}, \ldots, X_n \in [\mathfrak{g}, \mathfrak{g}]$. Let $P, Q \in U(\mathfrak{g})_C$ be canonically expressed in terms of the basis above as

$$P = \sum_\beta A_P^\beta X_{\ell+1}^{\beta_{\ell+1}} \ldots X_n^{\beta_n}$$

$$Q = \sum_\beta A_Q^\beta X_{\ell+1}^{\beta_{\ell+1}} \ldots X_n^{\beta_n}$$

where $\beta = (\beta_{\ell+1}, \ldots, \beta_n)$ and A_P^β, A_Q^β are of the form $\Sigma\, c_\alpha X_1^{\alpha_1} \ldots X_\ell^{\alpha_\ell}$.

We define P and Q to be <u>equivalent with respect to the basis $X_1, \ldots, X_n$</u> if A_P^β and A_Q^β have the same highest degree part for each β.

In particular, if $\mathfrak{g}$ is abelian, P and Q are equivalent if they have the same highest degree part (with respect to any basis).

Notice that once we fix a basis as above, the "equivalence" is really an equivalence relation.

Lemma 2.1.8

Let $\mathfrak{g}$ be a Lie algebra over R of dimension n. Let $0 \neq X_n \in \mathfrak{g}$ be a vector which spans one dimensional ideal of $\mathfrak{g}$. Then there exists an automorphism ϕ of $U(\mathfrak{g})_C$ such that $X_n\phi(P) = PX_n$ for all $P \in U(\mathfrak{g})_C$.

<proof>

Assume first of all, that for each $P \in U(\mathfrak{g})_C$ there exists an element $\phi(P)$ such that $X_n\phi(P) = PX_n$. Then such a $\phi(P)$ is unique because $U(\mathfrak{g})_C$ is an integral domain.

We now show that ϕ is an injective homomorphism. In fact that ϕ is injective follows immediately from the fact that $U(\mathfrak{g})_C$ is an integral domain. To show that ϕ is a homomorphism, take $P, Q \in U(\mathfrak{g})_C$. Then by the definition, $X_n\phi(PQ) = PQX_n = PX_n\phi(Q) = X_n\phi(P)\phi(Q)$.

So $\phi(PQ) = \phi(P)\phi(Q)$. Since the linearity of ϕ is clear, ϕ is actually a homomorphism of $U(\mathfrak{g})_C$. Now we will show that ϕ really exists. For this, we use induction on deg P. Assume $\deg P \geqq 2$ and that for any element of $U(\mathfrak{g})_C$ of degree less than deg P, ϕ is defined. Without loss of generality we may assume that $P = Q_1Q_2$ with $\deg Q_1 < \deg P$ $\deg Q_2 < \deg P$ because if we can define ϕ for such elements, we can define ϕ for a linear combination of such elements. Now $PX_n = Q_1(Q_2X_n) = Q_1X_n\phi(Q_2) = X_n\phi(Q_1)\phi(Q_2)$ by our induction hypothesis. So we put $\phi(P) = \phi(Q_1)\phi(Q_2)$. If $\deg P = 1$, we have

$$PX_n = X_nP + [P,X_n] = X_nP + cX_n \quad \text{for some} \quad c \in C$$

since X_n spans an ideal. So we put $\phi(P) = P + c$ in this case.

Therefore, by induction, ϕ is defined for all elements in $U(\mathfrak{g})_C$.

q.e.d.

The following Lemma has a generalization when N has more than one dimension. But for simplicity, we state it for $\dim N = 1$ because this will be sufficient for our purpose.

<u>Lemma 2.1.9</u>

Let G be a Lie group with Lie algebra $\mathfrak{g}$ of dimension n. Let N be one dimensional closed connected normal subgroup of G with Lie algebra $\mathfrak{n}$.

Then for any left-invariant differential operator P on G, we can define a left-invariant differential operator $\tilde{P}$ on G/N by restricting P to right N-invariant functions on G. $\tilde{}$ gives a homomorphism of $U(\mathfrak{g})_C$ onto $U(\mathfrak{g}/\mathfrak{n})_C$ which coincides with the homomorphism given as the extension of the Lie algebra homomorphism $d\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$ where $d\pi$ is the differential of the projection $\pi\colon G \to G/N$.

The kernel of $\tilde{}$ is $U(\mathfrak{g})_C\mathfrak{n}$.

<proof>

Let $P \in U(\mathfrak{g})_C$. We want to show first that P maps a right N-invariant C^∞ function on G to a right invariant one. Thus choose a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ so that X_n spans $\mathfrak{g}$. Suppose $f \in C^\infty(G)$ is right N-invariant. Then $X_n f \equiv 0$. On the other hand we can write

$$X_n P = Q X_n \quad \text{(See Lemma 2.1.8)}$$

for some $Q \in U(\mathfrak{g})_C$.
Therefore $X_n(Pf) = Q(X_n f) \equiv 0$ which implies that Pf is right N-invariant. So we can define a linear operator

$$\tilde{P}: C^\infty(G/N) \to C^\infty(G/N)$$

We claim that $\tilde{P}$ is a differential operator i.e. for $u \in C_0^\infty(G/N)$,

$$\text{supp } \tilde{P}u \subset \text{supp } u.$$

Let f denote the right N-invariant function on G corresponding to u. (By this, we simply mean that $f(x) = u(xN)$). By the definition of $\tilde{P}$, Pf is the right N-invariant function on G corresponding to $\tilde{P}u$ on G/N. Since P is a differential operator on G, we have

$\text{supp}\ Pf \subset \text{supp}\ f$ on G.

Hence we have

$\text{supp}\ \tilde{P}u \subset \text{supp}\ u$ on G/N.

Thus $\tilde{P}$ is actually a differential operator on G/N and its left-invariance follows from the left-invariance of P on G. Next we show that the map $\tilde{}: U(\mathfrak{g})_C \to U(\mathfrak{g}/\mathfrak{n})_C$ is a homomorphism. Let $P, Q \in U(\mathfrak{g})_C$. Let $u \in C^\infty(G/N)$ and let $f \in C^\infty(G)$ be the corresponding right N-invariant function on G.

We want to show

$\widetilde{QP}u = \tilde{Q}(\tilde{P}u)$.

But as remarked above $\tilde{P}u$ on G/N corresponds to Pf on G hence $\tilde{Q}(\tilde{P}u)$ on G/N corresponds to QPf on G. On the other hand $\widetilde{QP}u$ corresponds to QPf.

Hence $\widetilde{QP}u = \tilde{Q}(\tilde{P}u)$. So $\tilde{}$ is a homomorphism. Next, we want to show that $d\pi: \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$ and $\tilde{}$ coincide on $\mathfrak{g}$. Let $X \in \mathfrak{g}$. Let $u \in C^\infty(G/N)$ and let $f \in C^\infty(G)$ be the corresponding function to u. Then $d\pi(X)u(xN) = \frac{d}{dt}u(xN\cdot(\exp t\, d\pi(X)))|_{t=0}$

$= \frac{d}{dt}u(x(\exp tX)N)|_{t=0} = \frac{d}{dt}f(x \exp tX)|_{t=0} = Xf(x) = \tilde{X}u(xN)$.

Hence $\sim(X)$ and $d\pi(X)$ coincide for all $X \in \mathfrak{g}$.

Finally we want to show that the kernel of $\sim$ is $U(\mathfrak{g})_C \mathfrak{n}$. $U(\mathfrak{g})_C \mathfrak{n} \subset \ker \sim$ is clear. Suppose $P \in \ker \sim$. Take $X_1, \ldots, X_{n-1} \in \mathfrak{g}$ so that $X_1, \ldots, X_n$ form a basis of $\mathfrak{g}$. Let $P = \Sigma\, c_\alpha X_1^{\alpha_1} \ldots X_n^{\alpha_n}$ be the canonical expression of P with respect to the basis $X_1, \ldots, X_n$.

$$\sim(P) = \Sigma\, c_\alpha (\tilde{X}_1)^{\alpha_1} \ldots (\tilde{X}_{n-1})^{\alpha_{n-1}} (\tilde{X}_n)^{\alpha_n} = 0.$$

Since $\tilde{X}_1, \ldots, \tilde{X}_{n-1}$ is a basis of $\mathfrak{g}/\mathfrak{n}$, it is clear that $c_\alpha = 0$ implies $\alpha_n > 0$. So $\ker \sim \subset U(\mathfrak{g})_C \cdot \mathfrak{n}$.

Hence $\ker \sim = U(\mathfrak{g})_C \cdot \mathfrak{n}$.

q.e.d.

Lemma 2.1.10

Let $\mathfrak{g}$ be a split solvable Lie algebra over R of dimension n. Then we can find a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ satisfying the following: For each i,

(1) $\{X_i, X_{i+1}, \ldots, X_n\}$ spans an ideal of $\mathfrak{g}$.

(2) There exists an integer ℓ such that $\{X_{\ell+1} \ldots X_n\}$ is a basis for $[\mathfrak{g}, \mathfrak{g}]$.

<proof>

Let $\mathfrak{g} = \mathfrak{g}_0 \supsetneqq \mathfrak{g}_1 \supsetneqq \cdots \supsetneqq \mathfrak{g}_n = 0$ be a chain of ideals of $\mathfrak{g}$ such that $\dim(\mathfrak{g}_i/\mathfrak{g}_{i+1}) = 1$.

Obviously $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_0 \supseteqq [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_2 \supseteqq \cdots \supseteqq [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_n = 0$ is a chain of ideals of $[\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}$ such that one is of codimension at most one (possibly zero) in the preceeding one. Picking up a subsequence of $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_i$ $i = 0, \ldots, n$, and rename them $\mathfrak{s}_0, \mathfrak{s}_1, \ldots, \mathfrak{s}_{n-\ell}$. We thus get a chain of ideals of $\mathfrak{g}$ contained in $[\mathfrak{g}, \mathfrak{g}]$:

$$[\mathfrak{g}, \mathfrak{g}] = \mathfrak{s}_0 \supsetneqq \mathfrak{s}_1 \supsetneqq \cdots \supsetneqq \mathfrak{s}_{n-\ell} = \{0\}$$

$$\dim (\mathfrak{s}_i / \mathfrak{s}_{i+1}) = 1.$$

Choose a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ as follows. Take $X_{\ell+1} \in \mathfrak{s}_0 \backslash \mathfrak{s}_1$, $X_{\ell+2} \in \mathfrak{s}_1 \backslash \mathfrak{s}_2$, $\ldots$, $X_n \in \mathfrak{s}_{n-\ell-1} \backslash \mathfrak{s}_{n-\ell}$ and take any ℓ linearly independent vectors $X_1, \ldots, X_\ell$ from $\mathfrak{g} \backslash [\mathfrak{g}, \mathfrak{g}]$. Then $\{X_1, \ldots, X_n\}$ satisfies both (1) and (2).

q.e.d.

Lemma 2.1.11

Let $\mathfrak{g}$ be a split solvable Lie algebra over R of dimension n and let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ satisfying (1) and (2) of Lemma 2.1.10. Furthermore, assume that $\mathfrak{g}$ is not abelian.

Then for any two elements P and Q in $U(\mathfrak{g})_C$ which are equivalent with respect to $X_1, \ldots, X_n$, $\tilde{P}$ and $\tilde{Q}$ are equivalent in $U(\mathfrak{g}/\mathfrak{n})_C$ with respect to the basis $\tilde{X}_1, \ldots, \tilde{X}_{n-1}$ of $\mathfrak{g}/\mathfrak{n}$.

Here $\mathfrak{n} = RX_n$ and $\sim : U(\mathfrak{g})_C \to U(\mathfrak{g}/\mathfrak{n})_C$ is defined in Lemma 2.1.9.

<proof>

Let $\dim [\mathfrak{g}, \mathfrak{g}] = n - \ell$ so that $X_{\ell+1}, \ldots, X_n$ is a basis of $[\mathfrak{g}, \mathfrak{g}]$. If P and Q are equivalent with respect to $X_1, \ldots, X_n$ we have

$$P = \sum_{\beta} A_\beta^P X_{\ell+1}^{\beta_{\ell+1}} \cdots X_n^{\beta_n}$$

$$Q = \sum_{\beta} A_\beta^Q X_{\ell+1}^{\beta_{\ell+1}} \cdots X_n^{\beta_n}$$

with A_β^P and A_β^Q having the same highest degree part for each β. Since $\sim$ is a homomorphism, we have the following canonical expressions for $\tilde{P}$, $\tilde{Q}$ with respect to $\tilde{X}_1, \ldots, \tilde{X}_{n-1}$.

$$\tilde{P} = \sum_{\beta \in I} \tilde{A}_\beta^P X_{\ell+1}^{\beta_{\ell+1}} \cdots \tilde{X}_{n-1}^{\beta_{n-1}}$$

$$\tilde{Q} = \sum_{\beta \in I} \tilde{A}_\beta^Q \tilde{X}_{\ell+1}^{\beta_{\ell+1}} \cdots \tilde{X}_{n-1}^{\beta_{n-1}}$$

where $I = \{\beta \mid \beta = (\beta_1, \ldots, \beta_{n-1}, 0)\}$ and $\tilde{A}_\beta^P$, $\tilde{A}_\beta^Q$ are given by replacing X_i by $\widetilde{X_i}$ $(i = 1, \ldots, \ell)$ in the expressions of A_β^P, A_β^Q respectively. Then it is clear that $\tilde{A}_\beta^P$ and $\tilde{A}_\beta^Q$ have the same highest degree part for $\beta \in I$. Thus we see that $\tilde{P}$ and $\tilde{Q}$ are equivalent with respect to $\tilde{X}_1, \ldots, \tilde{X}_{n-1}$ by noting that

$\{\tilde{X}_{\ell+1}, \ldots, \tilde{X}_{n-1}\}$ spans $[\mathfrak{g}/\mathfrak{n}, \mathfrak{g}/\mathfrak{m}]$.

q.e.d.

Lemma 2.1.12

Let $\mathfrak{g}$ be a split solvable Lie algebra over R of dimension n and let $X_1, \ldots, X_n$ be a basis which satisfies (1) (2) of Lemma 2.1.10 and let $\phi\colon U(\mathfrak{g})_C \to U(\mathfrak{g})_C$ be given by Lemma 2.1.8 i.e.

$$X_n\phi(P) = PX_n \qquad P \in U(\mathfrak{g})_C.$$

Then for any $P \in U(\mathfrak{g})_C$, P and $\phi(P)$ are equivalent with respect to the basis $X_1, \ldots, X_n$.

<proof>

First we remark that $X_{\ell+1}, \ldots, X_n$ commute with X_n. ($\{X_{\ell+1}, \ldots, X_n\}$ is a basis of $[\mathfrak{g},\mathfrak{g}]$.) In fact we can define a linear functional $\phi'\colon \mathfrak{g} \to R$ by $[Z,X_n] = \phi'(Z)X_n$. By Jacobi Identity it is easily seen that $\phi'([\mathfrak{g},\mathfrak{g}]) = 0$.

So $[X_i, X_n] = \phi'(X_i)X_n = 0$ for $n \geqq i \geqq \ell + 1$. By linearity, without loss of generality, we may assume that $P = X_1^{\alpha_1} \ldots X_n^{\alpha_n}$. Now the remark above implies that

$$(1) \quad PX_n = X_1^{\alpha_1} \ldots X_\ell^{\alpha_\ell}\cdot(X_{\ell+1}^{\alpha_{\ell+1}} \ldots X_n^{\alpha_n})\cdot X_n$$

$$= X_1^{\alpha_1} \ldots X_\ell^{\alpha_\ell} X_n(X_{\ell+1}^{\alpha_{\ell+1}} \ldots X_n^{\alpha_n})$$

For each $i = 1, \ldots, \ell$, we shall show by induction on $m \in Z^+$ that

$$(2) \quad X_i^m X_n = X_n(X_i^m + a_1 X_i^{m-1} + a_2 X_i^{m-2} + \ldots + a_m)$$

for some $a_j \in R$ which depend on i, m.

If $m = 1$, $[X_i, X_n] = \phi'(X_i)X_n$ and (2) is true. Suppose $m > 1$ and that (2) is true for all power of order lower than m. Then

$$X_i^m X_n = X_i(X_i^{m-1} X_n)$$

$$= X_i X_n (X_i^{m-1} + a_1 X_i^{m-2} + \ldots + a_{m-1})$$

(by induction hypothesis)

$$= X_n(X_i + \phi'(X_i))(X_i^{m-1} + a_1 X_i^{m-2} + \ldots + a_{m-1})$$

$$= X_n(X_i^m + b_1 X_i^{m-1} + \ldots + b_m)$$

where $a_j, b_j \in R$.

So (2) is established. Now applying (2) for all $i = 1, \ldots, \ell$,

(3) $X_1^{\alpha_1} \cdots X_\ell^{\alpha_\ell} X_n = X_n(X_1^{\alpha_1} \cdots X_\ell^{\alpha_\ell}$ + terms of degree less than $\alpha_1 + \cdots + \alpha_\ell$ in $X_1, \ldots, X_\ell$).

Combining (1) and (3) the equivalence of P and $\phi(P)$ follows.

q.e.d.

§2. Semi-global solvability of semi-bi-invariant differential operators.

In this section, using the L^2-estimate for bi-invariant operators by Rouvière [15], we prove the semi-global solvability of semi-bi-invariant differential operators on simply connected solvable Lie group.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ of dimension n. By $d_r g$, $d_\ell g$ we denote fixed right-invariant, left-invariant measures on G respectively so that by the modular function Δ on G they are related by $d_\ell g = \Delta(g) d_r g$. Let $(\, , \,)_U$ denote the scalar product of $L^2(U, d_r g)$ where U is an open set of G. The corresponding L^2-norm is denoted by $\| \quad \|_U$. We have an injection from $C_0^\infty(U)$ into $\mathcal{D}'(U)$ given by $f \longrightarrow f d_r g$. The adjoint of a differential operator P with respect to $(\, , \,)_U$ is denoted by P^*. The pairing of $\mathcal{D}'(U)$ and $C_0^\infty(U)$ is denoted by $\langle \, , \, \rangle$ and the transpose of a differential operator P with respect to this pairing is denoted by ${}^tP$. In short,

$u, v \in C_0^\infty(U)$, $T \in \mathcal{D}'(U)$

$(P^*u,v)_U = (u,Pv)_U$

$\langle {}^tPT,u\rangle = \langle T,Pu\rangle$

We remark that ${}^tP = \overline{P}^*$. For $X \in \mathfrak{g}_C$, we have $X^* = -\overline{X}$, ${}^tX = -X$. The map $P \longrightarrow {}^tP$ gives an anti-automorphism of $U(\mathfrak{g})_C$. The map $P \longrightarrow {}^*P$ gives an anti-complex anti-automorphism of $U(\mathfrak{g})_C$. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$. We define the m-th Sobolev space $H^m(U)$ on U $(m \in Z^+ \cup \{0\})$ by $H^m(U) = \{u \in \mathcal{D}'(U) | X^\alpha u \in L^2(U,d_rg)$ for $|\alpha| \leqq m\}$ and the norm on it by

$$||u||_{m,U} = \Big(\sum_{|\alpha| \leq m} ||X^\alpha u||_U^2\Big)^{1/2}$$

where $X^\alpha = X_1^{\alpha_1} \ldots X_n^{\alpha_n}$. $||\ ,\ ||_{m,U}$ of course depends on the choice of basis but any two choices of basis give the equivalent norms. Hence $H^m(U)$ is well-defined. By $H_0^m(U)$ we denote the closure of $C_0^\infty(U)$ in $H^m(U)$. The following lemma shows that * and ^{t} map semi-bi-invariant operators to semi-bi-invariant ones.

<u>Lemma 2.2.1</u>

Let $\lambda \in \mathfrak{g}_C^*$, $P \in U(\mathfrak{g})_C^\lambda$

then $P^* \in U(\mathfrak{g})_C^{\overline{\lambda}}$, ${}^tP \in U(\mathfrak{g})_C^\lambda$.

<proof>

Let $P \in U(\mathfrak{g})^{\lambda}_{C}$. Then $[X,P] = \lambda(X)P$ for $X \in \mathfrak{g}_{C}$. Applying * to both sides, $(XP - PX)^* = \overline{\lambda(X)}P^*$. Since $*$ is an anti-complex anti-automorphism of $U(\mathfrak{g})_C$, the left-hand side becomes

$$P^*X^* - X^*P^* = P^*(-\overline{X}) + \overline{X}P^* = [\overline{X}, P^*].$$

This implies that for any $Y \in \mathfrak{g}^*_C$, $[Y,P^*] = \overline{\lambda(\overline{Y})}P^*$. Hence we have $P^* \in U(\mathfrak{g})^{\overline{\lambda}}_{C}$. On the other hand $[Y,{}^tP] = [Y,\overline{P}^*] = \overline{[\overline{Y},P^*]} = \overline{\overline{\lambda(Y)P}^*} = \lambda(Y)\,{}^tP$. So ${}^tP \in U(\mathfrak{g})^{\lambda}_{C}$.

q.e.d.

Now we state the fundamental L^2-inequality of bi-invariant operators due to Rouvière.

Proposition 2.2.2 (Rouvière [15] Proposition 3)

Let G be a simply connected solvable Lie group. Let P be a non-zero bi-invariant differential operator on G. Then for each relative compact open set U of G, we have a constant $c_U > 0$ such that

$$||Pu||_U \geqq c_U||u||_U \quad \text{for} \quad u \in C_0^{\infty}(U).$$

We will extend the above inequality to all semi-bi-invariant operators.

Proposition 2.2.3

Let G be a simply connected solvable Lie group and P a non-zero semi-bi-invariant differential operator on G. Then, for each relative compact open set U of G, we have a constant $c_U' > 0$ such that

$$||Pu||_U \geqq c_U' ||u||_U \quad \text{for} \quad u \in C_0^\infty(U).$$

<proof>

We shall use induction on dim G. Let $\mathfrak{g}$ be the Lie algebra of G. If dim $G = 1$, the statement is clear. Suppose dim $G > 1$ and assume that the statement holds for all simply connected solvable Lie group of dimension less than dim G. Let P be a non-zero semi-bi-invariant differential operator on G. By Proposition 2.2.2, we may assume that P is not bi-invariant. Then Lemma 2.1.4(1) implies that there exists an ideal $\mathfrak{h}$ of $\mathfrak{g}$ of codimension one such that $P \in U(\mathfrak{h})_C$. Let H denote the analytic subgroup of G with the Lie algebra $\mathfrak{h}$. H is simply connected solvable as remarked in §1. So for any relative compact open set V of H, there is a constant $c_V' > 0$ such that

(1) $\quad ||Pv||_V \geqq c_V' ||v||_V$ for $v \in C_0^\infty(V)$

(since P is semi-bi-invariant also on H, this inequality is the consequence of our induction hypothesis applied to H).

Now take any relative compact open set U of G. Take V to be a relative compact open set in H satisfying $U^{-1}\cdot U \cap H \subset V$. Then for $u \in C_0^\infty(U)$, $g \in U$, if we put $u_g(x) = u(gx)$, we have $u_g \in C_0^\infty(V)$. The inequality (1) above implies that

$$\int_H |Pu_g(x)|^2 d_r x \geq c_V'^2 \int_H |u_g(x)|^2 d_r x \quad \text{for } g \in U$$

where $d_r x$ is a right-invariant measure on H. Let $d_\ell x$, $\Delta_H(x)$ denote the left invariant measure of H, the modular function of H respectively so that $d_\ell x = \Delta_H(x) d_r x$. By the left invariance of P we have $Pu_g(x) = (Pu)_g(x)$ for $g \in U$.

Therefore we have

$$(2) \quad \int_H |(Pu)(gx)|^2 d_r x \geq c_V'^2 \int_H |u(gx)|^2 d_r x \quad \text{for } g \in U.$$

Since H is normal in G, we have a G-invariant measure dg_H on G/H such that

$$\int_G f(g) d_\ell g = \int_{G/H} dg_H \int_H f(gx) d_\ell x \quad \text{for } f \in C_0^\infty(G).$$

(See Helgason [8] Chap. X Theorem 1.7)

Now (2) implies that

$$\int_H |(Pu)(gx)|^2 \Delta_H^{-1}(x)\, d_\ell x \geq c'_V \int_H |u(gx)|^2 \Delta_H^{-1}(x)\, d_\ell x$$

Since V is relatively compact, e have constants $\alpha > 0$, $\beta > 0$ such that

$$\alpha \geqq |\Delta_H^{-1}(x)| \geqq \beta \quad \text{for} \quad x \in V.$$

Hence we get

$$\alpha \int_H |Pu(gx)|^2 d_\ell x \geqq \beta c'_V \int_H |u(gx)|^2 d_\ell x \quad \text{for all} \quad g \in U.$$

Integrating over G/H, we get

$$\alpha \int_{G/H} dg_H \int_H |Pu(gx)|^2 d_\ell x \geqq \beta c'_V \int_{G/H} dg_H \int_H |u(gx)|^2 d_\ell x$$

i.e.

$$\int_G |Pu|^2 d_\ell g \geqq C \int_G |u|^2 d_\ell g \quad \text{for} \quad u \in C_0^\infty(U)$$

where C is some positive constant depending only on U.

Again, using the fact that for the modular function Δ on G we have constants $a > 0$, $b > 0$ such that

$$a \geqq |\Delta(g)| \geqq b \quad \text{for} \quad g \in U,$$

we get

$$a\int_G |Pu|^2 d_r g \geqq bC\int_G |u|^2 d_r g$$

for $u \in C_0^\infty(U)$. So there exists a constant $c_U' > 0$ such that

$$||Pu||_U \geqq c_U' ||u||_U$$

q.e.d.

In order to conclude the semi-global existence of a fundamental solution, we need a lemma.

Lemma 2.2.4 (Rouvière [15], Lemma 3)

Let G be a simply connected solvable Lie group. Let U be a relative compact open set containing the origin of G. Then there exists $\ell \in Z^+$ such that the map $u \to u(e)$ from $C_0^\infty(U)$ to C is continuous where $C_0^\infty(U)$ is given the relative topology of $H^\ell(U)$.

Proposition 2.2.5

Let G be a simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. Let P be a non-zero semi-bi-invariant differential operator on G. Then for each relative compact open set U containing the origin of G we have a fundamental solution E for P on U i.e. $E \in \mathcal{D}'(U)$, $Pu = \delta$ on U where δ is the delta function at the identity.

<proof>

Let P, U be as given in the statement of the proposition. Then by Lemma 2.2.1 P^* is semi-bi-invariant. So we have a constant $C > 0$, such that

(1) $$||P^*u||_U \geqq C||u||_U \quad \text{for} \quad u \in C_0^\infty(U)$$

Take $\lambda \in \mathfrak{g}_C^*$ so that

$$[X,P^*] = \lambda(X)P^* \quad \text{for} \quad X \in \mathfrak{g}.$$

Then $P^*X = XP^* - \lambda(X)P^*$

By induction we can show that for each $m \in Z^+$,

$P^*X^m = (X^m + \text{polynomial in } X \text{ of degree less than } m)\cdot P^*$.

In fact if the above is true for m, then

$$\begin{aligned} P^*X^{m+1} &= (P^*X^m)X \\ &= (X^m + \text{polynomial in X of degree less than m})P^*X \\ &= (X^m + \text{polynomial in X of degree less than m})(X-\lambda(X))P^* \end{aligned}$$

Hence the same is true for $m+1$.

Take a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$. Then for each $\alpha = (\alpha_1, \ldots, \alpha_n)$, we have by the remark above,

$$\begin{aligned} P^*X^\alpha &= P^*X_1^{\alpha_1} \ldots X_n^{\alpha_n} \\ &= (X_1^{\alpha_1} \ldots X_n^{\alpha_n} + \text{polynomial in } X_1,\ldots,X_n \text{ of degree} \\ &\qquad \text{less than } |\alpha|)P^*. \end{aligned}$$

This shows that for each $\alpha = (\alpha_1, \ldots, \alpha_n)$, there exists a constant $c_\alpha > 0$ such that

(2) $\|P^*u\|_{|\alpha|,U} \geqq c_\alpha \|P^*X^\alpha u\|_U$ for $u \in C_0^\infty(U)$.

On the other hand by Proposition 2.2.3 we have

(3) $\|P^*X^\alpha u\|_U \geqq c'_U \|X^\alpha u\|_U$ for $u \in C_0^\infty(U)$

and for all α where c'_U is a positive constant.

Recalling Lemma 2.2.4 we see that (2) and (3) imply that the map $P^*u \to u(e)$ is continuous from $P^*(C_0^\infty(U))$ to C where $P^*C_0^\infty(U)$ is given the relative topology of $H^\ell(U)$ for some $\ell \in Z^+$. Therefore by Hahn-Banach theorem, there exists a distribution $E \in H^{-\ell}(U)$ = the dual of $H_0^\ell(U)$ such that
$\langle E, P^*u\rangle = u(e)$ for $u \in C_0^\infty(U)$

i.e. $PE = \delta$ on U

q.e.d.

Theorem 2.2.6

Let G be a simply connected solvable Lie group. Then every non-zero semi-bi-invariant differential operator is semi-globally solvable.

<proof>

Let P be a non-zero semi-bi-invariant differential operator on G, U a relative compact open set. We may assume that U contains the origin of G. Take a relative compact open set V of G so that $V \supset U^{-1}\cdot U$. Let E be a fundamental solution (Proposition 2.2.5) of P on V. For $f \in C_0^\infty(U)$, put

$u(g) = \langle E_x\ f(gx^{-1})\rangle$ where E_x denotes the distribution in the variable x. Then $u(g) \in C_0^\infty(U)$ and

$(Pu)(g)$

$= P_g\langle E(xg), f(x^{-1})\rangle$ (P_g is P acting on g-variable)

$= \langle (PE)(xg), f(x^{-1})\rangle$ (the left-invariance of P)

$= \langle \delta(xg), f(x^{-1})\rangle$

$= f(g)$ for $g \in U$.

q.e.d.

§3. P-convexity and global solvability.

In this section we obtain the main results on P-convexity and global solvability. First of all we show the following proposition which is a generalization of a proposition in Wigner [19].

Proposition 2.3.1

Let G be a simply connected solvable Lie group with Lie algebra $\mathfrak{g}$ of dimension n. Suppose there exists a non-zero element $X_n \in \mathfrak{g}$ which spans an ideal of $\mathfrak{g}$. Let $\mathfrak{n} = RX_n$, $N = \{\exp tX_n \mid t \in R\}$ and $\pi: G \to G/N$ be the projection. Take any $P \in U(\mathfrak{g})_C$ and put $P_1 = P$, $P_2 = \phi(P_1)$, ..., $P_{i+1} = \phi(P_i)$, ... where $\phi: U(\mathfrak{g})_C \to U(\mathfrak{g})_C$ is as in Lemma 2.1.8. (i.e. $X_n\phi(D) = DX_n$). Then for any compact set K of

G/N which is $\tilde{P}_i$-full for all $i = 1, 2, \ldots,$
$\pi^{-1}(K)$ is P-full where $\sim: U(\mathfrak{g})_C \to U(\mathfrak{g}/\mathfrak{n})_C$ is
as in Lemma 2.1.9.

Remark

1) In case X_n is central in $\mathfrak{g}$, all the P_i are the same. Hence the proposition reads "If $K \subset G/N$ is a compact $\tilde{P}$-full set, then $\pi^{-1}(K)$ is P-full".

2) One does not have to worry about the case $\tilde{P} = 0$ because then no set in G/N would be $\tilde{P}$-full.

3) To show the D-fullness of a closed set $A \subset G$ for $D \in U(\mathfrak{g})_C$, one only has to show
$u \in C_0^\infty(G)$ supp $Du \subset A \Rightarrow$ supp $u \subset A$ instead of working in distributions. In fact let $\rho \in C_0^\infty(G)$ and $u \in \mathcal{E}'(G)$ then

$$D_g\langle u(x), \rho(gx^{-1})\rangle = \langle (Du)(x), \rho(gx^{-1})\rangle.$$

Hence we can approximate u by a smooth function with compact support $\langle u(x), \rho(gx^{-1})\rangle$ and Du by a smooth function with compact support $\langle Du(x), \rho(gx^{-1})\rangle$, taking ρ to be a mollifier.

<proof>

We define for $u \in C_0^\infty(G)$,

$$\tilde{u}(xN) = \int_N u(xn)\,dn \in C_0^\infty(G/N).$$

Also at the same time we write

$$\tilde{u}(x) = \int_N u(xn)\,dn$$

but there will be no fear of confusion.

We claim that for $u \in C_0^\infty(G)$, $Q \in U(\mathfrak{g})_C$

(1) $$\widetilde{Qu} = \widetilde{\phi(Q)}\,\tilde{u}$$

To prove the claim (1) above, we use induction on deg Q. If deg $Q = 1$, we can write $Q = X + c$, $X \in \mathfrak{g}_C$, $c \in C$. For any $u \in C_0^\infty(G)$, we have

$$\begin{aligned}\widetilde{(X+c)u} &= \int_N (X+c)u(xn)\,dn \\ &= \int_N (Xu)(xn)\,dn + c\int_N u(xn)\,dn \\ &= \int_N \frac{d}{dt}u(xn \exp tX)\Big|_{t=0} dn + c\tilde{u}(xN) \\ &= \frac{d}{dt}\int_N u(x(\exp tX)(\exp -tX)n(\exp tX))\,dn\Big|_{t=0} + c\tilde{u}(xN)\end{aligned}$$

Write $n = \exp sX_n$. Then $(\exp -tX)\cdot n\cdot(\exp tX)$ $= \exp se^{-\alpha t}X_n$ where $[X, X_n] = \alpha X_n$. By the change of variables $n' = (\exp -tX)\cdot n\cdot(\exp tX)$ we have $dn = e^{\alpha t}dn'$. Hence the above expression becomes

$$\frac{d}{dt}e^{\alpha t}\int_N u(x(\exp\ tX)n')dn'\Big|_{t=0} + c\tilde{u}(xN)$$

$$= \alpha\int_N u(xn')dn' + \frac{d}{dt}\int_N u(x(\exp\ tX)\cdot n')dn'\Big|_{t=0} + c\tilde{u}(xN)$$

$$= \alpha\tilde{u}(xN) + \tilde{X}\tilde{u}(xN) + c\tilde{u}(xN).$$

So we have shown that

$$(2) \qquad \widetilde{(X + c)u} = (\tilde{X} + \alpha + c)\tilde{u}$$

Since $[X,X_n] = \alpha X_n$, we have

$$X_n(X + \alpha + c) = (X + c)X_n.$$

Therefore $\phi(X + c) = X + \alpha + C$.

By (2), we conclude

$$\widetilde{(X + c)u} = \widetilde{\phi(X + c)}\tilde{u}$$

Assume that $\Omega \in U(\mathfrak{g})_C$, deg $\Omega > 1$, and that (1) holds for all operators with degree less than deg Ω. In order to show (1) for Ω, by linearity, we may without loss of generality assume that $\Omega = \Omega_1\Omega_2$ for some $\Omega_1, \Omega_2 \in U(\mathfrak{g})_C$ with deg $\Omega_1 <$ deg Ω, deg $\Omega_2 <$ deg Ω. For $u \in C_0^\infty(G)$,

$$\widetilde{Qu} = \widetilde{Q_1 Q_2 u} = \widetilde{\phi(Q_1)}\widetilde{Q_2 u}$$

(induction hypothesis applied to Q_1)

$$= \widetilde{\phi(Q_1)}\widetilde{\phi(Q_2)}\tilde{u}$$

(induction hypothesis applied to Q_2)

$$= \widetilde{\phi(Q_1 Q_2)}\tilde{u}$$

(ϕ, ~ are homomorphisms).

Therefore (1) is completely proved. Now suppose $P \in U(\mathfrak{g})_C$ is given and a compact set K of G/N is $\widetilde{P}_i$-full for all $i = 1, 2 \ldots$. We intend to show the P-fullness of $\pi^{-1}(K)$ i.e.

(3) $u \in C_0^\infty(G)$, supp $Pu \subset \pi^{-1}(K)$

$\Rightarrow$ supp $u \subset \pi^{-1}(K)$

Now assume that $u \in C_0^\infty(G)$ and supp $Pu \subset \pi^{-1}(K)$.
Then

$$\int_N Pu(xn)\,dn = 0 \quad \text{for} \quad x \notin \pi^{-1}(K)$$

since $P = P_1$, for $x \notin \pi^{-1}(K)$ we have

$$\begin{aligned} 0 &= \int_N P_1 u(xn)\,dn \\ &= \widetilde{P_1 u}(xN) \\ &= \widetilde{\phi(P_1)}\tilde{u}(xN) \quad \text{by (1).} \end{aligned}$$

Since $P_2 = \phi(P_1)$ and K is $\widetilde{P}_2$-full, we have

(4) $\tilde{u}(xN) \equiv 0$ for $x \notin \pi^{-1}(K)$.

Choose $X_1, \ldots, X_{n-1} \in \mathfrak{g}$ so that $X_1, \ldots, X_n$ form a basis of $\mathfrak{g}$ and the map

$(\exp t_1X_1) \ldots (\exp t_nX_n) \to (t_1, \ldots, t_n)$ gives a diffeomorphism of G onto R^n (By Lemma 2.1.1 such a basis exists). We shall frequently identify G with R^n by this diffeomorphism. Choose a function $\phi' \in C_0^\infty(R)$ with $\int_R \phi'(x)\,dx = 1$ and put $b(\exp t_1X_1 \cdot \ldots \cdot \exp t_nX_n) = \phi'(t_n)$. Then $b \in C^\infty(G)$.

Define u_1 by

(5) $u_1(x) \equiv u(x) - \tilde{u}(x)b(x)$

Then $u_1 \in C_0^\infty(G)$ because $\operatorname{supp} b$ is bounded in t_n-direction and $\tilde{u}$ is 0 for large $t_1, \ldots, t_{n-1}$. (One remarks that in our identification of G with R^n, $\pi^{-1}(K) = \{t_1, \ldots, t_n) \in R^n \mid (t_1, \ldots, t_{n-1}) \in B \subset R^{n-1}\}$ for some compact set B of R^{n-1}.)

Also we have

(6) $\int_N u_1(xn)\,dn = \int_N u(xn)\,dn - \int_N \tilde{u}(xn)b(xn)\,dn$

$= \tilde{u}(x) - \tilde{u}(x)\int_N b(xn)\,dn$

$= \tilde{u}(x) - \tilde{u}(x)\int_R \phi'(x)\,dx = 0$ for all $x \in G$.

Hence we can find $u_1^* \in C_0^\infty(G)$ such that

(7) $X_n u_1^* \equiv u_1$ on G.

In fact, if we recall that X_n is identified with $\frac{d}{dt_n}$, then (7) is an immediate consequence of the following fact in Calculus:

On $(R^n, (x_1, \ldots, x_n))$, if $F \in C_0^\infty(R^n)$ satisfies $\int_{-\infty}^{\infty} F(x_1, \ldots, x_{n-1}, x_n)dx_n = 0$ for all $x_1, \ldots, x_{n-1}$, then there exists $F' \in C_0^\infty(R^n)$ such that $\frac{d}{dx_n}F' \equiv F$ on R^n.

Now (4), (5) imply that

(8) $u_1(x) = u(x)$ for $x \notin \pi^{-1}(K)$

So (7) gives

(9) $X_n u_1^*(x) = u(x)$ for $x \notin \pi^{-1}(K)$.

For $x \notin \pi^{-1}(K)$, by our assumption (3),

$0 = Pu(x)$

$= PX_n u_1^*(x)$ (by (9))

$= X_n P_2 u_1^*(x)$ (the definition of P_2).

The complement of $\pi^{-1}(K)$ is of the form

$\{(t_1, \ldots, t_n) \in R \mid (t_1, \ldots, t_{n-1}) \notin B \subset R^{n-1}\}$
for some B in R^{n-1}. Therefore the injectivity of $\frac{d}{dx}$ on $C_0^\infty(R^1)$ (of course this injectivity could be deduced as a corollary of Rouvière's estimate in Proposition 2.2.2) implies that

(10) $$P_2 u_1^*(x) = 0 \quad \text{for} \quad x \notin \pi^{-1}(K)$$

Now integrating (10) along N,

$$\widetilde{P_2 u_1^*}(xN) = \int_N (P_2 u_1^*)(xn)\,dn = 0 \quad \text{for} \quad x \notin \pi^{-1}(K)$$

Since $\widetilde{(P_2 u_1^*)}(xN) = \widetilde{\phi(P_2)}\widetilde{u_1^*}(xN)$ by (1), we have

$$\widetilde{\phi(P_2)}\widetilde{u_1^*}(xN) = 0 \quad \text{for} \quad x \notin \pi^{-1}(K).$$

Since $\phi(P_2) = P_3$ and K is $\widetilde{P_3}$-full,

(11) $$\tilde{u}_1^*(xN) = 0 \quad \text{for} \quad x \notin \pi^{-1}(K)$$

Suppose that for some $m \in Z^+$, we have defined $u_m, u_m^* \in C_0^\infty(G)$ with the following properties:

(12) $$P_{m+1} u_m^*(x) = 0 \quad \text{for} \quad x \notin \pi^{-1}(K)$$

(13) $$\tilde{u}_m^*(x) = 0 \quad \text{for} \quad x \notin \pi^{-1}(K)$$

(14) $$X_n^m u_m^*(x) = u(x) \quad \text{for} \quad x \notin \pi^{-1}(K)$$

Note that for $m = 1$, we have already done this. Namely (10), (11), (9) correspond to (12), (13), (14) respectively. Now we define u_{m+1} by

$$(15) \qquad u_{m+1}(x) = u_m^*(x) - \tilde{u}_m^*(x)b(x)$$

where b is defined right before (5). As in (5), (6), we see that

$$u_{m+1} \in C_0^\infty(G)$$

$$\int_N u_{m+1}(x_n)dx = 0 \quad \text{for all} \quad x \in G$$

Therefore there exists $u_{m+1}^* \in C_0^\infty(G)$ such that

$$(16) \qquad X_n u_{m+1}^* \equiv u_{m+1} \quad \text{on} \quad G.$$

By (13), (15), we see that

$$u_{m+1}(x) = u_m^*(x) \quad \text{for} \quad x \notin \pi^{-1}(K)$$

Together with (16), we get

$$(17) \qquad X_n u_{m+1}^*(x) = u_m^*(x) \quad \text{for} \quad x \notin \pi^{-1}(K)$$

So by (14), we have

$$(18) \qquad X_n^{m+1} u_{m+1}^* = u(x) \quad \text{for} \quad x \notin \pi^{-1}(K)$$

On the otherhand, for $x \notin \pi^{-1}(K)$

$$X_n P_{m+2} u^*_{m+1}(x) = P_{m+1} X_n u^*_{m+1}(x) \quad \text{(the definition of the } P_i\text{)}$$

$$= P_{m+1} u^*_m(x) \quad \text{(by (17))}$$

$$= 0 \quad \text{(by (12))}$$

Therefore

(19) $$P_{m+2} u^*_{m+1}(x) = 0 \quad \text{for} \quad x \notin \pi^{-1}(K)$$

Integrating over N, for $x \notin \pi^{-1}(K)$,

$$0 = \int_N P_{m+2} u^*_{m+1}(x)$$

$$= \widetilde{\phi(P_{m+2})}\, \tilde{u}^*_{m+1}(xN) \quad \text{(from (1))}$$

$\phi(P_{m+2}) = P_{m+3}$ and K is $\widetilde{P_{m+3}}$-full.

So we have

(20) $$\tilde{u}^*_{m+1}(xN) = 0 \quad \text{for} \quad x \notin \pi^{-1}(K)$$

(19), (20), (18) are the same as (12), (13), (14) respectively except that m is replaced by $m + 1$. Hence by induction, we conclude that for each $\ell \in Z^+$, there exists $u^*_\ell \in C_0^\infty(G)$ such that

(21) $$X_n^{\ell+1} u^*_\ell(x) = u(x) \quad \text{for} \quad x \notin \pi^{-1}(K).$$

This implies $u(x) = 0$ for $x \notin \pi^{-1}(K)$. In fact, let $p \notin K$. Let u_p denote the restirction of u to $\pi^{-1}(p)$. Then that $X_n^{\ell+1}u_\ell^*(x) = u(x)$ for all ℓ, $x \notin \pi^{-1}(K)$, implies that u_p, regarded as a compactly supported smooth function on R^1 $(\pi^{-1}(p) \cong R^1)$ has the Fourier image which is an analytic function with zeros of infinite order at 0. So $u_p \equiv 0$. Thus $u(x) = 0$ for $x \notin \pi^{-1}(K)$. So $\operatorname{supp} u \subset \pi^{-1}(K)$ as desired.

q.e.d.

Proposition 2.3.2

Let G be a simply connected solvable Lie group with Lie algebra $\mathfrak{g}$ of dimension n. Suppose that we have an element $X_n \neq 0$ in $\mathfrak{g}$ which spans an ideal of $\mathfrak{g}$. Let $N = \{\exp tX_n \mid t \in R\}$ and let $\pi: G \to G/N$ be the projection. Then for any non-zero $P \in U(\mathfrak{g})_C$ and for any compact set $K \subset G/N$, there exists a compact set $B \subset G$ and a real valued function $f \in C^\infty(G)$ such that

(1) $|\sigma(P)(df)| > M$ on $\pi^{-1}(K) \setminus B$

for some positive constant M.

(2) $X_n f \equiv 1$ on G

<proof>

There are two cases depending on if X_n is central in $\mathfrak{g}$ or not.

<u>Case I</u> X_n is central in $\mathfrak{g}$.

In this case we prove the following stronger statement:

(3) For any non-zero $P \in U(\mathfrak{g})_C$ and any compact set K of G/N there exists a real valued function $f \in C^{\infty}(G)$ such that

(a) $|\sigma(P)(df)| > M$ on $\pi^{-1}(K)$ for some positive constant M

(b) $X_n f \equiv 1$ on G.

We fix P and K as given above. By Lemma 2.1.1 we can choose $X_1, \ldots, X_{n-1} \in \mathfrak{g}$ so that $X_1, \ldots, X_n$ form a basis of $\mathfrak{g}$ and for each i, $\{x_i, \ldots, x_n\}$ spans a subalgebra of $\mathfrak{g}$ and the span of $\{X_{i+1}, \ldots, X_n\}$ is an ideal of the span of $\{X_i, \ldots, X_n\}$. In particular we have a diffeomorphism of G onto R^n:

$$(\exp t_1X_1) \ldots (\exp t_nX_n) \longrightarrow (t_1, \ldots, t_n)$$

We frequently identify G with R^n by this diffeomorphism.

We are going to prove (3) by induction on deg P.

Assume deg $P = 0$. Then P is simply a non-zero complex number and (a) is obviously satisfied regardless of f we choose. If we put $f(t_1, \dots, t_n) = t_n$ then $X_n f \equiv 1$ because X_n is identified with $\frac{d}{dt_n}$. Hence (3) holds in this case.

Now, assume $\deg P > 0$ and suppose that (3) holds for all operators of degree less than $\deg P$.

Let P_m denote the highest degree part of P in the canonical expression in terms of the basis $X_1, \dots, X_n$.

Write the canonical expression of P_m:

(4) $$P_m = X_\ell^k Q_k + X_\ell^{k-1} Q_{k-1} + \dots + X_\ell Q_1 + Q_0$$

$Q_k \not\equiv 0, \quad k \geqq 1,$

where each Q_i $(0 \leqq i \leqq k)$ is of the form

$$\Sigma\, c_\alpha X_{\ell+1}^{\alpha_{\ell+1}} \dots X_n^{\alpha_n} .$$

Since $\deg Q_k < \deg P$, applying our induction hypothesis to Q_k, we have a real valued function $u \in C^\infty(G)$ such that

(5) $$|\sigma(Q_k)(du)| > M' \quad \text{on} \quad \pi^{-1}(K)$$

for some positive constant M'

(6) $$X_n u \equiv 1 \quad \text{on} \quad G.$$

From (6) and the assumption that X_n is central, we have for each j,

$$X_n X_j u = X_j X_n u = X_j(1) = 0 \quad \text{on} \quad G.$$

Hence $X_j u(t_1, \ldots, t_n)$ is independent of t_n. Since $\pi^{-1}(K)$ is of the form

$$\{(t_1, \ldots, t_{n-1}, t_n) \in R^n \mid (t_1, \ldots, t_{n-1}) \in B \subset R^{n-1}\}$$

for some compact set B of R^{n-1}, we conclude that $(X_j u)(t_1, \ldots, t_n)$ is bounded on $\pi^{-1}(K)$.

For each $N \in R$, we define $u_N \in C^\infty(G)$ by $u_N(t_1, \ldots, t_n) = Nt_\ell$.

We claim that

(7) $\qquad X_\ell u_N \equiv N \qquad\qquad \text{on} \quad G$

(8) $\qquad X_j u_N \equiv 0 \quad \text{for} \quad j > \ell \quad \text{on} \quad G$

In fact,

$$X_\ell u_N(t_1, \ldots, t_n)$$

$$= \frac{d}{ds} u_N((\exp t_1 X_1) \ldots (\exp t_\ell X_\ell) \ldots (\exp t_n X_n)(\exp sX_\ell))\big|_{s=0}$$

$$= \frac{d}{ds} u_N((\exp t_1 X_1) \ldots (\exp(t_\ell+s) X_\ell)(\exp \phi_{\ell+1}(s,t) X_{\ell+1}) \ldots$$

$$\ldots (\exp \phi_n(s,t) X_n))\big|_{s=0}$$

where

$\phi_{\ell+1}, \ldots, \phi_n$ are functions in $s, t_{\ell+1}, \ldots, t_n$.

Therefore we get

$$X_\ell u_N = \frac{d}{ds} N(t_\ell + s)\big|_{s=0}$$

$$= N$$

Thus (7) is proved.

In order to show (8), we observe that for $j > \ell$,

$$X_j u_N(t_1, \ldots, t_n)$$

$$= \frac{d}{ds} u_N(\exp t_1 X_1) \ldots (\exp t_\ell X_\ell) \ldots (\exp t_j X_j) \ldots (\exp t_n X_n)(\exp sX_j))\big|_{s=0}$$

$$= \frac{d}{ds} u_N((\exp t_1 X_1) \ldots (\exp t_\ell X_\ell) \ldots (\exp (t_j + s) X_j)(\exp \psi_{j+1}(s,t)) \ldots$$

$$\ldots (\exp \psi_n(s,t) X_n))\big|_{s=0}$$

where $\psi_{j+1}, \ldots, \psi_n$ are functions in $s, t_{j+1}, \ldots, t_n$.

Therefore $X_j U_N = \frac{d}{ds} N t_\ell = 0$ for $j > \ell$. Thus (8) is proved.

Since $X_\ell u$ is bounded on $\pi^{-1}(K)$, by (7), for any $L > 0$ we can choose N_L so that

(9) $$|X_\ell (u + u_{N_L})| > L \text{ on } \pi^{-1}(K) .$$

On the other hand (8) and the boundedness of $X_j u$ imply that there is a constant R_1 such that

(10) $$|X_j(u+u_N)| < R_1 \quad \text{on} \quad \pi^{-1}(K)$$

for all N and $j > \ell$.

By (4) and Lemma 2.1.6 we have

(11) $$|\sigma(P_m)(d(u+u_{N_L}))| = |X_\ell(u+u_{N_L}))|^k |\sigma(Q_k)(d(u+u_{N_L}))$$
$$+ \frac{\sigma(Q_{k-1})(d(u+u_{N_L}))}{X_\ell(u+u_{N_L})} + \cdots + \frac{\sigma(Q_0)(d(u+u_{N_L}))}{X_\ell(u+u_{N_L})^k}| \; .$$

But Lemma 2.1.6, (10) above, and the fact that each Q_i is expressed only by $X_{\ell+1}, \ldots, X_n$ imply that there is a constant R_2 such that

(12) $$|\sigma(Q_i)(d(u+u_N))| < R_2 \quad \text{on} \quad \pi^{-1}(K)$$

for each i, N.

Now (9), (11), (12) yield

$$|\sigma(P_m)(d(u+u_{N_L}))|$$
$$\geqq L^k(\sigma(Q_k)(d(u+u_{N_L})) - \frac{R_2}{L} - \frac{R_2}{L^2} - \cdots - \frac{R_2}{L^k}) \quad \text{on} \quad \pi^{-1}(K).$$

But (5) and (8) imply that

$$|\sigma(Q_k)(d(u+u_N))| > M' > 0 \quad \text{on} \quad \pi^{-1}(K) \quad \text{for all } N.$$

Hence if we take L very large, we have

$$|\sigma(P_m)(d(u+u_{N_L}))| > M \quad \text{on} \quad \pi^{-1}(K)$$

for some positive constant M.

If $\ell = n$, we have by (6) and (7)

$$\frac{1}{N+1}X_n(u+u_N) = 1$$

so we put $f = \frac{1}{N+1}(u_N+u)$.

If $\ell \neq n$, we have

$$X_n(u+u_N) = X_n u = 1$$

so we put $f = u_N+u$.

Since $\sigma(P_m) = \sigma(P)$ both (a) and (b) of (3) are now satisfied for P using the f defined above.

q.e.d. for Case I

Case II X_n is non-central in $\mathfrak{g}$.

Assume that a compact set $K \subset G/N$ and non-zero $P \in u(\mathfrak{g})_C$ are given as in the statement of the proposition. We have a non-zero linear functional $\phi \in \mathfrak{g}^*$, such that $[X,X_n] = \phi(X)X_n$ for $X \in \mathfrak{g}$. By Jacobi Identity, $\phi([\mathfrak{g},\mathfrak{g}]) = 0$. So ker ϕ is an ideal of $\mathfrak{g}$ of codimension one. By Lemma 2.1.1 we can choose $X_2, \ldots, X_n \in \ker\phi$ so that $X_2, \ldots, X_n$ form a basis of ker ϕ and for each $i \geqq 2$, the span of $\{X_i, \ldots, X_n\}$ is a subalgebra of ker ϕ and the span of $\{X_{i+1}, \ldots, X_n\}$ is an ideal of the span of $\{X_i, \ldots, X_n\}$. Take the element $X_1 \in \ker \phi$ such that $[X_1, X_n] = X_n$ (i.e. $\phi(X_1) = 1$). Then the ordered basis $X_1, \ldots, X_n$ satisfies the condition in Lemma 2.1.1. Hence we can identify G with R^n by the diffeomorphism

$$(\exp t_1X_1)(\exp t_2X_2) \ldots (\exp t_nX_n) \longrightarrow (t_1, \ldots, t_n)$$

from G onto R^n.

Let G' denote the analytic subgroup of G with Lie algebra ker ϕ. Then G' is simply connected and we can identify G' with $R^{n-1} \subset R^n$ by the map

$$(\exp t_2X_2) \ldots (\exp t_nX_n) \longrightarrow (0, t_2, \ldots, t_n).$$

Let P_m be the highest degree part of P in the canonical expression with respect to $X_1, \ldots, X_n$.

Let

$$(13) \qquad P_m = X_1^kQ_k + X_1^{k-1}Q_{k-1} + \ldots + Q_0$$

be the canonical expression of P_m where $k \geqq 0$ (possibly zero!), $Q_k \neq 0$ and all the Q_i are of the form

$$\Sigma\, c_\alpha X_2^{\alpha_2} \ldots X_n^{\alpha_n}.$$

Observe that X_n is central in ker ϕ. Put

$S = \{(0, t_2, \ldots, t_n) \in G' \mid$ there exists t_1 such that

$(t_1, \ldots, t_n) \in \pi^{-1}(K)\}$.

Then it is clear that there exists a compact set $K' \subset G'/N$ such that $\pi'^{-1}(K') \supset S$, where $\pi' : G' \to G'/N$ is the projection. Regarding Q_k as an operator on G', we can apply the result (3) of Case I to Q_k:

(14) There exists a real valued function $g \in C^\infty(G')$ such that

(a) $|\sigma(Q_k)(dg)| > M'$ on S for some positive constant M'

(b) $X_n g \equiv 1$ on G'.

Now we extend g to a function $\tilde{g} \in C^\infty(G)$ by putting

$$\tilde{g}(t_1, \ldots, t_n) = g(t_2, \ldots, t_n)$$

For $i \geqq 2$, we have

$$(X_i\tilde{g})(t_1, t_2, \ldots, t_n)$$

$$= \frac{d}{ds}\tilde{g}(\exp t_1X_1 \ldots \exp t_nX_n \exp sX_i)|_{s=0}$$

$$= \frac{d}{ds}g(\exp t_2X_2 \ldots \exp t_nX_n \exp sX_i)|_{s=0}$$

$$= X_i g(t_2, \ldots, t_n)$$

So (14) (a) (b) give

(15) (a) $|\sigma(Q_k)(d\tilde{g})| > M'$ on $\pi^{-1}(K)$

(b) $X_n\tilde{g} \equiv 1$ on G.

Since X_n is central in $\ker \phi$, we have

$$X_nX_i\tilde{g} = X_iX_n\tilde{g} = X_i\,1 = 0 \quad \text{for} \quad i \geqq 2.$$

This means that $X_i\tilde{g}$ is independent of t_n for $i \geqq 2$. Therefore $X_i\tilde{g}$ is bounded on $\pi^{-1}(K)$ for $i \geqq 2$. On the other hand

$$\begin{aligned} X_nX_1\tilde{g} &= X_1X_n\tilde{g} - X_n\tilde{g} \quad ([X_1,X_n] = X_n) \\ &= X_1 1 - X_n\tilde{g} \\ &= 0 - 1 \end{aligned}$$

This means that $X_1\tilde{g}$ is of the form

$$X_1\tilde{g}(t_1, \ldots, t_n) = g_1(t_1, \ldots, t_{n-1}) - t_n$$

This implies that for any $L > 0$, we can choose $\delta_L > 0$ so that

$$|X_1\tilde{g}(t_1, \ldots, t_n)| > L \quad \text{for all} \quad (t_1, \ldots, t_n) \in \pi^{-1}(K)$$

with $|t_n| > \delta_L$.

Now we come to the final stage of our proof.

First assume $k = 0$. Then $P_m = Q_k$ and (15) (a) (b) give the desired conclusion. Next assume $k \geqq 1$. Then by Lemma 2.1.6

$$|\sigma(P_m)(d\tilde{g}(t_1, \ldots, t_n))|$$

$$= |X_1\tilde{g}(t_1, \ldots, t_n)|^k|\sigma(Q_k)(d\tilde{g}(t_1, \ldots, t_n))$$

$$+ \frac{\sigma(Q_{k-1})(d\tilde{g}(t_1, \ldots, t_n))}{X_1\tilde{g}(t_1, \ldots, t_n)} + \ldots + \frac{\sigma(Q_0)(d\tilde{g}(t_1, \ldots, t_n))}{X_1\tilde{g}(t_1, \ldots, t_n)^k}\Bigg|.$$

If $|t_n| > \delta_L$, due to the boundedness of $\sigma(Q_i)(d\tilde{g})$ on $\pi^{-1}(K)$ which follows from the boundedness of $X_j\tilde{g}$ on $\pi^{-1}(K)$ for $j \geqq 2$ indicated above, we have a constant M'' independent of L such that

$|\sigma(Q_i)(d\tilde{g})| < M''$ on $\pi^{-1}(K)$ for each i.

Hence

$$|\sigma(P_m)(d\tilde{g}(t_1, \ldots, t_n))| \geqq L^k(M' - \frac{M''}{L} - \frac{M''}{L^2} - \ldots - \frac{M''}{L^k})$$

for $(t_1, \ldots, t_n) \in \pi^{-1}(K)$ with $|t_n| > \delta_L$. Here we used (15)(a). Now taking L very large, we have a positive constant M and $\delta_L > 0$ such that

$|\sigma(P_m)(d\tilde{g})| > M$ for $(t_1, \ldots, t_n) \in \pi^{-1}(K)$ and $|t_n| > \delta_L$.

Recalling (15) (b), we see that $\tilde{g}$ can be taken as f in (1) (2) of the statement in Proposition 2.3.2 with the compact set B being

$\{(t_1, \ldots, t_n) \in \pi^{-1}(K) \mid |t_n| \leqq \delta_L\}$ for a sufficient large L.

<u>q.e.d.</u>

We are now ready to prove the following theorem which asserts the P-convexity of all simply connected solvable Lie groups for all semi-bi-invariant operators P.

<u>Theorem 2.3.3</u>

Let G be a simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. Let $P \in U(\mathfrak{g})_C$ be a non-zero semi-bi-invariant differential operator on G. Then for any compact set K in G, we can find a P-full compact set K' in G such that $K \subset K'$.

In particular G is Q-convex for all non-zero semi-bi-invariant operators $Q \in U(\mathfrak{g})_C$.

<proof>

The proof goes by induction on dim G. If G is abelian, by Theorem 1.7 the convex hull of K plays the role of K'. In particular, the theorem is true if dim G = 1. Assume that dim G > 1. We will consider two cases. First, the case the center of $\mathfrak{g}$ is zero, second the case the center of $\mathfrak{g}$ is non-zero.

Assume that the center of $\mathfrak{g}$ is zero. Then by Lemma 2.1.4 (2), we have an ideal $\mathfrak{h}$ of codimension one in $\mathfrak{g}$ such that all semi-bi-invariant operators are in $U(\mathfrak{h})_C$. Let H be the analytic normal subgroup correspodning to $\mathfrak{h}$. Notice that H is a simply connected solvable group and we have a diffeomorphism from $R \times H$ onto G given by

(1) $(t, h) \longrightarrow (\exp tX)h$

Here X is an arbitrarily chosen non-zero vector such that $X \notin \mathfrak{h}$.

Since K is compact, we can find a constant $M > 0$ and a compact set K_1 in H such that

$$K \subset \{(\exp tX)\cdot K_1 \mid |t| \leqq M\}$$

For each fixed $t_0 \in R$, we have a diffeomorphism of $(\exp t_0X)\cdot H$ onto H given by

(2) $(\exp t_0X)\cdot h \longrightarrow h$

By the induction hypothesis applied to H, we have a compact P-full set K_2 in H such that $K_2 \supset K_1$ where P is regarded as an operator on H.

We now claim that the set

$$B_M = \{(\exp tX)\cdot K_2 \mid |t| \leqq M\}$$

is a P-full set in G. For each $t_0 \in R$ and $f \in C_0^\infty(G)$, let f_{t_0} denote the function on H given by first restricting f to the subset $(\exp t_0X)\cdot H$ of G, then pushing it forward by the diffeomorphims (2). Clearly $f \in C_0^\infty(H)$. Now assume that $u \in C_0^\infty(G)$ and supp $Pu \subset B_M$.

Then for each $t_0 \in R$ we have $\text{supp}\,(Pu)_{t_0} \subset K_2$. By the left-invariance of P, it is clear that $(Pu)_{t_0} = P(u_{t_0})$ where on the left hand side P is regarded as an operator on G and on the right hand side P is regarded as an operator on H. Therefore the P-fullness of K_2 gives $\text{supp}\, u_{t_0} \subset K_2$. Thus we have $\text{supp}\, u \subset \{(\exp tX)\cdot K_2 \mid t \in R\}$. On the other hand by our assumption,

$$(Pu)_{t_0} \equiv 0 \quad \text{for} \quad |t_0| \geqq M.$$

Hence $P(u_{t_0}) \equiv 0$ for $|t_0| \geqq M$. Since P is semi-bi-invariant on H, the injectivity of semi-bi-invariant operators of H on the space $C_0^\infty(H)$ implies $u_{t_0} \equiv 0$ for $|t_0| \geqq M$. (The above mentioned injectivity is an immediate consequence of the L^2-inequality of Proposition 2.2.3). Therefore we conclude that

$$u \in C_0^\infty(G),\ \text{supp}\, Pu \subset B_M \Rightarrow \text{supp}\, u \subset B_M$$

This implies that B_M is P-full (see Remark 3 of Proposition 2.3.1). Since B_M contains K and is compact, the first case (the case when cetner of $\mathfrak{g}$ is 0) is settled. Next, we assume that the center of $\mathfrak{g}$ is non-zero. Let $\dim \mathfrak{g} = n$. We have a non-zero central element X_n. Let $\mathfrak{n} = RX_n$, $N = \{\exp tX_n \mid t \in R\}$ and let

$\pi: G \to G/N$ be the projection. Note that G/N is a simply connected solvable Lie group. We have $\ell \in Z^+ \cup \{0\}$ such that $P = P_1 \cdot X_n^\ell$ and $P_1 \notin U(\mathfrak{g})_C \cdot \mathfrak{n}$. By Lemma 2.1.9 $\tilde{P}_1 \neq 0$ where $\sim: U(\mathfrak{g})_C \to U(\mathfrak{g}/\mathfrak{n})_C$ is defined in Lemma 2.1.9. Applying our induction hypothesis to G/N, we have a $\tilde{P}_1$-full compact set K_1 of G/N such that $K_1 \supset \pi(K)$. Now Proposition 2.3.1 (see Remark (1) there) implies that $\pi^{-1}(K_1)$ is P_1-full. On the other hand by Case I (3) of the proof of Proposition 2.3.2, we have a real valued function $f \in C^\infty(G)$ such that

(3) $\sigma(P_1)(df) \neq 0$ on $\pi^{-1}(K_1)$

(4) $X_n f \equiv 1$ on G.

Applying Proposition 1.8 with $D = P_1$, $M = G$, $F = \pi^{-1}(K_1)$, $\phi = f$, $N = 0$ in the notation there, we conclude that the set $B_L = \{x \in \pi^{-1}(K_1) \mid |f(x)| \leqq L\}$ is P_1-full for all $L \geqq 0$. Since K is compact, we can choose M so that $K \subset B_M$.

We now claim that B_M is X_n-full. By choosing $X_1, \ldots, X_{n-1} \in \mathfrak{g}$ so that the map

$$(\exp t_1X_1) \ldots (\exp t_nX_n) \longrightarrow (t_1, \ldots, t_n)$$

is a diffeomorphism of G onto R^n, we identify G with R^n by the above diffeomorphism. Then $X_n f \equiv 1$ means that f is of the form

$$f(t_1, \dots, t_n) = f_1(t_1, \dots, t_{n-1}) + t_n$$

So for each fixed $t_1, \dots, t_{n-1}$, B_M is convex in t_n-direction. Since X_n is identified with $\frac{d}{dt_n}$, B_M is X_n-full. By the definition of "fullness" the X_n-fullness and P_1-fullness of B_M imply the $P_1 \cdot X_n^{\ell}$-fullness of B_M. B_M is clearly compact.
For the last statement in the theorem, we only have to remark that ^{t} (transpose with respect to right invariant measure) is an anti-automorphism of $U(\mathfrak{g})_C$ and sends semi-bi-invariant operators to semi-bi-invariant ones. (Lemma 2.2.1).

q.e.d.

Corollary 2.3.4

Every non-zero semi-bi-invariant differential operator on a simply connected solvable Lie group is globally solvable.

<proof>

Theorem 2.2.6 (semi-global solvability) and Theorem 2.3.3 (P-convexity) imply the global solvability by Theorem 1.6.

q.e.d.

The next result is the P-convexity of simply connected split solvable groups where P is an arbitrary non-zero left-invariant operator. The statement of the theorem takes a stronger form because we need a strong induction hypothesis.

Theorem 2.3.5

Let G be a simply connected split solvable Lie group with Lie algebra $\mathfrak{g}$ of dimension n. Let $X_1, \dots, X_n$ be a basis of $\mathfrak{g}$ such that $X_1, \dots, X_\ell \notin [\mathfrak{g}, \mathfrak{g}]$, $X_{\ell+1}, \dots, X_n \in [\mathfrak{g}, \mathfrak{g}]$ and for each i, $\{X_i, \dots, X_n\}$ spans an ideal of $\mathfrak{g}$. (See Lemma 2.1.10). Let $\{P_\lambda\}_{\lambda \in I}$ be a family of non-zero equivalent operators in $U(\mathfrak{g})_C$ with respect to the above basis $X_1, \dots, X_n$. Then for any compact set K in G, there exists a compact set K' in G such that $K \subset K'$ and K' is P_λ-full for all $\lambda \in I$.

In particular G is Q-convex for all non-zero $Q \in U(\mathfrak{g})_C$.

<proof>

The proof goes by induction on dim G. If G is abelian, then by Theorem 1.7, the statement of the theorem obviously holds. So we may assume that dim $G > 1$, G is non-abelian (i.e. $[\mathfrak{g}, \mathfrak{g}] \neq 0$) and that the statement of the theorem is true for groups of lower dimension. Let $\mathfrak{n} = RX_n$, $N = \{\exp tX \mid t \in R_n\}$ and assume that

$\{P_\lambda\}_{\lambda \in I}$, K are given as in the statement of the theorem. Since the P_λ are equivalent with respect to $X_1, \ldots, X_n$, by the definition of equivalence, we have $\ell \in Z^+ \cup \{0\}$ such that

(1) $\qquad P_\lambda = Q_\lambda X_n^\ell$ for all $\lambda \in I$

where the $Q_\lambda \in U(\mathfrak{g})_C$ satisfy $\tilde{Q}_\lambda \neq 0$. $\sim: U(\mathfrak{g})_C \to U(\mathfrak{g}/\mathfrak{n})_C$ was defined in Lemma 2.1.9.

(The reason why we assumed $\mathfrak{g}$ to be non-abelian is that we want (1) to hold and want to use Lemma 2.1.11). Again, by the definition of equivalence, $\{Q_\lambda\}_{\lambda \in I}$ is a family of non-zero equivalent operators with respect to the basis $X_1, \ldots, X_n$. Put for each $\lambda \in I$,

$$Q_{\lambda,1} = Q$$

$$Q_{\lambda,2} = \phi(Q_{\lambda,1})$$

$$\vdots$$

$$Q_{\lambda,i+1} = \phi(Q_{\lambda,i})$$

$$\vdots$$

where $\phi: U(\mathfrak{g})_C \to U(\mathfrak{g})_C$ was defined in Lemma 2.1.8. By Lemma 2.1.12, all the $Q_{\lambda,i}$ $(\lambda \in I,\ i = 1,2,\ldots)$ are equivalent with respect to $X_1, \ldots, X_n$. Hence by Lemma 2.1.11 all the $\tilde{Q}_{\lambda,i}$ $(\lambda \in I,\ i = 1,2,\ldots)$ are equivalent with respect to the basis $\tilde{X}_1, \ldots, \tilde{X}_{n-1}$ of $\mathfrak{g}/\mathfrak{n}$ and they are non-zero. Applying the induction hypothesis to G/N (Note this is again split solvable by Remark (4) after Definition 2.1.2), we get a compact set K_2 of G/N such that

$$\pi(K) \subset K_2$$

and

$$K_2 \text{ is } \tilde{Q}_{\lambda,i}\text{-full for all } \lambda \in I,\ i = 1,2,\ldots.$$

By Proposition 2.3.1, we conclude that $\pi^{-1}(K_2)$ is Q_λ-full for all $\lambda \in I$. Note that, by the definition of equivalence, all the Q_λ have the same highest degree part in the canonical expression with respect to $X_1, \ldots, X_n$. The Proposition 2.3.2 and its proof then show that there are a real valued function $f \in C^\infty(G)$ and a compact set B in G such that for all $\lambda \in I$

$$\sigma(Q_\lambda)(df) \neq 0 \quad \text{on} \quad \pi^{-1}(K_2) \setminus B$$

$$X_n f \equiv 1 \qquad \text{on} \quad G.$$

We can take M large so that

$$B_M = \{x \in \pi^{-1}(K_2) \mid |f(x)| \leqq M\}$$

is a compact set of G containing K and B. Now Proposition 1.8 with $M = G$, $D = Q_\lambda$, $F = \pi^{-1}(K_2)$,

$\phi = f$, $N = 2M$ shows that B_{2M} is Q_λ-full for all $\lambda \in I$. Again, as was indicated at the end of the proof of Theorem 2.3.3, B_M is X_n-full. Hence B_M is $Q_\lambda X_n^\ell$-full for all $\lambda \in I$. Thus B_M is P_λ-full for all $\lambda \in I$. For the last statement of the theorem, we have only to remark that ^{t} (transpose with respect to the right invariant measure on G) is an anti-automorphism of $U(\mathfrak{g})_C$.

q.e.d.

Corollary 2.3.6 (Helgason [10])

Let G/K be a symmetric space of non-compact type where G is a non-compact semisimple Lie group with finite center and K a maximal compact subgroup. Then G/K is D-convex for any G-invariant differential operator D on G/K.

<proof>

Let $G = ANK$ be an Iwasawa decomposition. Then G/K is diffeomorphic to the simply connected split solvable Lie group AN. (Remark 2) after Definition 2.1.2) Under this diffeomorphism, G-invariant operators on G/K correspond to some left invariant operators on AN. Now Theorem 2.3.5 gives the desired conclusion.

q.e.d.

Remark

This convexity result actually gives the global solvability on G/K since the semi-global solvability is known. (See Helgason [10]). Also note that Helgason's proof of the P-convexity gives a finer result. Namely, he showed that a ball of radius r ($r \geq 0$) in the Riemannian manifold G/K is convex with respect to invariant operators.

CHAPTER III

Symmetric Spaces

§1 Preliminaries

Let M be a pseudo-Riemannian manifold. The Laplacian P of M is defined as a differential operator which is in local coordinates $(x_1, \ldots, x_n)$ expressed by

$$Pf = \frac{1}{\sqrt{\bar{g}}} \sum_k \frac{\partial}{\partial x_k} \left(\sum_i g^{ik} \sqrt{\bar{g}} \frac{\partial f}{\partial x_i} \right) \quad \text{for} \quad f \in C^\infty(M),$$

where

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

$$\sum_j g_{ij} g^{jk} = \delta_{ik} \quad \text{(Kronecker's delta)}$$

$\bar{g} = |\det (g_{ij})|$ with g the pseudo Riemannian structure of M.

It is an operator invariant under all isometries of M. We shall show that if M is a non-compact pseudo-Riemannian symmetric space of a certain type, P is globally solvable.

<u>Definition 3.1</u> <u>A non-compact semisimple symmetric space</u> is a homogeneous space G/H where G is a non-compact semisimple Lie group and H is an open subgroup of the fixed point group of an involution θ of G.

<u>Remark</u>

(1) Such a G/H becomes a G-invariant pseudo-Riemannian

manifold by the non degenerate bilinear form on $\mathfrak{m}$ given by the restriction of the Killing form of $\mathfrak{g}$. Here $\mathfrak{g}$ denotes the Lie algebra of G and $\mathfrak{m}$ denotes the (-1)-eigenspace of $d\theta$, the differential of θ so that $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (orthogonal direct sum), where $\mathfrak{h}$ is the Lie algebra of H. We identify $\mathfrak{m}$ with the tangent space at the origin of G/H.

(2) The pseudo-Riemannian structure of G/H mentioned above induces the canonical affine connection on G/H. (See Nomizu [12] for a detailed study of such connections). In the sequel, we shall use the following important fact: With respect to the canonical affine connection, the geodesics of G/H are the G-translates of $\{\pi(\exp tX) \mid t \in R\}$, $X \in \mathfrak{m}$, where π is the projection $G \to G/H$.

(3) G/H defined as above, are actually non-compact. (Berger [1]).

For the general theory of non-compact semisimple symmetric spaces, the reader is referred to Berger [1], Rossman [14], Flensted-Jensen [7].

Example 3.2

1) A symmetric space of non-compact type G/K, where G is a non-compact semisimple Lie group with finite center and K is a maximal compact subgroup. In this case, the involution whose fixed points group is K is called a

Cartan involution. Helgason [10] showed that not only the Laplacian, but all the G-invariant operators of G/K are globally solvable.

2) A non-compact semisimple Lie group G. Define an involution θ on $G \times G$ by $\theta(x,y) = (y,x)$. The fixed point group of θ is the diagonal subgroup: $G^* = \{(x,x) \mid x \in G\}$. G is diffeomorphic to $G \times G/G^*$ and Laplacian of $G \times G/G^*$ corresponds to the Casimir operator on G. Rauch-Wigner [13] proved the global solvability of the Casimir operator when G has finite center.

3) There are various other kinds of non-compact semisimple symmetric spaces e.g. complex semisimple Lie group mod its real form, $SO_0(p,q)/SO_0(p,q-1)$, etc. ....

We prove the global solvability of the Laplacian of a non-compact semisimple symmetric space when G is connected and has finite center. (So far, this restriction does not seem easily removable). The first thing we do is to show that a bicharacteristic of the Laplacian of pseudo-Riemannian manifolds is a geodesics. This is a well known fact which is almost as old as the notion of bicharacteristics. But I would like to give a complete proof here.

Let M be an arbitrary pseudo-Riemannian manifold with the pseudo-Riemannian structure g. Let $(x_1, \ldots, x_n)$

be local coordinates. Then locally, the Laplacian P is expressed as

$$P = \Sigma g^{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + (\text{differential operator of degree} \leqq 1).$$

In the induced coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$, the principll symbol $p(x,\xi)$ of P is given by

(1) $$p(x,\xi) = \Sigma g^{ij}(x)\xi_i\xi_j$$

So we have by noting $g^{ij} = g^{ji}$,

(2) $$\frac{\partial p}{\partial \xi_k}(x,\xi) = 2 \sum_j g^{kj}\xi_j.$$

(3) $$\frac{\partial p}{\partial x_k}(x,\xi) = \sum_{i,j} \frac{\partial g^{ij}}{\partial x_k} \xi_i\xi_j.$$

A bicharacteristic strip of P is a curve in $T^*M \setminus 0$ (the cotangent bundle of M minus zero section) which is in the local coordinates described as a solution $(x(t), \xi(t)) = (x_1(t), \ldots, x_n(t), \xi_1(t), \ldots, \xi_n(t))$ of the following equations.

(4)
$$\frac{d}{dt}x_i(t) = \frac{\partial p}{\partial \xi_i}(x(t), \xi(t))$$
$$\frac{d\xi_i(t)}{dt} = \frac{-\partial p}{\partial x_i}(x(t), \xi(t)) \qquad i = 1, 2, \ldots, n,$$

By (2), (3), the above equations become

$$(5)\quad \begin{aligned} \frac{dx_i}{dt}(t) &= 2 \sum_j g^{ij}\xi_j(t) \\ \frac{d\xi_i}{dt}(t) &= - \sum_{kj} \frac{\partial g^{kj}}{\partial x_i}\xi_k(t)\xi_j(t) \end{aligned}$$

A bicharacteristic curve is the projection of a bicharacteristic strip form T^*M to M. Let Γ^{ℓ}_{ik} denote the Christoffel symbols in our local coordinates:

$$\nabla_{\left(\frac{\partial}{\partial x_i}\right)} \frac{\partial}{\partial x_k} = \sum_{\ell} \Gamma^{\ell}_{ik} \frac{\partial}{\partial x_\ell} \quad \text{where } \nabla \text{ is the canonical}$$

affine connection on M induced from g.
The relation between Γ^{ℓ}_{ik} and g is given by

$$(6)\quad \Gamma^{\ell}_{ik} = \sum_m g^{\ell m}\left(\frac{1}{2}\right)\left\{\frac{\partial g_{im}}{\partial x_k} + \frac{\partial g_{mk}}{\partial x_i} - \frac{\partial g_{ik}}{\partial x_m}\right\}$$

(See Wolf [18] page 49)
We want to show that a bicharacteristic curve of P is geodesic. Namely we want to show:

$$\frac{d^2x_i(t)}{dt^2} + \sum_{j,k} \Gamma^{i}_{jk} \frac{dx_j(t)}{dt}\frac{dx_k(t)}{dt} = 0$$

$i = 1, \ldots, n$ for solutions of (5). (See Helgason [8] page 30 (3)).

Let $x(t) = (x_1(t), \ldots, x_n(t))$, $\xi(t) = (\xi_1(t), \ldots, \xi_n(t))$ be a solution of (5). Then

$$(7)\quad \frac{d^2}{dt^2}x_i(t) = \frac{d}{dt}\left(\frac{d}{dt}x_i(t)\right) = \frac{d}{dt}\left(2\sum_j g^{ij}\xi_j\right) \qquad \text{(by (5))}$$

$$= 2\sum_j \left(\sum_k \frac{\partial g^{ij}}{\partial x_k}\frac{dx_k}{dt}\xi_j + g^{ij}\frac{d\xi_j}{dt}\right)$$

$$= 2\sum_j\left(\sum_k \frac{\partial g^{ij}}{\partial x_k}\left(2\sum_q g^{kq}\xi_q\right)\xi_j\right)$$

$$+ 2\sum_j g^{ij}\left(-\sum_{pq}\frac{\partial g^{pq}}{\partial x_j}\xi_p\xi_q\right) \qquad \text{(by (5))}$$

Note that $\dfrac{\partial g^{ij}}{\partial x_m} = -\sum_{k,\ell} g^{i\ell}\dfrac{\partial g_{\ell k}}{\partial x_m}g^{kj}$ (for all i,j,m).

(In fact, $\sum_j g^{ij}g_{jk} = \delta_{ik}$

$$\Rightarrow \sum_j\left(\frac{\partial g^{ij}}{\partial x_m}g_{jk} + g^{ij}\frac{\partial g_{jk}}{\partial x_m}\right) = 0$$

$$\Rightarrow \sum_j \frac{\partial g^{ij}}{\partial x_m}g_{jk} = -\sum_j g^{ij}\frac{\partial g_{jk}}{\partial x_m}$$

$\Rightarrow$ the desired equality)

Using this, we get from (7)

$$(8)\quad \frac{d^2}{dt^2} x_i(t) = 2 \sum_j \sum_k (- \sum_{m,p} g^{im} \frac{\partial g_{mp}}{\partial x_k} g^{pj})(2 \sum_q g^{kq}\xi_q)\xi_j$$

$$+ 2 \sum_j g^{ij}(- \sum_{p,q} (\sum_{m,k} (-g^{pm} \frac{\partial g_{mk}}{\partial x_j} g^{kq})\xi_p\xi_q)$$

If one interchanges j and p in the first term, j and m in the second term, then (8) becomes

$$(9)\quad \frac{d^2}{dt^2} x_i(t) = 2 \sum_p \sum_k (- \sum_{m,j} g^{im} \frac{\partial g_{mj}}{\partial x_k} g^{jp})(2 \sum_q g^{kq}\xi_q)\xi_p$$

$$+ 2 \sum_m g^{im}(- \sum_{pq} \sum_{jk} (-g^{pj} \frac{\partial g_{jk}}{\partial x_m} g^{kq})\xi_p\xi_q$$

On the other hand we have by (5) and (6),

$$(10)\quad \sum_{j,k} \Gamma^i_{jk} \frac{dx_j}{dt} \frac{dx_k}{dt}$$

$$= \frac{1}{2}(\sum_{mjk} g^{im}(\frac{\partial g_{jm}}{\partial x_k} + \frac{\partial g_{mk}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_m}))$$

$$\times (2 \sum_p g^{ip}\xi_p)(2 \sum_q g^{kq}\xi_q)$$

$$= \frac{1}{2} \sum_{mjk} g^{im}(\frac{\partial g_{jm}}{\partial x_k} + \frac{\partial g_{mk}}{\partial x_j})(2 \sum_p g^{jp}\xi_p)(2 \sum_q g^{kq}\xi_q)$$

$$- \frac{1}{2} \sum_{mjk} g^{im}(\frac{\partial g_{jk}}{\partial x_m})(2 \sum_p g^{jp}\xi_p)(2 \sum_q g^{kq}\xi_q)$$

$$= \frac{1}{2} \sum_{mjk} g^{im}\, 2\left(\frac{\partial g_{jm}}{\partial x_k}\right)\left(2 \sum_p g^{jp}\xi_p\right)\left(2 \sum_q g^{kq}\xi_q\right)$$

$$- \frac{1}{2} \sum_{mjk} g^{im}\left(\frac{\partial g_{jk}}{\partial x_m}\right)\left(2 \sum_p g^{jp}\xi_p\right)\left(2 \sum_q g^{kq}\xi_q\right)$$

Recalling that $g_{ij} = g_{ji}$, $g^{ij} = g^{ji}$ for all i, j, we see that (9) + (10) = 0.

Therefore

$$\frac{d^2}{dt^2} x_i(t) + \sum_{jk} \Gamma^i_{jk} \frac{dx_i}{dt} \frac{dx_k}{dt} = 0$$

for any solution of (5). Hence a bicharacteristic curve of the Laplacian on a pseudo-Riemannina manifold is a geodesic.

§2 Null bicharacteristics

In this section we prove that no null bicharacteristic curve of the Laplacian P of our non-compact semisimple symmetric space G/H stays inside a compact set. (Here, by "a null bicharacteristic curve" we mean the projection of a bicharacteristic strip on which the principal symbol of the differential operator vanishes).

From now on, G/H shall always denote a connected non-compact semisimple symmetric space where G is a connected non-compact semisimple Lie group with finite center,

θ an involution on G and H an open subgroup of the fixed point group of θ. Let $\mathfrak{g}$, $\mathfrak{h}$ respectively denote the Lie algebras of G, H. $d\theta$ shall denote the differential of θ. By P, we denote the Laplacian of G/H. Let $\mathfrak{m}$ be the (-1)-eigenspace of $d\theta$ so that

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$$

is a direct sum decomposition. We shall keep to this notation hereafter. First of all we need an elementary lemma.

Lemma 3.2

If $X \in \mathfrak{m}$ is such that $\{\pi(\exp tX) | t \in R\}$ is relatively compact in G/H, then $\{\exp tX | t \in R\}$ is relatively compact in G where $\pi: G \to G/H$ is the projection.

<proof>

Let $X \in \mathfrak{m}$. If $\{\pi(\exp tX) | t \in R\} \subset G/H$ is relatively compact, then there exists a compact set B in G such that $\{\pi(\exp tX) | t \in R\} \subset \pi(B)$. Therefore, for any $t \in R$, there exists $b \in B$, $h \in H$ such that

(1) $\exp tX = bh$

Applying the involution θ, we get

(2) $\theta(\exp tX) = \theta(b)\theta(h)$

But since $X \in \mathfrak{m}$ = (-1)-eigenspace of $d\theta$ and $\theta(h) = h$, we have

(3) $\exp(-tX) = \theta(b)h$.

Multiplying (1) by the inverse of (3) we have

$$\exp 2tX = b\theta(b)^{-1} \in B\cdot\theta(B)^{-1}$$

Since $B\cdot\theta(B)^{-1}$ is relatively compact in G, $\{\exp 2tX \mid t \in R\}$ lies in a compact set.

q.e.d.

It is well-known that there exists a Cartan involution τ of G which commutes with θ (Berger [1]).

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition corresponding to $d\tau$, the differential of τ. Then

(4) $\mathfrak{g} = (\mathfrak{h} \cap \mathfrak{k}) + (\mathfrak{h} \cap \mathfrak{p}) + (\mathfrak{m} \cap \mathfrak{k}) + (\mathfrak{m} \cap \mathfrak{p})$

is a direct sum decomposition. Let $m = \dim(\mathfrak{m} \cap \mathfrak{k})$ and $\ell = \dim(\mathfrak{m} \cap \mathfrak{p})$. Take a basis $X_1, \ldots, X_m, Y_1, \ldots, Y_\ell$ of $\mathfrak{m}$ so that

(5) $B(X_i, X_j) = -\delta_{ij} \qquad 1 \leqq i, j \leqq m$

$B(Y_i, Y_j) = \delta_{ij} \qquad 1 \leqq i, j \leqq \ell$

$B(X_i, Y_j) = 0 \qquad 1 \leqq i \leqq m, 1 \leqq j \leqq \ell$

We take local coordinates $(x_1, \ldots, x_m, y_1, \ldots, y_\ell)$ around o so that o corresponds to $(0, \ldots, 0, 0, \ldots, 0)$ and the $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}$ correspond to the X_i, Y_j respectively at o. (Here o denotes the origin of G/H). Then by the definition of the pseudo-Riemannian structure of G/H, we have

$$(6)\quad \begin{aligned} g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)(o) &= B(X_i, X_j) = -\delta_{ij} \\ g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right)(o) &= B(Y_i, Y_j) = \delta_{ij} \\ g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right)(o) &= B(X_i, Y_j) = 0 \end{aligned}$$

Here we used (5).

So by (1) of §1, the principal symbol $p(x,y,\xi,\eta)$ of the Laplacian P satisfies

$$(7)\quad p(0,0,\xi,\eta) = -\sum_{i=1}^{m} \xi_i^2 + \sum_{i=1}^{\ell} \eta_i^2$$

where $(x,y,\xi,\eta) = (x_1, \ldots, x_m, y_1, \ldots, y_\ell, \xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_\ell)$ are the induced coordinates of $T^*(G/H)$.

On the other hand, (5) of §1 implies that if a bicharacteristic strip of P passes through $(0,0,\xi,\eta) \neq (0,0,0,0)$ then the corresponding bicharacteristic curve (which is simply the projection of the bicharacteristic strip) has the tangent vector

$$(8) \quad -2\sum_{i=1}^{m} \xi_i X_i + 2\sum_{i=1}^{\ell} \eta_i Y_j \in \mathfrak{m}$$

at the origin.

If the bicharacteristic curve is null, then by (7)

$$(9) \quad -\sum_{i=1}^{m} \xi_i^2 + \sum_{i=1}^{\ell} \eta_i^2 = 0$$

But if (9) holds, then

$$B(-2\Sigma\xi_i X_i + 2\Sigma\eta_i Y_j, -2\Sigma\xi_i X_i + 2\Sigma\eta_i Y_i) = 4(-\Sigma\xi_i^2) + 4(\Sigma\eta_i^2) = 0$$

hence the set $\{\exp t(-2\Sigma\xi_i X_i + 2\Sigma\eta_i Y_i) \mid t \in R\}$ can not be contained in a compact set of G. (Recall that if $Z \in \mathfrak{g}$ is non-zero, and the one parameter subgroup $t \longrightarrow \exp tZ$ of G stays inside a compact set then $B(Z,Z) < 0$.) Recall now that the geodesic emanating from $o \in G/H$ with the tangent vector $Z \in \mathfrak{m}$ is given by $t \longrightarrow \pi(\exp tZ)$ where π is the projection from G onto G/H. Now Lemma 2.1 implies that no null bicharacteristic curve of the Laplacian passing through o stays inside a compact set of G/H.

By the G-invariance of the Laplacian, we conclude that no null bicharacteristic curve of the Laplacian stays inside a compact set of G/H.

§3 Construction of a function

In this section, we construct a non-negative real valued function $f \in C^\infty(G/H)$ such that

(1) The set $\{x \in G/H \mid f(x) = 0\}$ does not contain any open subset of G/H.

(2) For any $M > 0$, the set $B_M = \{x \in G/H \mid f(x) \leqq M\}$ is compact and P-full.

Once we have an f which satisfies (1), (2), we shall have the following consequences.

(3) For any compact set C_1 in G/H, we have a compact P-full set C_2 containing it.

(4) $Pu \equiv 0$, $u \in C_0^\infty(G/H) \Rightarrow u \equiv 0$

In fact (3) follows from the fact that for any compact set C_1, $N = \sup_{x \in C_1} f(x) < \infty$ and B_N works as C_2.

To see (4), suppose $u \in C_0^\infty(G/H)$ and $Pu \equiv 0$. Then for any $M > 0$, supp $Pu \subset B_M$. Since by (2) B_M is assumed to be P-full, we have supp $u \subset B_M$ for all $M > 0$. This implies supp $u \subset B_0 = \{x \in G/H \mid f(x) = 0\}$. But by (1), B_0 contains no open subset of G/H. Since $u \in C_0^\infty(G/H)$, this implies that $u \equiv 0$. So (4) follows.

Before establishing (1), (2), we briefly summarize some basic facts about symmetric spaces. For our purpose Flensted-Jensen [7] §4 is the best reference and we reproduce a part of it.

Let us go back to the decomposition (4) of §2.

$$\mathfrak{g} = (\mathfrak{h} \cap \mathfrak{k}) + (\mathfrak{h} \cap \mathfrak{p}) + (\mathfrak{m} \cap \mathfrak{k}) + (\mathfrak{m} \cap \mathfrak{p}).$$

Put $\mathfrak{g}_0 = \mathfrak{h} \cap \mathfrak{k} + \mathfrak{m} \cap \mathfrak{p}$. Then $\mathfrak{g}_0$ is reductive and $\mathfrak{g}_0' = [\mathfrak{g}_0, \mathfrak{g}_0]$ is semisimple. Let $\mathfrak{a}_0'$ be a maximal abelian in $\mathfrak{g}_0' \cap \mathfrak{p}$, then $\mathfrak{a}_0 = \mathfrak{a}_0' + \mathfrak{c}_0 \cap \mathfrak{p}$ is maximal abelian in $\mathfrak{m} \cap \mathfrak{p}$, where $\mathfrak{c}_0$ is the center of $\mathfrak{g}_0$. Choose a positive Weyl chamber $\mathfrak{a}_0'^{+}$ in $\mathfrak{a}_0'$ and define $\mathfrak{a}_0^{+} = \mathfrak{a}_0'^{+} + \mathfrak{c}_0 \cap \mathfrak{p}$. Let W_0 be the Weyl group of $(\mathfrak{g}_0, \mathfrak{a}_0)$ and put $A_0 = \exp \mathfrak{a}_0$, $A_0^{+} = \exp \mathfrak{a}_0^{+}$. Since G has finite center, the analytic subgroup K corresponding to $\mathfrak{k}$ is compact.

We have the following important facts. (See [7] Theorem 4.1).

(5) For any $x \in G$, there exists a unique $a \in \overline{A_0^{+}}$ such that $x \in KaH$ where $\overline{A_0^{+}}$ = the closure of A_0^{+} in G.

(6) There is a bijective correspondence $C^{\infty}(K \backslash G/H) \cong C^{\infty}_{W_0}(A_0)$ given by the restriction to A_0.

Here $C^\infty(K\backslash G/H)$ = smooth functions on G/H left-invariant under K, $C^\infty_{W_0}(A_0)$ = smooth functions on $A_0 \neq 0$ invariant under W_0. (Remark $A_0 \neq 0$ since G/H is assumed to be non-compact).

Now take an orthonormal basis $H_1, \ldots, H_p$ of $\mathfrak{a}_0$ with respect to the restriction of the Killing form B of $\mathfrak{g}$ to $\mathfrak{a}_0 \times \mathfrak{a}_0$. The Weyl group W_0 acts as a group of linear isometries on $\mathfrak{a}_0$ with respect to the metric given by the restriction of B. We identify $\mathfrak{a}_0$ with A_0 via the exponential map. Define a function ϕ on A_0 by

$$(7) \quad \phi: \sum_{i=1}^{p} a_i H_i \longrightarrow \sum_{i=1}^{p} a_i^2$$

Then $\phi \geqq 0$ and $\phi \in C^\infty_{W_0}(A_0)$ because ϕ is invariant under all linear isometries. By (6), we can extend ϕ to $f \in C^\infty(K\backslash G/H)$ so that $f(aH) = \phi(a)$ for $a \in A_0$. Then $f \geqq 0$ and by (5), (7) we have

$$\{x \in G/H | f(x) = 0\} = \{kaH | k \in K,\ a \in \overline{A_0^+}\ \ \phi(a) = 0\}$$

$$= \pi(K)$$

where $\pi: G \to G/H$ is the projection. $\pi(K) \subset G/H$ does not contain any open subset of G/H. Therefore (1) of this section is established. Also note that

$B_M = \{x \in G/H | f(x) \leqq M\} = \{kaH | \phi(a) \leqq M,\ a \in \overline{A_0^+},\ k \in K\}$ is compact for each $M > 0$. Next, we want to show that $\sigma(P)(df) \neq 0$ outside $\pi(K)$.

First of all, remark that the compact group K acts as a group of isometries on G/H and satisfies for all $a \in A_0^+$,

(8) $\quad Ka \cap A_0^+ = \{a\}$

(9) $\quad (G/H)_a = (K \cdot a)_a \oplus (A_0^+)_a \qquad$ (orthogonal direct sum)

where $(M)_x$ denotes the tangent space of the manifold M at x.

In fact (8) follows from (5). On the other hand (9) can be verified as follows. Let $X \in \mathfrak{k}$, and $a \in A_0^+$ be written as $a = \exp A$, $A \in \mathfrak{a}_0^+$. Then

(10) $\quad (\exp tX)a = a \exp e^{-\mathrm{ad}A}tX$

where $(\mathrm{ad}\ X_1)X_2 = [X_1, X_2]$.

Let $X = X_{\mathfrak{h}} + X_{\mathfrak{m}}$ be the decomposition of X such that $X_{\mathfrak{h}} \in \mathfrak{h}$, $X_{\mathfrak{m}} \in \mathfrak{m}$. Then

(11) $$e^{-\mathrm{ad}A}X = e^{-\mathrm{ad}A}(X_{\mathfrak{h}} + X_{\mathfrak{m}})$$

$$= (X_{\mathfrak{h}} - (\mathrm{ad}\ A)X_{\mathfrak{m}} + \frac{(\mathrm{ad}\ A)^2}{2!}X_{\mathfrak{h}} - \frac{(\mathrm{ad}\ A)^3}{3!}X_{\mathfrak{m}} + \ldots)$$

$$+ (X_{\mathfrak{m}} - (\mathrm{ad}\ A)X_{\mathfrak{h}} + \frac{(\mathrm{ad}\ A)^2}{2!}X_{\mathfrak{m}} - \frac{(\mathrm{ad}\ A)^3}{3!}X_{\mathfrak{h}} + \ldots)$$

is the decomposition of $e^{-\mathrm{ad}A}X$ into its $\mathfrak{h}$ and $\mathfrak{m}$ components. Here we used the fact that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

Hence by (10), we have

(12) $(\exp tX)a\cdot H = a(\exp e^{-\mathrm{ad}A}tX)\cdot H$

$$= a(\exp e^{-\mathrm{ad}A}tX)(\exp t(X_{\mathfrak{h}} - (\mathrm{ad}\ A)X_{\mathfrak{m}} + \frac{(\mathrm{ad}\ A)^2}{2!}X_{\mathfrak{h}} - \ldots))\cdot H$$

$$= a\ \exp\{t(X_{\mathfrak{m}} - (\mathrm{ad}\ A)X_{\mathfrak{h}} + \frac{(\mathrm{ad}\ A)^2}{2!}X_{\mathfrak{m}} - \ldots) + O(t^2)\}\cdot H$$

for small $t \in R$, where $O(t^2)$ denotes a vector such that $\lim_{t\to 0} \frac{1}{t^2} O(t^2) < \infty$.

Since $X \in \mathfrak{k}$, we have $X_{\mathfrak{m}} \in \mathfrak{k}$ by §2(4). So we have $B(X_{\mathfrak{m}}, \mathfrak{a}_0) = 0$ because $\mathfrak{a}_0 \subset \mathfrak{p}$ and $B(\mathfrak{k}, \mathfrak{p}) = 0$. On the other hand for any $Z \in \mathfrak{g}$,

$B((\mathrm{ad}\ A)\cdot Z, \mathfrak{a}_0)$

$= B(Z, -(\mathrm{ad}\ A)\cdot \mathfrak{a}_0))$

$= B(Z, 0)$ ($\mathfrak{a}_0$ is abelian)

$= 0.$

Thus we get

(13) $B(X_{\mathfrak{m}} - (\mathrm{ad}\ A)X_{\mathfrak{h}} + \frac{(\mathrm{ad}\ A)^2}{2!}X_{\mathfrak{m}} - \ldots, \mathfrak{a}_0) = 0.$

(12) and (13) imply the desired orthogonality (9).

Since (8), (9) are satisfied we can apply Theorem 2.11 of Helgason [9] (See the remark after it which says that the theorem holds for all pseudo-Riemannian manifolds).

Therefore, for any left K-invariant smooth function u on G/H and for $a_0 \in A_0^+$,

$$(14) \qquad Pu(a_0) = L\bar{u}(a_0) + L'\bar{u}(a_0)$$

where $\bar{u}$ is the restriction of u to A_0^+, L the Laplacian on A_0 and L' is a differential operator of degree less than two on A_0^+. Although L' can have singularities along the walls of Weyl chambers, those singularities do not influence our computations of the principal symbol of P below.

Take $k_0 \in K$, $a_0 \in A_0^+$. then

$$(15) \quad \sigma(P)(df(k_0a_0H))$$
$$= \frac{1}{2!}P(f - f(k_0a_0H))^2|_{k_0a_0H} \qquad \text{(Definition 1.1)}$$
$$= \frac{1}{2}\{L(\phi - \phi(a_0H))^2|_{a_0} + L'(\phi - \phi(a_0H))^2|_{a_0}\} \quad \text{(by (14))}$$
$$= \frac{1}{2}L(\phi - \phi(a_0H))^2|_{a_0} \qquad \text{(since deg } L' \leqq 1).$$

In terms of the coordinates of A_0

$$\sum_{i=1}^{p} a_i H_i \longrightarrow (a_1, \ldots, a_p), \quad L = \sum_{i=1}^{p} \frac{d^2}{da_i^2} .$$

If $a_0 = \sum_{i=1}^{p} \alpha_i H_i$ then

$$\tfrac{1}{2} L(\phi - \phi(a_0 H))^2 \big|_{a_0}$$

$$= \frac{1}{2} \sum_{i=1}^{p} \frac{d^2}{da_i^2} \left(\sum_{i=1}^{p} a_i^2 - \sum_{i=1}^{p} \alpha_i^2 \right)^2 \Big|_{a_i = \alpha_i}$$

$$= 4 \sum_{i=1}^{p} \alpha_i^2 .$$

Hence we get

$$\text{(16)} \qquad \sigma(P)(df(k_0 a_0 H)) = 4 \sum_{j=1}^{p} \alpha_j^2$$

for $a_0 \in A_0^+$, $k_0 \in K$ where $\Sigma\, \alpha_j H_j = a_0$.

But $\sigma(P)(df(x))$ is continuous in x everywhere in G/H. So (16) holds for all $k_0 \in K$ and $a_0 \in \overline{A_0^+}$. Hence $\sigma(P)(df) \neq 0$ outside $\pi(K)$. Since $f(x) \neq 0$ implies $x \notin \pi(K)$, by applying Proposition 1.8 with $M = G/H$, $D = P$, $F = G/H$, $\phi = f$, N = an arbitrary positive constant, we get the P-fullness of $B_M = \{x \in G/H \mid f(x) \leq M\}$ for any positive constant M. Thus (2) of this section is established.

§4 Global solvability

In this section we conclude the global solvability of the Laplacian P on G/H.

Theorem 3.4

Let G/H be a connected non-compact semisimple symmetric space where G is a connected non-compact semisimple Lie group with finite center and H is an open subgroup of the fixed point group of an involution of G. Then the Laplacian P of G/H is globally solvable.

<proof>

Since $P = {}^tP$ (${}^tP$ = the transpose of P with respect to the G-invariant Riemannian measure on G/H), (4) of §3 implies that:

(1) ${}^tP$ is injective on $C_0^\infty(G/H)$.

Also in §2 we proved that:

(2) No null bicharacteristic curve of P stays inside a compact set in G/H.

According to Theorem 6.3.1 of Duistermaat-Hörmander [6], (1) and (2) imply the semi-global solvability of P. On the other hand (3) of §3 implies the P-convexity of G/H (again noting $P = {}^tP$). Therefore by Theorem 1.6 we have the global solvability of P.

q.e.d.

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BIOGRAPHY

Weita Chang was born on November 24, 1951, in Tokyo. However, he is Chinese (Taiwan) and he is not Japanese. But anyway, he was educated in Japan. His first publication was a poem titled " PIG " which appeared in a children-oriented newspaper in 1963. He entered Osaka City University in 1970 and graduated from there in 1974. He loves Osaka and Nara very much and these places are his spiritual home. In September, 1974,he came to M.I.T. as a graduate student in mathematics and has been an inhabitant of Room 2-229. He is a member of Cambridge Little Orchestra. Besides the orchestra, he is playing String Quartet with friends. Unfortunately, it seems that he is the cause of the out-of-tuneness which is the weekly routine in the quartet. His favorite mathematician is K.Oka. His dream is to understand Oka's work in its original form(not in the context of current mathematics). But of course this is a dream inside mathematics. It would be much better if he became able to play his Viola as well as Primrose.