2-loop Perturbative Invariants of Lens Spaces and a Test of Chern-Simons Quantum Field Theory

by

Richard Stone

B.Sc. (Hons.) Australian National University (1990)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mathematics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1996

Author ....
Department of Mathematics
May 23, 1996

Certified by ....
Scott Axelrod and Isadore Singer
Professors of Mathematics
Thesis Supervisors

Accepted by ....
David A. Vogan
Chairman, Departmental Committee on Graduate Students

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Abstract

We calculate the 2-loop invariants of a one-parameter family of lens spaces , $L[p]$, as defined by Axelrod-Singer's perturbation theory for the $SU(2)$ Chern-Simons action around the trivial connection. We show that our values agree with those expected on the basis of the sub-leading asymptotics of the exact Witten-TQFT solution for the partition function of Chern-Simons quantum field theory. This extends, for the first time beyond the semi-classical setting to higher loops, existing "experimental" tests of the validity of the path integral defining the partition function and of Witten's "exact," physics-based analysis of it. In doing so, it verifies consistency, at least to two loops for these spaces, between the exact and perturbative treatments of Chern-Simons quantum field theory, and provides the first non-trivial evaluations of higher-loop invariants for the Axelrod-Singer theory.

A key element in the working is the derivation of a completely explicit form of the propagator for the theory on S^3. This should be an important ingredient in any future effort to undertake the theoretically important evaluation of all the higher-loop invariants of S^3. Certain integral identities concerning this propagator which arise in our evaluation of the $L[p]$ 2-loop invariants may also be useful in any such effort.

Thesis Supervisors: Scott Axelrod and Isadore Singer
Title: Professors of Mathematics

To my parents, and in fondest memory of Heidi, Tenny, Bebe and Pooge

Acknowledgements

I would like to express my gratitude to the many people whose friendship and support have been so important to me during my time at MIT. Within the mathematics department I think especially of Kirsten Bremke, Giuseppe Castellaci, Radu Constantinescu, Peter Dodds, Bruce Fischer, Gustavo Granja, Colin Ingalls, Edith Mooers, Peter Trapa, Chris Woodward, Dan Zanger and Dana Pascovici, but also of many others over the last five years. I think also of many people outside the department, among whom I would like particularly to single out Alan Blair, Andrew Hassell, Veronika Requat, Alice Carlberger, John Matz, Serena Keswani, Diane Ho, Tom Lee and above all Papa Rao.

To my parents and family I owe more than I can express for their unfailing love and understanding.

But most of all I would like to thank my joint thesis advisors, Professors Scott Axelrod and Is Singer, without whose interest, patience, and mathematical guidance, especially on the several occasions when I feared I had encountered insurmountable difficulties, this thesis would not have been possible. I feel very fortunate to have obtained my Ph.D. under their supervision.

Contents

Chapter 1

Introduction and Background

1.1 Background

In his paper [W] Witten defined a new class of differential invariants of 3-manifolds, one for each integer k, using Chern-Simons quantum field theory. His invariant, for each "level" k, is the value of the partition function for the theory, which is a formal functional integral over the space of all connections (gauge-fields) in the theory. Using physical intuition about the meaning and behaviour of this partition function and deep links with conformal field theory, he was able to evaluate it, and thus to obtain both the values of the invariants exactly for S^3, and a "sewing formula" for how the invariants change under surgery on the 3-manifold. Since any 3-manifold can be constructed by surgeries along a set of links in S^3, his invariants are thus theoretically calculable for *all* 3-manifolds. They have now been obtained explicitly in this way for a number of such families of 3-manifolds, chief among them the *lens spaces* (see [FG1] and [J1]).

From a mathematician's viewpoint, however, the use of the Feynman functional integral to define the Chern-Simons-Witten invariants is problematic. This is because, despite extensive efforts, a general way of rigorously defining such integrals and justifying the formal properties of them invoked by Witten is still unknown. To a mathematician therefore, it is not *apriori* clear that his invariants are even well-defined.

One way of resolving this difficulty is to find an alternative, mathematically rigorous definition of them for which the value of the invariants on S^3 is the same and for which the same sewing formula can be derived. Such a program has been successfully carried out from two different viewpoints. On the one hand an axiomatic formulation of *topological quantum field theory* (TQFT), encoding the physical insight about the behaviour of the partition function used by Witten without formally introducing functional integrals, has been used by various authors, beginning with Atiyah, to rigorise Witten's invariants. On the other, they have been shown by Walker ([Wal]), following work of Kirby and Melvin ([KM]), to arise from a more algebraic theory due to Reshetikhin and Turaev ([RT]) which uses quantum groups to generalise the original work of Jones on knots in S^3.

In abandoning Witten's quantum field theoretic starting point, however, neither of these alternatives quite manages to capture the full power of his heuristic Feynman integral approach. For example, by evaluating the semi-classical limit of the partition function using a formal stationary phase argument for the functional integral, Witten was able to obtain a formula giving the asymptotic behaviour of his invariants as $k \to \infty$. It is not known how to obtain this formula in either the TQFT or Reshetikhin-Turaev frameworks.

Given this, and the enormous interest of Chern-Simons quantum field theory in its own right, we instead focus in this thesis on a third strategy, due to Axelrod and Singer (also investigated by Kontsevich). This strategy retains the partition function path integral as the centrepiece and attacks the problem of understanding it head on, by showing that in this theory it *can* be interpreted in a mathematically rigorous way. This is done by approaching the partition function from the point of view of *perturbation* theory. Using this they define a different class of "perturbative Chern-Simons-Witten invariants" whose content should be essentially equivalent to that of the invariants introduced by Witten; i.e. roughly speaking, a knowledge of either set *in toto* should permit one to pass readily to the other set. In their papers [AS1] and [AS2] they succeed in proving that these perturbative invariants are rigorously well-defined and give finite, differential invariants of 3-manifolds.

Unfortunately, however, computation in their perturbative framework is considerably more difficult than in either Witten's original heuristic approach or the two subsequent alternative formulations of it. Indeed, to date, no calculations of *any* perturbative invariants for *any* 3-manifold have been performed, other than the observation, by Axelrod and Singer themselves, that the "even-loop" invariants of S^3 vanish identically for trivial symetry reasons. More fundamentally, neither a computation of the full set of perturbative invariants for S^3 (i.e. odd loop aswell as even), nor a derivation of the sewing formula have been obtained, and indeed, both tasks currently appear formidable. Thus it remains open as to whether Axelrod-Singer's rigorous perturbative version of Chern-Simons quantum field theory is consistent with Witten's original heuristic version and its later, non-field-theoretic rigorisations.

We note in passing, however, how significant a successful demonstration of this expected consistency would be. For it would not only constitute an index-type theorem of great importance in 3-manifold theory, relating the purely topological and representation-theoretic TQFT class of invariants to their geometric/analytic perturbative counterparts, it would also show that, in Chern-Simons theory at least, the traditional perturbative treatment of the partition function can be mathematically made sense of in a way that reproduces Witten's exact solution and so justifies his formal, physics-based working. This would add greatly to the mathematical credibility of quantum field theory and of the functional integral heuristics used routinely in its study both by physicists and, increasingly, by topologists and geometers.

In this thesis, therefore, it is precisely this consistency question with which we are concerned, albeit with a much more modest and experimental goal than proving the desired consistency. Our aim is simply to use the Axelrod-Singer perturbative theory to extend, in a direction that was not previously accessible, the strong computational evidence which already exists to support the validity of Witten's path integral analysis. At present this evidence is all in the form of checks of his exact computations, or rather their rigorous TQFT versions, against his *semi-classical* asymptotic formula for his invariants in the limit $k \to \infty$; i.e. for certain classes of 3-manifolds, namely the lens spaces (numerically in [FG1] and exactly in [J1]) and Brieskhorn spheres (only numerically in [FG1]),the Chern-Simons-Witten invariants have been computed in the rigorous TQFT formulations and their asymptotic behaviour found to agree with Witten's path-integral predictions. This amounts to verifying the mathematical validity of the Witten/TQFT solution for the partition function, Z_k, to *leading order* in its expansion as a perturbation series in k, for these classes of 3-manifolds. Roughly speaking our goal in this thesis is then to use the Axelrod-Singer perturbative definition of Z_k to extend this computational test, in the case of lens spaces, down below the leading term to the *sub-leading* , or "2-loop", coefficient also. It is not possible even to attack this question outside the perturbative setting.

To explain more precisely what we mean by all of this we need now to turn from generalities to give a brief mathematical introduction to Chern-Simons quantum field theory and, in particular, Axelrod-Singer's perturbative version.

1.2 Sketch of Chern-Simons Quantum Field Theory and Axelrod-Singer's work

We do not attempt here a fully self-contained exposition of these topics. The reader is referred to the original papers, [W] and [AS1],[AS2] for this, and to the general literature for discussion of the perturbation theory framework in which Axelrod and Singer operate. What follows is only the broadest outline necessary for our purposes.

The basic data of Chern-Simons quantum field theory consists of a compact, oriented, boundaryless 3-manifold, M^3, and a choice of compact, simple Lie group G. We form the trivial G-principal bundle $P = M^3 \times G$, and from it the associated adjoint vector bundle $adP = P \times_{ad} \underline{g}$ determined by the adjoint representation of G on $\underline{g}$. We identify adP with $M^3 \times \underline{g}$ via the canonical trivialisation of $P = M^3 \times G$.

The Chern-Simons action is defined on $\mathcal{A}$, the space of all connections on P, by using the canonical trivialisation of P to pull back any $A \in \mathcal{A}$ to an element (which we also denote A) of $\Omega^1(M^3; \underline{g})$ and setting

$$CS(A) = \frac{1}{8\pi^2} \int_{M^3} Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \,. \tag{1.1}$$

Here "Tr" is some multiple of the Killing form on $\underline{g}$ which we will tie down in a moment.

This action is invariant under gauge-transformations connected to the identity in the gauge-group $\mathcal{G}$, but can vary in discrete steps for arbitrary gauge-transformations due to a "winding number" factor arising from $\pi_3(G) \cong \mathbf{Z}$. We choose the normalisation of "Tr" mentioned above precisely so that these steps are "quantised" in increments of 1. In the case of $G = SU(N)$, which is all that will concern us, this makes "Tr" simply the ordinary trace in the standard N-dimensional representation.

This then allows us to formally quantise the theory "at level k," $k \in \mathbf{Z}$, by defining the partition function to be the Feynman integral over gauge orbits

$$Z_k = \int_{\mathcal{A}/\mathcal{G}} e^{2\pi i k CS(A)} \mathcal{D}A \,, \tag{1.2}$$

since now the integrality of k means that the integrand is well-defined on $\mathcal{A}/\mathcal{G}$ despite the lack of strict gauge-invariance at the classical level. Note that, without loss of generality, we may restrict our attention to $k \geq 0$ since changing k to $-k$ is equivalent simply to reversing the orientation of M^3 and thus the sign of $CS(A)$.

1.2.1 The Semi-Classical Limit

The "level" k plays the role here of $\frac{1}{\hbar}$ in real physical theories, and so the semi-classical limit corresponds to considering Z_k as $k \to \infty$. In Witten's heuristic treatment of 1.2 this semi-classical limit is evaluated by a formal stationary-phase analysis after introducing first a metric on M^3, to perform gauge-fixing to regularise Z_k, and then a counterterm to remove the resulting anomalous metric dependence of the solution. In the limit $k \to \infty$, contributions to Z_k come only from the stationary fields of the classical Chern-Simons action, i.e. the (gauge-equivalence classes of) flat connections. Assuming these form a discrete set, $\{A_i\}$, then Witten obtains an asymptotic formula for Z_k as a sum over the A_i, involving certain highly non-trivial geometric and topological invariants of these flat connections — their Chern-Simons invariants, $CS(A_i)$, their Reidemeister-Ray-Singer torsions, $\tau(M^3, A_i)$, and certain spectral invariants whose metric dependences cancel out.

1.2.2 Going Beyond the Semi-Classical Setting

The rest of Witten's analysis of Z_k, outside the asymptotic regime, is, however, independent of his semi-classical analysis. Rather than try to extend his asymptotic formula beyond leading order, he instead shows how to exactly evaluate Z_k for any fixed (arbitrary) value of k. He does this, as remarked in section 1.1, by using heuristic properties of the path integral 1.2, later formalised in the rigorous, axiomatic TQFT treatment.

The *perturbative approach* to 1.2, by contrast, seeks to build the semi-classical approximation into an expression for Z_k as a full perturbation series in the parameter k; i.e. rather than analyse Z_k for each value of k independently, the perturbative approach treats Z_k as a *function* of k and seeks a series expansion for it around $k = \infty$, after factoring out the semi-classical piece which represents the leading term. Such an expression should then yield the asymptotics to arbitrary order of the exact/TQFT Chern-Simons-Witten invariants, Z_k, allowing one to relate the exact and perturbative classes of invariants.

The content of these alternative approaches obviously *should* be equivalent, but the viewpoint is different — Witten's topological invariants are defined as the values of Z_k for each k, while the perturbative Chern-Simons-Witten invariants are defined as the *coefficients* in the perturbation series for Z_k.

There is, moreover, one further important difference between the exact/TQFT and perturbative approaches. Witten's treatment obtains invariants Z_k depending only on the parameter k. But perturbative theory, as we shall describe in the next section, involves an expansion of the integrand in 1.2 around a fixed choice of stationary solution, i.e. flat connection. It thus yields, not a single perturbation series, but one for each of our discrete set of flat connections, whose coefficients are topological invariants of M^3 *together with the flat connection*. We denote the perturbation series for Z_k around the flat connection A_i by $Z_k(M^3, A_i)$. To get from these more refined invariants to invariants only of M^3, which could then be compared with Witten's, requires adding the series for all the different flat connections.

1.2.3 The Perturbative Theory

Let us now focus on the perturbative theory and discuss it more fully, since it is the basis of all our work in this thesis.

As remarked, we make at the outset a choice of flat connection, say $A^{(0)}$, on M^3. If $d^{(0)}$ is the exterior covariant derivative twisted by $A^{(0)}$, acting on $\Omega^\star(M^3; adP) \cong \Omega^\star(M^3; \underline{g})$ to form the complex

$$\Omega^0(M^3; adP) \stackrel{d^{(0)}}{\rightarrow} \Omega^1(M^3; adP) \stackrel{d^{(0)}}{\rightarrow} \Omega^2(M^3; adP) \stackrel{d^{(0)}}{\rightarrow} \Omega^3(M^3; adP)\ , \qquad (1.3)$$

then, as in [AS1] and [AS2], we shall consider only the case where $H^1_{A^{(0)}}$, the first cohomology group of this complex, vanishes. This is equivalent to assuming that $A^{(0)}$ is isolated up to gauge-transformations, as we have done throughout the discussion so far in taking the moduli space of flat connections to be a discrete set.

We shall not, however, impose the extra assumption in [AS1] and [AS2] that $H^0_{A^{(0)}}$ is also zero, since, for all the 3-manifolds we wish to consider, this will not be the case. It has been shown in [FG1] (or alternatively [J1] or [R]) how Witten's semi-classical formula must be adapted to take account of this non-vanishing of $H^0_{A^{(0)}}$. We will see that the necessary amendments to Axelrod-Singer's "higher-loop" analysis are also easily made.

To define now the perturbative expansion of 1.2 around $A^{(0)}$, the first step as always is to make a choice of gauge-fixing (just as was done in the semi-classical analysis of 1.2 in [W]). In [AS1] this is done by introducing a metric, g, on M^3 and using it to perform BRS gauge-fixing in the Lorentz gauge. Note that because we are working perturbatively around $A^{(0)}$ here, our basic space of fields has changed from being $\mathcal{A}$ to being $T_{A^{(0)}}\mathcal{A}$ and it is this latter space to which the gauge condition, which cuts out a subspace complementary to $T_{A^{(0)}}(\mathcal{G}A^{(0)})$ in $T_{A^{(0)}}\mathcal{A}$, is applied.

Axelrod and Singer then show that, by introducing the supermanifold TM_{-}, the initial dynamical field and the ghost fields which arise in this BRS approach can all be combined into a single fermionic superfield, $\tilde{A}$, with respect to which the action appearing in the gauge-fixed path integral for $Z_k(M^3, A^{(0)}, g)$ has a particularly simple form. Since, moreover, there is a natural correspondence between superfunctions on TM_{-} and differential forms on M^3, they are further able to translate this simple form of the gauge-fixed action, S_{gf}, succinctly back into the more familiar language of differential forms.

In this language S_{gf} then splits naturally into a "free" kinetic piece, $\tilde{A} \wedge d^{(0)}\tilde{A}$ and only a single "interaction term" which is cubic in $\tilde{A}$. Consequently they are able to read off relatively easily the two key ingredients for generating the perturbative expansion, namely **(i)** the propagator of the theory, and **(ii)** the types of vertices allowed in building Feynman graphs in the theory, together with their associated Feynman rules.

(i) The propagator, as always, is the "Green's function," i.e. Schwartz kernel of the inverse, of the kinetic operator $d^{(0)}$, but where, as derived in [AS1], the domain of $d^{(0)}$ has been restricted to the orthogonal complement of its kernel by a Lagrange multiplier condition on the space of fields, $\tilde{A}$, over which the gauge-fixed path integral occurs, so that discussion of its inverse, and the associated Schwartz kernel, makes sense. We discuss this *Hodge-theoretic* inverse of $d^{(0)}$ in more detail in chapters 2 and 3. For the moment, the only observation we make is that since the new field, $\tilde{A}$, in S_{gf} has pieces of all form-degrees in $\Omega^{\star}(M^3; \underline{g})$ (due to the incorporation of the ghosts along the way) we must take the Schwartz kernel of $(d^{(0)})^{-1}$ on *all* of $(ker d^{(0)})^{\perp} \subset \Omega^{\star}(M^3; \underline{g})$, not just on the degree 1 subspace as one might have expected from the fact that the original dynamical fields *before* gauge-fixing were elements of $T_{A^{(0)}}\mathcal{A} \cong \Omega^1(M^3; \underline{g})$ only. This is what is meant mathematically by Axelrod-Singer's statement that "we need to sum over all particle types before integrating."

(ii) The presence of only the single cubic interaction term means that the theory has only one trivalent vertex, whose Feynman rule involves the structure constants, ${f_{ab}}^c$, of $\underline{g}$ (w.r.t. an orthonormal basis of $\underline{g}$ in the normalisation determined by Tr) together with the usual imposition of an integration over the spatial variable labelling the vertex.

The translation into the language of differential forms yields, moreover, one further benefit. It leads naturally to a point-splitting regularisation scheme introduced by Axelrod and Singer in [AS1], which, as always in perturbative analysis, is needed to handle the diagonal singularities of the propagator which arise in computing the amplitudes of Feynman diagrams.

With this background in place we can now finally give Axelrod and Singer's perturbative definition of Z_k. It has the standard form of a product of the semi-classical term and the higher-loop series,

$$Z_k(M^3, A^{(0)}, s) \equiv Z_k^{SC}(M^3, A^{(0)}, s)\ Z_{k+h}^{hl}(M^3, A^{(0)}, s)\ , \qquad (1.4)$$

where here s is a choice of framing (i.e. homotopy class of trivialisations of the tangent bundle) of M^3 and, as usual, the higher-loop series is defined as a graphical expansion in inverse powers of $k+h$,

$$Z_{k+h}^{hl}(M^3, A^{(0)}, s) \equiv exp\left(\sum_{l=2}^{\infty}\left(\frac{-i(k+h)}{2\pi}\right)^{1-l} \tilde{I}_l^{conn}(M^3, A^{(0)}, s)\right)\ . \qquad (1.5)$$

Two important issues require further explanation here, however. The first is the shift in parameter from k to $k+h$ in the higher-loop series. Here h is the dual Coxeter number of the group G ($h=2$ for $SU(2)$). At the moment this is done purely "by hand" in order to obtain agreement with the exact Witten/TQFT solution, where $k+h$ is the natural parameter. A variety of rationales have been given for this, principally in the physics literature (e.g. [ALR]), but also in [AS1]. However no rigorous justification of it is known at present, and so even though we adopt it, it does represent a current gap in our understanding of the perturbation theory.

The second is the graphical expansion in 1.5, which of course requires further discussion to define what is meant by the symbols $\tilde{I}_l^{conn}(M^3, A^{(0)}, s)$ giving the coefficients. These are in fact precisely the "perturbative Chern-Simons-Witten invariants" referred to in section 1.1. In line with standard perturbation theory one's first guess at their definition would be as the sum of the amplitudes of the connected l -loop Feynman diagrams in the theory; these diagrams all being constructed from our single trivalent vertex with no external edges, and their Feynman amplitudes computed using the propagator and the Feynman rules just outlined, along with the point-splitting regularisation scheme just described.

This, however, would leave the $\tilde{I}_l^{conn}$ metric-dependent, due to anomalies arising from the use of the metric in defining the gauge-fixing, in particular the consequent metric-dependence of the propagator. To get quantities that are pure differential invariants (of M^3 together with $A^{(0)}$), it is necessary therefore to change our definition of $\tilde{I}_l^{conn}$ to include "counterterms" that will cancel the metric-dependence. This is a familiar predicament in perturbative quantum field theory. In [W] Witten showed how this could be done to remove the anomalous metric-dependence at the 1-loop level and so obtain Z_k^{SC}, by introducing a framing, s, of the manifold and using a counterterm involving the "gravitational" Chern-Simons invariant of the Levi-Civita connection in the framing s. In [AS1] and [AS2] Axelrod and Singer show how Z_{k+h}^{hl} can be handled in a similar fashion using the same framing-dependent gravitational counterterm introduced by Witten. Specifically, they prove that there exist constants β_l, $l \geq 2$, such that

$$\tilde{I}_l^{conn}(M^3, A^{(0)}, s) \equiv I_l^{conn}(M^3, A^{(0)}, g) - \beta_l CS_{grav}(g, s) \tag{1.6}$$

is a rigorously well-defined, finite, differential invariant for each l, where $I_l^{conn}(M^3, A^{(0)}, g)$ now refers to the metric-dependent sum of Feynman amplitudes of l-loop graphs just discussed. And they compute the value of β_2 explicitly and provide heuristic arguments suggesting that $\beta_l = 0$ for all $l \geq 3$.

This then completes the explanation and discussion that was needed in order to fully make sense of Axelrod-Singer's perturbative definition of Z_k. Even so, however, this is not quite the final form of the perturbative definition of Z_k that we will use; by considering more detail the role of the framing, s, in 1.4 and 1.5 we will be led to one trivial change which will be slightly better adapted to our purposes.

The framing was introduced to allow us to cancel the metric dependence of the $I_l^{conn}(M^3, A^{(0)}, g)$ and obtain instead the $\tilde{I}_l^{conn}(M^3, A^{(0)}, s)$. At first glance this seems to be simply replacing one extraneous dependence with another, and certainly it does seem to undermine our hope of obtaining invariants only of M^3 itself and $A^{(0)}$. In fact, however, this extra framing-dependence of the $\tilde{I}_l^{conn}$ is not really a problem. This is because both Axelrod-Singer and Witten resolve explicitly in their work how the quantities Z_k, Z_k^{SC} and $\tilde{I}_l^{conn}$ vary on changing the choice of (homotopy class of) framing. Thus although the original Chern-Simons-Witten invariants and their perturbative counterparts in 1.4 and 1.5 *are* currently dependent on a choice of framing, we know their framing dependence exactly, and this is effectively as good as having invariants only of M^3 and $A^{(0)}$ as desired.

Indeed Atiyah has given an alternative, more elegant, way of sidestepping this issue altogether. In [A] he notes that instead of using TM^3 and a framing s to define the counterterms above, we could equally work on $2TM^3 \equiv TM^3 \oplus TM^3$, which has a natural Spin(6)-structure, and use a trivialisation of this vector bundle, known as a *biframing*. Since, in contrast to the case of framings, a *canonical* biframing of any 3-manifold does exist, defined by having its "signature defect" (see [A], [FG1], or [J1]) zero, we can thus define a differential invariant of M^3 and $A^{(0)}$ only, simply by taking the relevant invariant (Z_k or $\tilde{I}_l^{conn}$) with counterterm evaluated in the canonical biframing.

We choose to adopt this convention of Atiyah's in this thesis and it is this which necessitates the minor adjustment to Axelrod-Singer's perturbative definition of Z_k mentioned above. This choice is better adapted to our needs in this thesis, however, because it renders easier the comparisons we ultimately have to make with the work in [FG1], [J1], and [R] in the TQFT setting, where this approach is standard and

Chern-Simons-Witten invariants are always given in the canonical biframing.

To be specific now about our adjusted definition of Z_k, let the canonical biframing of M^3 that we will use throughout in defining our invariants, be denoted by σ. Then our invariants are $Z_k(M^3, A^{(0)}, \sigma)$, $Z_k^{SC}(M^3, A^{(0)}, \sigma)$ and $\tilde{I}_l^{conn}(M^3, A^{(0)}, \sigma)$, and the *final* metric-independent perturbative definition of Z_k that we use in place of 1.4 and 1.5 to replace the arbitrary s everywhere by the canonicalσ, is simply

$$Z_k(M^3, A^{(0)}, \sigma) \equiv Z_k^{SC}(M^3, A^{(0)}, \sigma)\, Z_{k+h}^{hl}(M^3, A^{(0)}, \sigma)\ , \tag{1.7}$$

where

$$Z_{k+h}^{hl}(M^3, A^{(0)}, \sigma) \equiv exp\left(\sum_{l=2}^{\infty}\left(\frac{-i(k+h)}{2\pi}\right)^{1-l} \tilde{I}_l^{conn}(M^3, A^{(0)}, \sigma)\right)\ . \tag{1.8}$$

With this definition now in final form we conclude this section, giving an overview of Chern-Simons quantum field theory and especially the work of Axelrod-Singer, with one last observation. It is that in 1.8 the higher loop series clearly has leading expansion

$$Z_{k+h}^{hl} = 1 + (2\pi i)\, \tilde{I}_2^{conn}(M^3, A^{(0)}, \sigma)\, (k+h)^{-1} + O((k+h)^{-2})\ . \tag{1.9}$$

In 1.7 this then explains our remark in section 1.1 that the checks on $Z_k^{SC}(M^3, A^{(0)}, \sigma)$ in [FG1] and [J1] constitute tests of the path integral against the exact Witten/TQFT solution at *leading* order, and that our new 2-loop calculations for lens spaces in this thesis represent an extension of these tests to *sub-leading* order.

We now leave generalities and end this introductory chapter by stating in precise terms the goal and organisation of the remainder of the thesis.

1.3 Outline of Thesis

1.3.1 The Precise Goal

In [J1] Lisa Jeffrey, following up the initial numerical work in [FG1], uses the TQFT definition of the Chern-Simons-Witten invariants, $Z_k(M^3, \sigma)$ to derive explicitly the closed form of these invariants for the case of lens spaces with $G = SU(2)$. She verifies that they match the leading-order asymptotics of Witten's semi-classical formula as $k \to \infty$.

In this thesis we also restrict to $G = SU(2)$ and to lens spaces (in fact, a subclass of lens spaces which we shall define shortly), but we instead compute explicitly the simplest of the *perturbative* Chern-Simons-Witten invariants in these cases, namely the 2-loop invariants, $\tilde{I}_2^{conn}$, around the *trivial flat connection* . Our aim, as remarked on several occasions, is then to use these to check agreement between the perturbative and exact TQFT definitions of Z_k to sub-leading order in the asymptotics as $k \to \infty$, thus extending the experimental semi-classical tests of the path integral undertaken by Jeffrey.

In stating this program, however, we have glossed over the fact that, as we pointed out earlier, Jeffrey's non-perturbative TQFT invariants do not depend on a choice of flat connection on the base 3-manifold, unlike our perturbative 2-loop invariants, $\tilde{I}_2^{conn}$, which are only defined relative to the trivial connection around which our perturbative expansion is being performed. This discrepancy cannot be ignored since the trivial SU(2) principal bundle, P, over a lens space generally has many (gauge-equivalence classes of) flat connections.

Fortunately, however, this difficulty is not serious. This is because Jeffrey herself, in [J2], showed how a trick involving Fourier resummation can be used to decompose her solution asymptotically into contributions from each of the different flat connections. We thus just have to be careful, when comparing our 2-loop invariants with those predicted by the sub-leading asymptotics of the exact/TQFT solution, to base the comparison *only* on the component of this solution due to the trivial connection.

1.3.2 Organisation

In chapter 2 we quote the exact definition of the 2-loop invariant, $\tilde{I}_2^{conn}(M^3, A_{triv}, \sigma)$, from [AS1] and explain the different terms present. This involves examining the Feynman graphs which arise in computing $I_2^{conn}(M^3, A_{triv}, g)$, and determining formulae for their amplitudes, aswell as giving a precise definition of the canonical biframing, σ, and the counterterm in which it appears. We end the chapter with a section introducing a variety of geometric objects and results about S^3 that we will use extensively, and defining the class of lens spaces whose 2-loop invariants we will calculate.

In chapter 3 we begin this calculation by performing the lengthy computation of the propagator for the perturbative theory on S^3, and deducing quickly from it the propagators on our lens spaces. The relevant results are proposition 3.15 and proposition 3.16 (or 3.17) respectively. Their derivation involves intensive computations on S^3 exploiting symmetries to reduce to a boundary value ODE problem, but, for those interested in simply passing directly to these key results, their correctness could alternatively be verified after the fact just by checking them against the properties (PL 0)–(PL 3) in [AS1] which uniquely characterise the propagator (after obvious adjustments to take account of the non-vanishing of cohomology in dimensions 0 and 3 in the present case).

In chapter 4 we then use these propagators in our formula from chapter 2 to calculate the graphical components of $\tilde{I}_2^{conn}$ for our lens spaces. Again this involves intensive computations and simplifications, some dependent on our restriction of the class of lens spaces under consideration. We show that these graphical contributions are made up of two non-trivial integral terms (proposition 4.9). The first of these we evaluate exactly (proposition 4.10), but for the second we are compelled to turn to numerical computations to obtain insight; these computations indicate very clearly a certain conjectural formula (conjecture 4.11), on the basis of which we then calculate the value of the second integral term exactly. Our final result (proposition 4.12) is that the graphical contributions to $\tilde{I}_2^{conn}$ turn out to be identically zero for all our lens spaces, due to exact cancellation of these two non-trivial pieces.

In chapter 5 we turn to the counterterm. We compute it easily (proposition 5.1) by invoking a relationship between it and another well-known metric invariant, the eta-invariant of Atiyah-Patodi-Singer, whose value for lens spaces can be found in the literature. In light of our results in chapter 4 this then gives us the full 2-loop invariants, $\tilde{I}_2^{conn}(M^3, A_{triv}, \sigma)$, for our lens spaces with $G = SU(2)$. Having obtained these we then finally perform the desired comparison between our values and those expected on the basis of the sub-leading asymptotics of the trivial connection contribution to the exact TQFT solution, as extracted from either [J2] or [R]. Our main result in this thesis is that *these values agree* , providing the further "experimental" support for the validity of the objects and techniques of quantum field theory discussed earlier in this chapter.

Chapter 2

The 2-loop Invariant in detail and Some Computational Preliminaries

2.1 Precise Definition of $\tilde{I}_2^{conn}(M^3, A_{triv}, \sigma)$

Quoting from Corollary 5.6 of [AS1], with a trivial adjustment to accomodate our use of the

canonical biframing rather than the arbitrary framing s, the 2-loop invariant $\tilde{I}_2^{conn}(M^3, A_{triv}, \sigma)$ is defined by

$$\tilde{I}_2^{conn}(M^3, A_{triv}, \sigma) = I_2^{conn}(M^3, A_{triv}, g) - \frac{h\, dim(G)}{48} CS_{grav}(g, \sigma) . \qquad (2.1)$$

We refer to $I_2^{conn}(M^3, A_{triv}, g)$ as the graphical piece of this formula, and $\frac{h\, dim(G)}{48} CS_{grav}(g, \sigma)$ as the counterterm. Note that we are using here that Axelrod-Singer's formula for the 2-loop invariant, obtained in the acyclic setting in [AS1], needs no amendment at all to take account of non-vanishing H^0. This fact is remarked upon in [AS1] (section 6, part II, remark (i)), with an unpublished proof by Axelrod (personal communication).

2.1.1 The Graphical piece

As defined in chapter 1, $I_2^{conn}(M^3, A_{triv}, g)$ is the sum of the Feynman amplitudes of the connected vacuum-vacuum 2-loop graphs in the theory, evaluated using the propagator and the Feynman rules, which incorporate Axelrod-Singer's point-splitting regularisation.

Since we only have a single trivalent vertex in the theory there are precisely two such graphs, the well-known dumbbell graph and sunset graph (for those unfamiliar with these terms, see [AS1] where they are depicted explicitly). We now discuss them in turn.

Let us consider the sunset graph first. An expression for its amplitude, I_{sunset}, has already been computed in the explicit formula in equation (5.89) of [AS1]. Note, however, that in quoting this formula we will be implicitly adopting, as we shall throughout the remainder of the thesis to facilitate easy reference, two conventions from [AS1] that need remark. The first is the obvious notational convention of distinguishing a particular copy of M^3, and objects associated with it, by adding the name of a variable parametrizing that copy as a subscript. The second, however, is the very *unusual* policy (adopted to be compatible with their superspace conventions) of equipping $M_x^3 \times M_y^3$ with the non-standard orientation so that the positive volume form is $\mathrm{vol}_{M_y^3} \wedge \mathrm{vol}_{M_x^3}$ not $\mathrm{vol}_{M_x^3} \wedge \mathrm{vol}_{M_y^3}$. In order to retain the identity $\int_{M_x^3 \times M_y^3} = \int_{M_x^3} \int_{M_y^3}$, this in turn necessitates choosing the sign convention that

$$\int_{M_y^3} [\psi(y) \wedge \chi(x)] \equiv \left[\int_{M_y^3} \psi(y)\right] \chi(x) \quad \text{for } \chi \in \Omega^\star(M_x^3) \text{ and } \psi \in \Omega^3(M_y^3)\,, \qquad (2.2)$$

which is the opposite of the usual one, and means that the exterior derivative operator, d_x, anticommutes with $\int_{M_y^3}$ rather than commuting.

With these conventions, the promised formula for I_{sunset} from [AS1] is

$$I_{sunset} = \frac{1}{12} \int_{M_x^3 \times M_y^3} f_{a^1 b^1}{}^{c^1} f_{a^2 b^2}{}^{c^2} L_{a^1 a^2}(x,y) \wedge L_{b^1 b^2}(x,y) \wedge L_{c^1 c^2}(x,y)\,. \qquad (2.3)$$

Here the $f_{ab}{}^c$ are the structure constants of $\underline{g}$ with respect to an orthonormal basis $\{T_a\}$, i.e. $[T_a, T_b] = f_{ab}{}^c T_c$, and the $L_{ab}(x,y)$ are copies of the propagator, which, from its definition as the Green's form of $d_{A_{triv}}$ on $\Omega^\star(M^3; adP)$, is a differential form on the product space $M_x^3 \times M_y^3$ with value at (x,y) in $\mathrm{Hom}((adP)_y; (adP)_x)$. The a,b indices in $L_{ab}(x,y)$ denote matrix indices on using the isomorphism $\mathrm{Hom}((adP)_y; (adP)_x) \cong \underline{g}_x \otimes \underline{g}_y$ (discussed in detail shortly) to write the value of the propagator at (x,y) as a matrix with respect to the T_a. Finally, we are, of course, using, as we shall for the remainder of the thesis, the Einstein summation convention of summing over repeated indices.

Equation 2.3 for I_{sunset} can, however, be further simplified. This is because in [AS1] it has been derived for an *arbitrary* flat connection and group G, whereas we are dealing only with the simplest possible flat connection, A_{triv}, and $G = SU(2)$. To see how this makes matters easier let us focus first on the propagator.

As discussed in chapter 1, this is the Hodge-theoretic Green's form for the exterior covariant derivative, $d_{A_{triv}}$, on $\Omega^\star(M^3; adP)$. To understand it better we start by describing the action of $d_{A_{triv}}$ on $\Omega^\star(M^3; adP)$ more concretely.

Let $\tilde{s} : M^3 \to P : x \mapsto (x, e)$ be the canonical trivialisation of P and use it together with the $\{T_a\}$ to trivialise adP by sections $\tilde{s}_a : M^3 \to adP : x \mapsto [\tilde{s}(x), T_a]$. Then, since $\tilde{s}$ is *globally* horizontal with respect to A_{triv} on P, it follows that if we write a general element, ν, of $\Omega^\star(M^3; adP)$ in the form $\nu^a \otimes \tilde{s}_a$ with each $\nu^a \in \Omega^\star(M^3)$, the action of $d_{A_{triv}}$ becomes simply the action of the ordinary exterior derivative d on the coefficient forms; i.e.

$$d_{A_{triv}}(\nu) = (d\nu^a) \otimes \tilde{s}_a \,. \tag{2.4}$$

But the trivialisation of adP via $\tilde{s}$ and the $\{T_a\}$ that we have used here to get 2.4 is precisely just the standard identification of adP with $M^3 \times \underline{g}$ that we referred to at the beginning of section 1.2 and have used frequently already. In particular it is exactly the identification that we used above in 2.3 in writing the propagator as a matrix-valued form.

It thus follows immediately from 2.4 that for A_{triv} this matrix form of the propagator simplifies greatly, splitting *globally* into a tensor product of spatial and matrix pieces

$$L_{ab}(x,y) = L(x,y) \otimes \delta_{ab} \tag{2.5}$$

where the spatial piece , $L(x,y)$, is the Hodge-theoretic Green's form of just the ordinary exterior derivative d on $\Omega^\star(M^3)$, and the matrix component, δ_{ab}, is simply the Kronecker delta symbol.

Substituting this back into 2.3 then gives the first simplification promised in our expression for I_{sunset}, due to specialisation to the case of A_{triv}, namely

$$I_{sunset} = \frac{1}{12}\int_{M^3_x \times M^3_y} f_{a^1 b^1}{}^{c^1}\, f_{a^2 b^2}{}^{c^2}\, \delta_{a^1 a^2}\, \delta_{b^1 b^2}\, \delta_{c^1 c^2}\, L(x,y) \,\wedge\, L(x,y) \,\wedge\, L(x,y)\ . \tag{2.6}$$

The final simplification then comes from the specialisation to $G = SU(2)$. For, with the inner product on $\underline{su}(2)$ discussed in Section 1.2, it is easily seen that its structure constants with respect to an orthonormal basis are given by

$$f_{ab}{}^c = \sqrt{2}\varepsilon_{ab}{}^c\ . \tag{2.7}$$

In 2.6, our *final* expression for I_{sunset} in our setting thus reduces to simply

$$I_{sunset} = \int_{M^3_x \times M^3_y} L(x,y) \,\wedge\, L(x,y) \,\wedge\, L(x,y)\ , \tag{2.8}$$

with the group-theoretic pieces having been calculated out of the integrand.

As for the other graph, the dumbbell, for a general flat connection its amplitude is not zero. But once again, for the trivial flat connection we are using, the situation simplifies and its Feynman amplitude *is* identically zero.

This is because the point-splitting regularisation we use, described in detail in [AS1], involves an antisymmetrization in the group-theoretic indices in the propagator. In light of the global splitting of the propagator for A_{triv} in 2.5 and the symmetry of its group-theoretic piece, δ_{ab}, this leads to the regularised propagator being identically zero on the diagonal in $M^3_x \times M^3_y$. This immediately forces the integral for $I_{dumbbell}$ (See [AS1], eq. (5.88)) to vanish.

This then completes our discussion of the graphical piece in 2.1.

2.1.2 The Counterterm

This is easier to describe. To begin with, as noted in Chapter 1, the Coxeter number h is 2 for $G = SU(2)$ so that $\frac{h\,dim(G)}{48} = \frac{1}{8}$.

As for the term $CS_{grav}(g, \sigma)$, recall from chapter 1 that for this we are working not on TM^3, but on $2TM^3 \equiv TM^3 \oplus TM^3$, which we are imbuing with a natural Spin(6)-structure using the lifting to Spin(6) of the diagonal embedding: $SO(3) \hookrightarrow SO(6)$ (See [A]). On this vector bundle we now first take the connection given by the direct sum of two copies of the "gravitational" Levi-Civita connection for the metric g on TM^3. And secondly we form the canonical biframing, σ, defined uniquely by the index-theoretic requirement that, for any 4-manifold, Y, making M^3 its boundary we have

$$Sign(Y) = \frac{1}{6}\, p_1(2TY, \sigma)\,, \tag{2.9}$$

where $Sign(Y)$ is the Hirzebruch signature of Y and $p_1(2TY, \sigma)$ is the relative Pontrjagin number of $2TY$ with respect to the biframing σ on the boundary (Again see [A] for more details). Then $CS_{grav}(g, \sigma)$ refers simply to the Chern-Simons invariant of this connection evaluated in this canonical biframing σ.

2.1.3 A final expression for $\tilde{I}_2^{conn}(M^3, A_{triv}, \sigma)$

We conclude this section now by drawing the discussion in sections 2.1.1 and 2.1.2 together to obtain our *final* definition of $\tilde{I}_2^{conn}$, simplifying 2.1.

Definition 2.1 *The 2-loop invariant* $\tilde{I}_2^{conn}(M^3, A_{triv}, \sigma)$, *in the case* $G = SU(2)$, *is given by simply*

$$\tilde{I}_2^{conn}(M^3, A_{triv}, \sigma) = \int_{M_x^3 \times M_y^3} L(x,y) \wedge L(x,y) \wedge L(x,y) \; - \; \frac{1}{8} CS_{grav}(g, \sigma) \tag{2.10}$$

where $L(x,y)$ *is the Hodge-theoretic Green's form of the ordinary exterior derivative* d *on* $\Omega^*(M^3)$ *and* $CS_{grav}(g, \sigma)$ *is as described in section 2.1.2.*

The bulk of the remainder of this thesis is then concerned with using this definition to actually compute $\tilde{I}_2^{conn}(M^3, A_{triv}, \sigma)$ for S^3 and lens spaces, in order to extend existing tests of the partition function path integral, as discussed in Chapter 1. With this in mind, we end this chapter with a section preparing for these computations. In it we describe various geometric objects and results regarding these spaces that we will use repeatedly.

2.2 Preliminaries on S^3 and Lens Spaces

Since even our computations for lens spaces will generally be reduced to computations on S^3, we start by setting up a variety of different coordinate systems, identifications, and associations between objects on S^3 that we will need. We turn to lens spaces themselves only at the end of the section.

2.2.1 Coordinates

We take standard coordinates on $\Re^4$ as $w^1, \ldots, w^4$ and S^3 as the submanifold $\left\{(w^1, w^2, w^3, w^4) \mid (w^1)^2 + (w^2)^2 + (w^3)^2 + (w^4)^2 = 1\right\}$, with metric g the standard metric induced from $\Re^4$.

Two sets of coordinates on S^3 will be used. The first is standard spherical polars (α, ϕ, θ) ; $\alpha \in [0, \pi]$ is the angle down from the "North pole" $N = (0,0,0,1)$, and for any given $\alpha_0 \in (0, \pi)$, (ϕ, θ) are the standard polar coordinates on the 2-sphere of radius $\sin \alpha_0$ obtained by slicing S^3 at height $w^4 = \cos \alpha_0$.

In many ways these are not good coordinates. They are singular at $\alpha = 0$ and π (the North and South poles, N and S), and, for any $\alpha \in (0, \pi)$, at $\phi = 0$ and π. Thus they break down as coordinates on the entire great circle $\{(w^1, w^2, w^3, w^4) \in S^3 \mid w^1 = w^2 = 0\}$. Nonetheless, by taking appropriate care, we will use them extensively.

The second set of coordinates is stereographic rectangular coordinates on $S^3 \backslash \{S\}$, obtained by stereographic projection from S onto $T_N S^3 \equiv \Re^3$. We denote these coordinates v^1, v^2, v^3 and define $\tilde{r} \equiv \sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2}$.

These are good coordinates on the entire patch $S^3 \setminus \{S\}$, but they are somewhat more cumbersome than spherical polars, which is why we often still opt to work in these latter coordinates.

The relationship of these two coordinate systems to each other and to the ambient w^i-coordinates can then easily be deduced. We have, on the one hand, that

$$\begin{aligned} w^1 &= \sin\alpha\sin\phi\cos\theta\ , \\ w^2 &= \sin\alpha\sin\phi\sin\theta\ , \\ w^3 &= \sin\alpha\cos\phi\ , \ \textit{and} \\ w^4 &= \cos\alpha\ , \end{aligned} \tag{2.11}$$

and on the other that

$$\tilde{r} = 2\tan\left(\frac{\alpha}{2}\right) \quad \textit{and} \quad \begin{cases} v^1 &= \tilde{r}\sin\phi\cos\theta \\ v^2 &= \tilde{r}\sin\phi\sin\theta \\ v^3 &= \tilde{r}\cos\phi\ . \end{cases} \tag{2.12}$$

Note that $\tilde{r} = 2\tan\left(\frac{\alpha}{2}\right)$ implies that

$$\alpha = 2\arctan\left(\frac{\tilde{r}}{2}\right)\ , \ \textit{and thus} \quad \cos\alpha = \frac{4-\tilde{r}^2}{\tilde{r}^2+4} \quad \textit{and} \quad \sin\alpha = \frac{4\tilde{r}}{\tilde{r}^2+4}\ . \tag{2.13}$$

Expressions for the metric g in index notation with respect to each of our two coordinate systems will also prove useful. In spherical polars g is easily seen to be given by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\alpha & 0 \\ 0 & 0 & \sin^2\alpha\sin^2\phi \end{pmatrix} \quad \textit{and} \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \csc^2\alpha & 0 \\ 0 & 0 & \csc^2\alpha\csc^2\phi \end{pmatrix}\ , \tag{2.14}$$

while, on observing that $\frac{4}{\tilde{r}^2+4} = \frac{1+\cos\alpha}{2}$, it is equally easy to deduce that in stereographic coordinates g is just given by

$$g_{ij} = \left(\frac{4}{\tilde{r}^2+4}\right)^2 \delta_{ij} \quad \textit{and} \quad g^{ij} = \left(\frac{\tilde{r}^2+4}{4}\right)^2 \delta^{ij}\ . \tag{2.15}$$

Note that stereographic projection is conformal.

Finally, from these expressions for the metric we can also write down at once expressions that we will use for the volume form, vol_{S^3}, and Hodge-star operator, $*$, in each coordinate system. We have that

$$\mathrm{vol}_{S^3} = \sin^2\alpha \sin\phi \; d\alpha\wedge d\phi\wedge d\theta = \left(\frac{4}{\tilde{r}^2+4}\right)^3 dv^1\wedge dv^2\wedge dv^3 \; , \tag{2.16}$$

while $*$ is given, in spherical polars, by

$$*1 = \mathrm{vol}_{S^3}, \quad \begin{array}{l} *d\alpha = \sin^2\alpha\sin\phi\; d\phi\wedge d\theta \\ *d\phi = \sin\phi\; d\theta\wedge d\alpha \\ *d\theta = \csc\phi\; d\alpha\wedge d\phi \end{array} \quad , \quad \begin{array}{l} *(d\alpha\wedge d\phi) = \sin\phi\; d\theta \\ *(d\phi\wedge d\theta) = \csc^2\alpha\csc\phi\; d\alpha \\ *(d\theta\wedge d\alpha) = \csc\phi\; d\phi \end{array} \; , \; *\mathrm{vol}_{S^3} = 1 \; , \tag{2.17}$$

and in stereographic coordinates by the same formulae for 0 and 3-forms and

$$*dv^i = \frac{1}{2}\left(\frac{4}{\tilde{r}^2+4}\right)\varepsilon^i{}_{jk}\, dv^j\wedge dv^k, \; *\left(dv^i\wedge dv^j\right) = \left(\frac{\tilde{r}^2+4}{4}\right)\varepsilon^{ij}{}_k\, dv^k \; . \tag{2.18}$$

2.2.2 Group Structure

We will also use crucially in our computations that S^3 has a natural group structure,

$$S^3 \cong SU(2) \equiv \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in GL_2(\mathbf{C}) \; : \; |a|^2+|b|^2 = 1 \right\}. \tag{2.19}$$

The explicit identification of S^3 with $SU(2)$ that we choose is

$$S^3 \ni (w^1,w^2,w^3,w^4) \longleftrightarrow \begin{pmatrix} (w^4+iw^3) & (w^1+iw^2) \\ (-w^1+iw^2) & (w^4-iw^3) \end{pmatrix} \in SU(2). \tag{2.20}$$

This is not, perhaps, the easiest or most natural identification. It has been arranged in this way simply so that the identity element of $SU(2)$ corresponds to the North pole, $N = (0,0,0,1)$, while preserving orientation (relative to the standard orientations on SU(2) and S^3). Under it, inversion in SU(2) corresponds to the simple map $(w^1,w^2,w^3,w^4) \mapsto (-w^1,-w^2,-w^3,w^4)$, and even more importantly, the metric g turns out to be bi-invariant. We will return to this in a moment.

First, however, we use the group structure on S^3 to introduce global left-invariant vector fields and dual 1-forms.

Definition 2.2 *Let* $\{X_i\}_{i=1}^3$ *be the left-invariant vector fields on* S^3 *obtained by defining* $(X_i)_N \equiv \frac{\partial}{\partial w^i} \in T_N S^3$ *for* $i = 1, 2, 3$ *and left-translating, and let* $\{\theta^i\}_{i=1}^3$ *be the dual left-invariant 1-forms.*

Note that our choice here to introduce *left-invariant* objects in utilising the group structure of S^3 is, of course, arbitrary. We could equally well use the corresponding right-invariant quantities in everything that follows.

We will use the X_i and θ^i extensively in our calculations because they are *globally* defined. They thus avoid the singularity problems which arise with the coordinate vector fields and 1-forms in either of our coordinate systems, making computation immeasurably easier.

As elements of the Lie algebra $\underline{su}(2)$, it is easy to see that the X_i correspond under 2.20 to

$$X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \textit{ and } \quad X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \tag{2.21}$$

and so satisfy the structure relations

$$[X_i, X_j] = 2\varepsilon_{ij}{}^k X_k \; . \tag{2.22}$$

Dually, it then follows from the Maurer-Cartan relations that our left-invariant 1-forms satisfy

$$d\theta^i = -\varepsilon^i{}_{jk}\theta^j \wedge \theta^k \; . \tag{2.23}$$

We would like, now, to relate group-structure and coordinates by expressing these left-invariant objects, X_i and θ^i, in terms of the coordinate systems discussed in the previous section. Such expressions will frequently prove useful in our computations. They can be obtained by a series of routine, though rather long and tedious, computations. We simply quote the relevant formulae here, leaving their derivations to the reader.

Lemma 2.3 *Under the identification in 2.20 the vector fields* $\{X_i\}_{i=1}^3$ *and 1-forms* $\{\theta^i\}_{i=1}^3$ *are given in terms of spherical polars and stereographic coordinates (on the* *domains where each of these coordinate systems is well-defined) by*

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} (\sin\phi\cos\theta) & \begin{pmatrix} \cot\alpha\cos\phi\cos\theta \\ +\sin\theta \end{pmatrix} & \begin{pmatrix} \cot\phi\cos\theta - \\ \cot\alpha\csc\phi\sin\theta \end{pmatrix} \\ (\sin\phi\sin\theta) & \begin{pmatrix} \cot\alpha\cos\phi\sin\theta \\ -\cos\theta \end{pmatrix} & \begin{pmatrix} \cot\phi\sin\theta + \\ \cot\alpha\csc\phi\cos\theta \end{pmatrix} \\ \cos\phi & (-\cot\alpha\sin\phi) & -1 \end{pmatrix} \begin{pmatrix} \partial_\alpha \\ \partial_\phi \\ \partial_\theta \end{pmatrix} \tag{2.24}$$

$$= \begin{pmatrix} (1-\frac{\bar{r}^2}{4}+\frac{(v^1)^2}{2}) & (v^3+\frac{1}{2}v^1v^2) & (-v^2+\frac{1}{2}v^1v^3) \\ (-v^3+\frac{1}{2}v^1v^2) & (1-\frac{\bar{r}^2}{4}+\frac{(v^2)^2}{2}) & (v^1+\frac{1}{2}v^2v^3) \\ (v^2+\frac{1}{2}v^1v^3) & (-v^1+\frac{1}{2}v^2v^3) & (1-\frac{\bar{r}^2}{4}+\frac{(v^3)^2}{2}) \end{pmatrix} \begin{pmatrix} \partial_{v^1} \\ \partial_{v^2} \\ \partial_{v^3} \end{pmatrix}, \tag{2.25}$$

and therefore, dually,

$$\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} = \begin{pmatrix} (\sin\phi\cos\theta) & \begin{pmatrix} \cot\alpha\cos\phi\cos\theta \\ +\sin\theta \end{pmatrix} & \begin{pmatrix} \cot\phi\cos\theta - \\ \cot\alpha\csc\phi\sin\theta \end{pmatrix} \\ (\sin\phi\sin\theta) & \begin{pmatrix} \cot\alpha\cos\phi\sin\theta \\ -\cos\theta \end{pmatrix} & \begin{pmatrix} \cot\phi\sin\theta + \\ \cot\alpha\csc\phi\cos\theta \end{pmatrix} \\ \cos\phi & (-\cot\alpha\sin\phi) & -1 \end{pmatrix} \begin{pmatrix} d\alpha \\ \sin^2\alpha\, d\phi \\ \sin^2\alpha \\ \sin^2\phi\, d\theta \end{pmatrix} \tag{2.26}$$

$$= \left(\frac{4}{\bar{r}^2+4}\right)^2 \begin{pmatrix} \left(1-\frac{\bar{r}^2}{4}+\frac{(v^1)^2}{2}\right) & \left(v^3+\frac{1}{2}v^1v^2\right) & \left(-v^2+\frac{1}{2}v^1v^3\right) \\ \left(-v^3+\frac{1}{2}v^1v^2\right) & \left(1-\frac{\bar{r}^2}{4}+\frac{(v^2)^2}{2}\right) & \left(v^1+\frac{1}{2}v^2v^3\right) \\ \left(v^2+\frac{1}{2}v^1v^3\right) & \left(-v^1+\frac{1}{2}v^2v^3\right) & \left(1-\frac{\bar{r}^2}{4}+\frac{(v^3)^2}{2}\right) \end{pmatrix} \begin{pmatrix} dv^1 \\ dv^2 \\ dv^3 \end{pmatrix}. \tag{2.27}$$

This lemma, in turn, then yields as an immediate corollary our important earlier assertion that *the metric g is bi-invariant*. For equation 2.24 for the X_i, together with 2.14, implies at once that the X_i are orthonormal *everywhere* on S^3. Thus left-translation preserves the orthonormality of the basis $\{(\partial_{w^i})_N\}_{i=1}^3$ in $T_N S^3$, which proves the left-invariance of the metric. And identical computations can likewise be performed for the corresponding right-invariant quantities to establish the metric's right-invariance.

Two useful consequences, moreover, then flow directly from this;

i) Since the X_i are orthonormal, so are the θ^i, and so the volume-form and Hodge-star are particularly simple when expressed in terms of the θ^i, namely

$$\mathrm{vol}_{S^3} = \theta^1 \wedge \theta^2 \wedge \theta^3 \ , \tag{2.28}$$

and

$$*1 = \mathrm{vol}_{S^3}, \quad *\theta^i = (\theta \wedge \theta)^{(i)} \ , \quad *(\theta \wedge \theta)^{(i)} = \theta^i \ , \ \textit{and} \quad * \, \mathrm{vol}_{S^3} = 1 \ , \tag{2.29}$$

where, in 2.29, we are adopting, as we shall for the rest of the thesis, the natural cyclic notation of writing

$$(\theta \wedge \theta)^{(i)} \equiv \frac{1}{2} \varepsilon^i{}_{jk} \theta^j \wedge \theta^k \ , i = 1, 2, 3. \tag{2.30}$$

ii) For any $h \in SU(2)$, the two natural maps of left-translation and right-translation by h are *isometries* of $S^3 \equiv SU(2)$. Since the isometry group of S^3 is $SO(4)$, we can therefore think of them as elements of $SO(4)$, denoting them by $\mathcal{L}_h$ and $\mathcal{R}_h$ respectively. We thus have two natural embeddings,

$$\mathcal{L} : SU(2) \hookrightarrow SO(4) : h \mapsto \mathcal{L}_h \ , \tag{2.31}$$

and

$$\mathcal{R} : SU(2) \hookrightarrow SO(4) : h \mapsto \mathcal{R}_h \ . \tag{2.32}$$

We denote the images of $SU(2)$ inside $SO(4)$ under these embeddings by $SU(2)_L$ and $SU(2)_R$. Note that clearly $SU(2)_L$ and $SU(2)_R$ commute inside $SO(4)$ since left and right translation commute on SU(2).

A quick calculation using the identification 2.20, moreover, gives us these embeddings explicitly; if $h = (w^1, w^2, w^3, w^4)$, then we find that

$$\mathcal{L}_h = \begin{pmatrix} w^4 & -w^3 & w^2 & w^1 \\ w^3 & w^4 & -w^1 & w^2 \\ -w^2 & w^1 & w^4 & w^3 \\ -w^1 & -w^2 & -w^3 & w^4 \end{pmatrix} \quad \textit{and} \quad \mathcal{R}_h = \begin{pmatrix} w^4 & w^3 & -w^2 & w^1 \\ -w^3 & w^4 & w^1 & w^2 \\ w^2 & -w^1 & w^4 & w^3 \\ -w^1 & -w^2 & -w^3 & w^4 \end{pmatrix}. \quad (2.33)$$

We shall use these concrete forms of these embeddings crucially at several different steps in our calculations. As a first example, note that they encapsulate the following useful general expressions for the $SU(2)$ product in terms of the ambient coordinates;

$$\begin{aligned} w^1_{xy} &= w^4_x w^1_y - w^3_x w^2_y + w^2_x w^3_y + w^1_x w^4_y\,, \\ w^2_{xy} &= w^3_x w^1_y + w^4_x w^2_y - w^1_x w^3_y + w^2_x w^4_y\,, \\ w^3_{xy} &= -w^2_x w^1_y + w^1_x w^2_y + w^4_x w^3_y + w^3_x w^4_y\,, \textit{and} \\ w^4_{xy} &= -w^1_x w^1_y - w^2_x w^2_y - w^3_x w^3_y + w^4_x w^4_y\,. \end{aligned} \quad (2.34)$$

We now conclude this section on group structure with one final computational lemma that will also prove extremely useful. It gives the directional derivatives of the ambient-coordinate functions, treated as elements of $C^\infty(S^3)$, in the directions of the X_i vector fields. Again its proof is by direct, if somewhat long, computation (this time applying equations 2.24 for the X_i to equations 2.11 for the ambient coordinates), and so we once more leave the details to the reader.

Lemma 2.4 *For $i, j \in \{1,2,3\}, i \neq j$, we have*

$$X_i(w^i) = w^4 \quad (\textit{no sum}), \quad X_i(w^j) = \varepsilon_i{}^j{}_k w^k, \quad \textit{and} \quad X_i(w^4) = -w^i \quad (2.35)$$

and, as a corollary of the last relation,

$$X_i(\alpha) = \csc\alpha\, w^i\,. \quad (2.36)$$

2.2.3 Lens spaces

We have now introduced all of the geometric structure on S^3 that we will use in our computations. The time has finally come to consider the lens spaces whose 2-loop invariants we aim to compute.

For this thesis we do not consider the full phylum of lens spaces (i.e. all $L(p,q)$ with $p,q \in \mathbf{Z}$, as described in, for example, [J1] or [FG1,2]), but rather just a single species, involving only one integer parameter, p, which are particularly well-adapted to our $SU(2)$ group structure on S^3. We denote these by $L[p]$, $p \geq 1$. They are defined as quotient spaces of S^3 by the action of a finite cyclic group.

Specifically, take S^3 as $SU(2)$ via 2.20, and consider the copy of $\mathbf{Z}_p$ embedded in the diagonal subgroup, generated by the element

$$z_p \equiv \begin{pmatrix} e^{\frac{2\pi i}{p}} & 0 \\ 0 & e^{\frac{-2\pi i}{p}} \end{pmatrix}. \tag{2.37}$$

We define the lens space $L[p]$ as the quotient of $SU(2)$ by the left-action of this copy of $\mathbf{Z}_p$; i.e. $L[p] \equiv SU(2)/\mathbf{Z}_p$ with elements being *left*-cosets $\mathbf{Z}_p h$, $h \in SU(2)$.

Alternatively, we can use the left-embedding in 2.33 to understand $L[p]$ in less group-theoretic terms; namely $L[p]$ is obtained by taking $S^3 \subset \Re^4$ and gluing together points related by either

$$L_{z_p} = \begin{pmatrix} c_{1,p} & -s_{1,p} & 0 & 0 \\ s_{1,p} & c_{1,p} & 0 & 0 \\ 0 & 0 & c_{1,p} & s_{1,p} \\ 0 & 0 & -s_{1,p} & c_{1,p} \end{pmatrix} \in SO(4) \tag{2.38}$$

or any power of L_{z_p}. Here $c_{1,p}$ denotes $\cos(\frac{2\pi}{p})$ and $s_{1,p}$ denotes $\sin(\frac{2\pi}{p})$.

Remarks and Notation:

(i) These lens spaces are particularly well-adapted to our $SU(2)$ group structure on S^3 because the copy of $\mathbf{Z}_p$ by which we are quotienting lies not only as a subgroup inside $SO(4)$ (using such subgroups we can get all lens spaces $L(p,q)$), but in fact lies in $SU(2)_L$ inside $SO(4)$. Hence, for example, all our left-invariant objects on S^3, such as the X_i, θ^i and g, descend naturally to any $L[p]$, where we will continue to denote them by the same symbols.

This last fact provides our first indication of the advantages of restricting our attention to the $L[p]$ family of lens spaces in this thesis. We shall defer a completely thorough discussion of this issue until chapter 4, however, where we will be better positioned to explain in full detail our reasons for limiting the class of lens spaces under consideration.

(ii) For reference, we note that in terms of the more standard notation for lens spaces as $L(p,q)$, our lens spaces $L[p]$ correspond to the spaces $L(p,p-1)$. This is easy to see from the quotient definition of the $L(p,q)$ given in [FG2] and our characterisation of $L[p]$ using L_{z_p} given in 2.38. We shall need to keep this in mind in chapter 5, when we come to extracting from the literature the exact TQFT predictions for the 2-loop invariants for comparison with our own calculated values.

(iii) Although we shall not notationally distinguish between such quantities as X_i, θ^i or g on S^3 and down on $L[p]$, it *will* prove useful to explicitly distinguish points on $L[p]$ from points on S^3, in order to avoid confusion between a point on S^3 and the point on $L[p]$ determined by its $\mathbf{Z}_p$-coset. To this end, we shall adopt a convention of writing points on $L[p]$ with a "bar" over them; i.e. for any point $x \in S^3$, the "point" $\mathbf{Z}_p x \in L[p]$ will be denoted $\bar{x}$.

Having now defined the lens spaces $L[p]$ and made these remarks, we conclude this section and the chapter by considering one final issue of obvious importance, namely the relationship between the $L[p]$ and S^3.

Clearly, from our definition, we have a natural covering map from S^3 down onto $L[p]$. Denoting this by π_p, it is then evident that π_p^* is an injection from $\Omega^\star(L[p])$ back into $\Omega^\star(S^3)$, whose image consists of those forms, μ, on S^3 which are invariant under the left-action of $\mathbf{Z}_p$, i.e. $\mathcal{L}^*_{z_p^k}\mu = \mu$ for all $k = 0, 1, \ldots, p-1$. Denoting these $\mathbf{Z}_p$-invariant forms on S^3 by $\Omega^\star_{\mathbf{Z}_p}(S^3)$, we can thus define an inverse map

$$\rho : \Omega^\star_{\mathbf{Z}_p}(S^3) \longrightarrow \Omega^\star(L[p]) \tag{2.39}$$

such that $\rho \circ \pi_p^* = id$.

It is easy to see, moreover, what the definition of ρ is in explicit terms. Namely, for any $\bar{x} \in L[p]$, letting x be any one of the p preimages in $\pi_p^{-1}(\bar{x})$, then ρ is uniquely defined by the following three properties;

$$\begin{aligned}
&\text{(i) } (\rho(\mu))(\bar{x}) = \mu(x) \textit{ for all } \mu \in \Omega^0_{\mathbf{Z}_p}(S^3)\ , \\
&\text{(ii) } (\rho(\theta^i))(\bar{x}) = \theta^i_{\bar{x}}\ ,\ \textit{and} \\
&\text{(iii) } \rho(\mu_1 \wedge \mu_2) = \rho(\mu_1) \wedge \rho(\mu_2)\ .
\end{aligned} \tag{2.40}$$

Note that clearly ρ is well-defined, independent of the choice of preimage $x \in \pi_p^{-1}(\bar{x})$ used in property (i) here, since the function μ in this property is $\mathbf{Z}_p$-invariant.

The introduction of π_p^* and ρ will prove very useful in our later working in allowing us to move computations easily back and forth between $L[p]$ and S^3. In particular, we will use them in the final section of the next chapter to obtain quicklyly, from a computation of the Green's form, L, of d just on S^3, the corresponding Green's forms on each $L[p]$, $p \geq 2$; these, of course, being a key ingredient in computing the perturbative 2-loop invariants $\tilde{I}_2^{conn}(L[p], A_{triv}, \sigma)$ from 2.10, which is our whole goal in this thesis.

In this context then, we end by noting one final property of ρ and π_p^* that will be essential at a certain point in the derivation just mentioned. It is that, since π_p is a local diffeomorphism (since it is a covering map) and d a local operator, so clearly, just as $[d, \pi_p^*] = 0$, we also have

$$[d, \rho] = 0 \tag{2.41}$$

as operators on forms.

This then completes our discussion of preliminaries in this chapter.

Chapter 3

Computation of the Propagator

We now begin the actual computation of $\tilde{I}_2^{conn}(M^3, A_{triv}, \sigma)$ for the lens spaces $L[p]$ from the expression in 2.10. Starting with the "graphical" integral in 2.10 the first step is obviously to compute explicitly the Green's form, $L(x, y)$, of d on $\Omega^\star(L[p])$. This chapter is devoted entirely to performing this lengthy computation.

3.1 Initial Reduction of the Computation

It will be easy to obtain the Green's form $L(x, y)$ for the lens spaces $L[p]$ from the corresponding Green's form on S^3. So for the moment we focus on $L(x, y)$ on S^3.

Recall what is meant by this (Hodge-theoretic) Green's form for d on $\Omega^\star(S^3)$: Using the standard metric g on S^3 (note that a metric has been a key ingredient in defining $L(x, y)$ from the start, having been introduced to perform gauge-fixing on the Feynman integral and hence deduce the perturbative expansion in the first place) we have a Hodge decomposition of $\Omega^\star(S^3)$ as

$$\Omega^\star(S^3) = Imd \oplus Im\delta \oplus \mathcal{H}^\star(S^3)\ . \tag{3.1}$$

Here δ is the adjoint of d on L^2 forms relative to the metric g, and $\mathcal{H}^\star(S^3)$ is the space of harmonic forms (which on S^3 occur only in dimensions 0 and 3, where they are the 1-dimensional spaces of constant functions and constant multiples of the volume form respectively). Relative to this decomposition, the exterior derivative acts by annihilating Imd and $\mathcal{H}^\star(S^3)$, while acting as a bijective linear operator between $Im\delta$ and Imd. L is taken to be the Schwartz kernel of the composite operator $d^{-1} \circ \hat{\pi}_d$ where $\hat{\pi}_d$ is orthogonal projection onto Imd in $\Omega^\star(S^3)$. It thus implements d^{-1} from Imd onto $Im\delta$ while annihilating $Im\delta \oplus \mathcal{H}^\star(S^3)$; i.e. it satisfies the defining equations

$$\int_{S^3_y} L(x,y) \wedge d\nu(y) \;=\; \nu(x) \quad \textit{for all } x \in S^3 \textit{ and for all } \nu \in Im\delta \tag{3.2}$$

and

$$\int_{S^3_y} L(x,y) \wedge \mu(y) \;=\; 0 \quad \textit{for all } x \in S^3 \textit{ and for all } \mu \in Im\delta \oplus \mathcal{H}^\star(S^3)\,. \tag{3.3}$$

It is easy to see from these equations that, as an element of $\Omega^\star(S^3_x \times S^3_y)$, L has total degree 2 and so consists of three pieces

$$L = L_{0,2} + L_{1,1} + L_{2,0} \tag{3.4}$$

where each $L_{i,j} \in \Omega^{i,j}(S^3_x \times S^3_y)$, $i+j = 2$, is the Schwartz kernel of d^{-1} on the subspace $\Omega^{3-j}(S^3)$.

To obtain L, however, we shall work indirectly. Motivating our approach is the observation in [AS1] that the Hodge theory inverse, d^{-1}, just described is given concretely by

$$d^{-1} = \delta \circ \Delta^{-1} \tag{3.5}$$

where $\Delta = d\delta + \delta d$ is the Hodge Laplacian for the metric g. This observation allows us to reduce the problem of finding the Green's form of the operator d (requiring delicate handling of its infinite-dimensional kernel) to the easier problem of finding the Green's form of Δ, which is elliptic and whose kernel, $\mathcal{H}^\star(S^3)$, consisting just of the trivial one-dimensional subspaces described above, can be handled relatively straightforwardly. Once we have the Green's form of Δ on S^3, which we denote $G(x,y)$, we can then obtain $L(x,y)$ simply by applying δ to it in the x-variable (with some care as to what is meant by this last statement).

We thus turn now to computing $G(x,y)$.

3.2 Computing the Green's form $G(x,y)$ of $\triangle$ on S^3

In analogy with the preceding discussion of L we are really considering G precisely as the Schwartz kernel of the composite operator $\triangle^{-1} \circ \hat{\pi}_{\mathcal{H}^\star(S^3)^\perp}$ where $\hat{\pi}_{\mathcal{H}^\star(S^3)^\perp}$ is projection onto the orthogonal complement of $\mathcal{H}^\star(S^3)$ (i.e. onto $Imd \oplus Im\delta$). Just as for L, it is then easy to see that, as an element of $\Omega^\star(S^3_x \times S^3_y)$, G has total degree 3 and so is a sum of four pieces

$$G = G_{0,3} + G_{1,2} + G_{2,1} + G_{3,0} \tag{3.6}$$

with each $G_{i,j} \in \Omega^{i,j}(S^3_x \times S^3_y)$, $i+j=3$, being the Schwartz kernel of $\triangle^{-1}$ on the subspace $\Omega^{3-j}(S^3)$. Its defining equations are

$$\int_{S^3_y} G(x,y) \wedge \triangle\nu(y) \;=\; \nu(x) \quad for\, all\, x \in S^3\, and\, for\, all\, \nu \in Imd \oplus Im\delta \tag{3.7}$$

and

$$\int_{S^3_y} G(x,y) \wedge \mu(y) \;=\; 0 \quad for\, all\, x \in S^3\, and\, for\, all\, \mu \in \mathcal{H}^\star(S^3). \tag{3.8}$$

Note, however, that with the ultimate goal of determining L in mind we can ignore equation 3.8. This is because 3.7 already determines G completely up to an element of $ker(\triangle_y)$ (as an operator on $\Omega^\star(S^3_x \times S^3_y)$), i.e. up to a constant multiple of $vol_{S^3_x}$ or $vol_{S^3_y}$. Since we will in any case be acting on G by δ in the x-variable to obtain L, and δ annihilates these free terms, we need not worry about tying down these free terms using 3.8. We thus consider only how to solve equation 3.7 and we start with the 0,3 piece $G_{0,3}(x,y)$.

3.2.1 Computing $G_{0,3}(x,y)$ on S^3

Here we may simply consult the literature regarding the Green's function of $\triangle$ on functions on a Lie group, since $S^3 \equiv SU(2)$. Fixing $x = N$ initially, it follows by a very short computation from the formula in [H], pp 316 that the fundamental solution

$G_{0,3}(N,y)$ is given explicitly by

$$G_{0,3}(N,y) = \begin{cases} \frac{1}{4\pi^2}\left[(\pi - \alpha_y)\cot(\alpha_y) + 1\right] vol_{S^3_y}\,, & y \in S^3 \setminus \{N,S\} \\ 0\,, & y = S = (0,0,0,-1)\,. \end{cases} \tag{3.9}$$

Remarks:

(i) Note that $G_{0,3}(N,y)$ depends only on the α_y coordinate of y in spherical polars. This reflects the invariance of Δ under the isotropy subgroup of N inside the full isometry group, $SO(4)$, of S^3, in light of the following well-known result;

Theorem 3.1 *Let M be a smooth, oriented manifold, P an operator on $\Omega^\star(M)$ with Green's form $p(x,y) \in \Omega^\star(M_x \times M_y)$, and ϕ a diffeomorphism of M. Suppose that P commutes with pull-back by ϕ as operators on $\Omega^\star(M)$; i.e.*

$$[P, \phi^*] = 0\,. \tag{3.10}$$

It follows that

(a) If ϕ is orientation-preserving then p is invariant under pull-back by the diffeomorphism $\phi \times \phi$ of $M_x \times M_y$; i.e.

$$(\phi \times \phi)^* p = p \tag{3.11}$$

(b) If ϕ is orientation-reversing then p is anti-invariant under pull-back by the diffeomorphism $\phi \times \phi$ of $M_x \times M_y$; i.e.

$$(\phi \times \phi)^* p = -p\,. \tag{3.12}$$

We shall use this theorem again many times in this chapter.

(ii) Note also that $G_{0,3}(N,y)$ has the correct asymptotic singularity as $y \to N$. Specifically, as $y \to N, \alpha_y \to 0$ and $G_{0,3}(N,y) \sim \left[\frac{1}{4\pi\alpha_y} + O(\alpha_y^0)\right] vol_{S^3_y} \sim \left[\frac{1}{4\pi\tilde{r}} + O(\tilde{r}^0)\right] vol_{\Re^3_y}$ where $\tilde{r}$ is distance from the origin in stereographic $\Re^3$, as defined in chapter 2. Thus $G_{0,3}(N,y)$ has the same asymptotic singularity as the fundamental solution centred at 0 for the flat Laplacian on stereographic $\Re^3$, namely $\frac{1}{4\pi\tilde{r}} vol_{\Re^3_y}$. This is as it should be since in a neighbourhood of N the expansion of

$\triangle$ in stereographic coordinates has leading order term equal to the flat Laplacian on $\Re^3$.

As in remark (i) we shall also use asymptotic comparisons with the Green's form of the flat Laplacian on $\Re^3$, similar to the one just performed, on many other occasions in the remainder of this chapter.

(iii) It is easy to check that the fundamental solution $G_{0,3}(N,y)$ in 3.9 is smooth across $y = S$, as it has to be. This can be seen by expanding $G_{0,3}(N,y)$ around $y = S$ in $\tilde{\alpha}_y \equiv (\pi - \alpha_y)$ and observing that it is an even power series in $\tilde{\alpha}_y$ starting at $O(\tilde{\alpha}_y^2)$.

(iv) Finally, observe that $G_{0,3}(N,y)$ satisfies the equation

$$\triangle_{S^3_y} G_{0,3}(N,y) = \left[\frac{-1}{2\pi^2} + \delta_N(y)\right] vol_{S^3_y} \tag{3.13}$$

where $\delta_N(y)$ is the delta-function at N. The extra $\frac{-1}{2\pi^2}$ in this standard Green's function equation arises because of the presence of cohomology, i.e. harmonic 0-forms, on S^3 and the fact that $\triangle$ is only invertible on the orthogonal complement of $\mathcal{H}^0(S^3)$ in $\Omega^0(S^3)$. More concretely, the $\frac{-1}{2\pi^2}$ represents $\frac{-1}{Volume(S^3)}$ and guarantees that we still get the right result on applying integration by parts to the left-hand side of the equation

$$\int_{S^3_y} G_{0,3}(N,y) \wedge \triangle(1) \quad = \quad 0 \tag{3.14}$$

which expresses the non-invertibility of $\triangle$ on $\mathcal{H}^0(S^3)$.

Now, returning to the main task, it is easy to go from the fundamental solution 3.9 to the general expression for $G_{0,3}(x,y)$ when x is arbitrary. For given any $(x,y) \in S^3_x \times S^3_y$ take $\phi \in SO(4)$ such that $\phi(x) = N$; e.g. for convenience, take $\phi = \mathcal{L}_{x^{-1}}$. Then $(\phi \times \phi)^* vol_{S^3_y} = vol_{S^3_y}$ and so, observing that $[\triangle, \phi^*] = 0$ since ϕ is an isometry, we obtain by applying theorem 3.1 that

Result 3.2 *For x, $y \in S^3$, $x \neq y$, the 0,3-piece of G is given by*

$$G_{0,3}(x,y) = (d(\phi \times \phi)_{(x,y)})^t G_{0,3}(N, x^{-1}y) = \frac{1}{4\pi^2}\left[(\pi - \alpha_{x^{-1}y})\cot(\alpha_{x^{-1}y}) + 1\right] vol_{S^3_y} \tag{3.15}$$

where, in view of remark (iii) above, this expression is understood to be smooth everywhere away from the diagonal $x = y$ and hence to be defined by extension as 0 when $\alpha_{x^{-1}y} = \pi$ (where the expression 3.15 is not formally well-defined).

Remark: Note that although the choice of ϕ here is by no means unique, for all such ϕ we have $\alpha_{\phi(y)}$ the same, namely $\alpha_{x^{-1}y}$. This is because all such angles represent the geodesic distance between x and y on S^3, which is invariant under $SO(4)$-action. In light of our previous remark (i), the non-uniqueness of ϕ does not, therefore, introduce any ambiguity into the definition of $G_{0,3}(x,y)$.

We now turn to $G_{1,2}(x,y)$, the Green's form of the Laplacian on 1-forms on S^3.

3.2.2 Computing $G_{1,2}(x,y)$ on S^3

As far as we know the computation of $G_{1,2}(x,y)$ on S^3 is only tackled in the literature from a representation-theoretic point of view, in [F], where an expression is obtained by describing precisely all the eigenvalues and eigenforms of $\triangle$ on $\Omega^1(S^3)$ using the representation theory of $SO(4)$. We shall not use this work at all, however, since we need a more concrete, closed-form expression in terms of spherical polar coordinates and left-invariant 1-forms (along the lines of the one just obtained for $G_{0,3}$), in order to facilitate subsequent calculation of the integral in 2.10.

Unfortunately we will find that obtaining such a closed-form expression for $G_{1,2}$ (or rather getting sufficiently close to one to permit derivation of L — more on this later) is much harder than was obtaining $G_{0,3}$ and will require long and intensive computations. We break these computations into a sequence of steps.

As before, we begin by fixing $x = N$ and trying to compute the fundamental solution $G_{1,2}(N,y)$ satisfying the defining equation (from 3.7);

$$\int_{S^3_y} G_{1,2}(N,y) \wedge \triangle\nu(y) \;=\; \nu(N) \quad \textit{for all } \nu \in \Omega^1(S^3). \tag{3.16}$$

Step 1: Rewriting Equation 3.16 in Component Form

We expand the forms appearing in equation 3.16 in terms of their components with respect to the global basis of left-invariant 1-forms on S^3, $\{\theta^i\}_{i=1}^3$, introduced in chapter 2. We thus write

$$\nu(y) = \nu_i(y)\theta_y^i \,,\ \triangle\nu(y) = (\triangle\nu)_i(y)\theta_y^i \ \ and\ \ G_{1,2}(N,y) = A_{ij}(y)\theta_N^i\wedge(\theta_y\wedge\theta_y)^{(j)} \quad (3.17)$$

with each $\nu_i \in C^\infty(S^3)$ and with the notation $(\theta_y \wedge \theta_y)^{(i)}$ introduced in chapter 2. Writing the 1-forms as column vectors, and recalling the unorthodox sign convention for mixed integrals adopted in chapter 1, 3.16 then becomes the matrix integral equation

$$\int_{S_y^3} \begin{pmatrix} A_{11}(y) & A_{12}(y) & A_{13}(y) \\ A_{21}(y) & A_{22}(y) & A_{23}(y) \\ A_{31}(y) & A_{32}(y) & A_{33}(y) \end{pmatrix} \begin{pmatrix} (\triangle\nu)_1(y) \\ (\triangle\nu)_2(y) \\ (\triangle\nu)_3(y) \end{pmatrix} vol_{S_y^3} = - \begin{pmatrix} \nu_1(N) \\ \nu_2(N) \\ \nu_3(N) \end{pmatrix} \quad (3.18)$$

for all $\nu \in \Omega^1(S^3)$. To complete the recasting of 3.16 in components it now only remains to express each of the components $(\triangle\nu)_i$ in 3.18 just in terms of the component functions ν_i. This is accomplished by the following lemma using the left-invariant vector fields, X_i, dual to the θ^i;

Lemma 3.3 *The Laplacian on $C^\infty(S^3)$ is given by*

$$\triangle f = -X_i(X_i(f)) \quad for\ all\ f \in C^\infty(S^3)\,, \quad (3.19)$$

and hence the Laplacian on $\Omega^1(S^3)$ is given in matrix form by

$$\begin{pmatrix} (\triangle\nu)_1 \\ (\triangle\nu)_2 \\ (\triangle\nu)_3 \end{pmatrix} = \begin{pmatrix} (\triangle+4) & 2X_3 & -2X_2 \\ -2X_3 & (\triangle+4) & 2X_1 \\ 2X_2 & -2X_1 & (\triangle+4) \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \ for\ all\ \nu = \nu_i\theta^i \in \Omega^1(S^3). \quad (3.20)$$

Proof: Using our basic results from chapter 2 we have that for any $f \in C^\infty(S^3)$,

$$\begin{aligned}\triangle f &= \delta df = - * d * df \\ &= - * d * [X_i(f)\theta^i] = - * d\left[X_i(f)(\theta\wedge\theta)^{(i)}\right] \\ &= - * \left\{[X_i(X_i(f))]\, vol_{S_y^3}\right\} = -X_i(X_i(f))\end{aligned}$$

proving the first formula.

To prove the second, suppose initially that ν just has the form $\nu = \nu_1\theta^1$. Then the formula just proven and our results from chapter 2 once again yield that

$$\begin{aligned}
\triangle(\nu_1\theta^1) &= (d\delta + \delta d)(\nu_1\theta^1) = (-d*d* + *d*d)(\nu_1\theta^1) \\
&= -d*d\left[\nu_1(\theta\wedge\theta)^{(1)}\right] \; + \; *d*\begin{bmatrix} -X_2(\nu_1)(\theta\wedge\theta)^{(3)} + X_3(\nu_1)(\theta\wedge\theta)^{(2)} \\ -2\nu_1(\theta\wedge\theta)^{(1)} \end{bmatrix} \\
&= -d*\left[X_1(\nu_1)vol_{S^3_y}\right] \; + \; *d\left[-X_2(\nu_1)\theta^3 + X_3(\nu_1)\theta^2 - 2\nu_1\theta^1\right] \\
&= -d\left[X_1(\nu_1)\right] \; + \\
&\quad *\begin{bmatrix} X_1(X_2(\nu_1))(\theta\wedge\theta)^{(2)} & - & X_2(X_2(\nu_1))(\theta\wedge\theta)^{(1)} & + & 2X_2(\nu_1)(\theta\wedge\theta)^{(3)} \\ X_1(X_3(\nu_1))(\theta\wedge\theta)^{(3)} & - & X_3(X_3(\nu_1))(\theta\wedge\theta)^{(1)} & - & 2X_3(\nu_1)(\theta\wedge\theta)^{(2)} \\ 2X_2(\nu_1)(\theta\wedge\theta)^{(3)} & - & 2X_3(\nu_1)(\theta\wedge\theta)^{(2)} & + & 4\nu_1(\theta\wedge\theta)^{(1)} \end{bmatrix} \\
&= -\left[X_1(X_1(\nu_1))\theta^1 + X_2(X_1(\nu_1))\theta^2 + X_3(X_1(\nu_1))\theta^3\right] \\
&\qquad + \begin{bmatrix} X_1(X_2(\nu_1))\theta^2 & - & X_2(X_2(\nu_1))\theta^1 & + & 2X_2(\nu_1)\theta^3 \\ X_1(X_3(\nu_1))\theta^3 & - & X_3(X_3(\nu_1))\theta^1 & - & 2X_3(\nu_1)\theta^2 \\ 2X_2(\nu_1)\theta^3 & - & 2X_3(\nu_1)\theta^2 & + & 4\nu_1\theta^1 \end{bmatrix} \\
&= \left[(\triangle + 4)\nu_1\right]\theta^1 \; + \; \left[([X_1, X_2] - 4X_3)\nu_1\right]\theta^2 \; + \; \left[(4X_2 - [X_3, X_1])\nu_1\right]\theta^3 \\
&= \left[(\triangle + 4)\nu_1\right]\theta^1 \; - \; 2X_3(\nu_1)\theta^2 \; + \; 2X_2(\nu_1)\theta^3.
\end{aligned}$$

In identical fashion we likewise find that

$$\triangle(\nu_2\theta^2) = 2X_3(\nu_2)\theta^1 \; + \; \left[(\triangle + 4)\nu_2\right]\theta^2 \; - \; 2X_1(\nu_2)\theta^3, \qquad \text{and}$$

$$\triangle(\nu_3\theta^3) = -2X_2(\nu_3)\theta^1 \; + \; 2X_1(\nu_3)\theta^2 \; + \; \left[(\triangle + 4)\nu_3\right]\theta^3.$$

The required formula then follows simply by adding these three computations using the linearity of $\triangle$.

♣

Substituting 3.20 into 3.18 now completes the promised recasting of 3.16 in component form; namely, the fundamental solution $G_{1,2}(N, y) = A_{ij}(y)\theta^i_N \wedge (\theta_y \wedge \theta_y)^{(j)}$

satisfies the defining equation

$$\int_{S^3_y} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} (\Delta+4)\nu_1 + 2X_3(\nu_2) - 2X_2(\nu_3) \\ -2X_3(\nu_1) + (\Delta+4)\nu_3 + 2X_1(\nu_3) \\ 2X_2(\nu_1) - 2X_1(\nu_2) + (\Delta+4)\nu_3 \end{pmatrix} vol_{S^3_y} = - \begin{pmatrix} \nu_1(N) \\ \nu_2(N) \\ \nu_3(N) \end{pmatrix} \tag{3.21}$$

for all $\nu = \nu_i\theta^i \in \Omega^1(S^3)$. This equation can, of course, be further broken down as nine coupled integral equations determining the nine unknown functions $A_{ij}(y)$ making up $G_{1,2}(N,y)$: i.e.

$$\begin{aligned} \textstyle\int_{S^3_y} (A_{11}(y)((\Delta+4)u) - 2A_{12}(y)X_3(u) + 2A_{13}(y)X_2(u))\, vol_{S^3_y} &= -u(N), \textit{ and} \\ \textstyle\int_{S^3_y} (2A_{11}(y)X_3(u) + A_{12}(y)((\Delta+4)u) - 2A_{13}(y)X_1(u))\, vol_{S^3_y} &= 0, \textit{ and} \\ &\vdots \\ \textstyle\int_{S^3_y} (-2A_{31}(y)X_2(u) + 2A_{32}(y)X_1(u) + A_{33}(y)((\Delta+4)u))\, vol_{S^3_y} &= -u(N) \end{aligned} \tag{3.22}$$

for all $u \in C^\infty(S^3)$. This completes step 1.

Step 2: Applying $SO(4)$ Invariance

Trying to solve the nine coupled equations in 3.22 directly is too hard. Instead we need first to simplify the problem by using $SO(4)$ invariance to obtain *apriori* restrictions on our nine unknown functions. This will reduce us from nine such functions on S^3 to only three, each of which is, moreover, a function only of the single polar variable α_y.

To do this, recall first that, as remarked in Section 3.2.1, the Laplacian clearly commutes with the full group of isometries of S^3, namely $SO(4)$, and so, by Theorem 1 applied to the 1,2 piece of the Green's form, we have

$$G_{1,2}(x,y) = (d(B\times B)_{(x,y)})^t G_{1,2}(B(x),B(y)) \quad \textit{for all} \quad B \in SO(4). \tag{3.23}$$

Now for the moment we are just trying to solve for $G_{1,2}(N, y)$, i.e. with the x-variable fixed at N. Hence in 3.23 we cannot consider arbitrary $B \in SO(4)$, but rather only $B \in I_N$ where I_N is the isotropy subgroup of N inside $SO(4)$. Noting that clearly I_N is given by

$$I_N = \left\{ \left(\begin{array}{ccc|c} & & & 0 \\ & (\tilde{B}) & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array} \right) : \tilde{B} \in SO(3) \right\} , \tag{3.24}$$

we want to know what restrictions 3.23 places on $G_{1,2}(N, y)$, i.e. on the $A_{ij}(y)$. To answer this we clearly first need a result telling us how the left-invariant 1-forms θ^i pull back under arbitrary $B \in I_N$.

Result 3.4 *If $B \in I_N$ is as shown in 3.24 with $(\tilde{B}) = \tilde{B}^i_j$ (relative to the ambient coordinates w^1, w^2, w^3) then*

$$B^* \theta^i = \tilde{B}^i_j \, \theta^j \quad \textit{for all } i = 1, 2, 3. \tag{3.25}$$

Remark: This seems at first a surprisingly simple formula given that the $\tilde{B}^i_j$ are defined with respect to the ambient coordinates in $\Re^4$ while the θ^i are defined using left-translation with respect to the $SU(2)$ group structure on S^3. The underlying reason is that I_N is (up to $\mathbf{Z}_2$ kernel) really just $SU(2)$ itself acting by the adjoint action. Nonetheless, to build familiarity with our geometric structures and notation, and for the sake of concreteness, we shall prove result 3.4 without reference to this fact. As we will see, however , it is at the heart of the key idea (lemma 3.5) in the proof, namely the remarkably simple intertwining of the action of I_N with the group structure on S^3 under the identification 2.20.

Proof of result: Fix $y \in S^3$ (arbitrary) and let $M^i_j(y)$ be the matrix of the transformation $(dB_y)^t : T^*_{B(y)} S^3 \to T^*_y S^3$ with respect to the bases $\left\{\theta^i_{B(y)}\right\}_{i=1}^3$ and $\left\{\theta^i_y\right\}_{i=1}^3$, i.e.

$$(dB_y)^t \, \theta^i_{B(y)} = M^i_j(y) \, \theta^j_y \, . \tag{3.26}$$

Then, since $\theta^j_y = (d\mathcal{L}_{y^{-1}})^t_y \theta^j_N$ and $\theta^i_{B(y)} = (d\mathcal{L}_{B(y)^{-1}})^t_{B(y)}\theta^i_N$, equation 3.26 becomes

$$(dB_y)^t (d\mathcal{L}_{B(y)^{-1}})^t_{B(y)}\theta^i_N = M^i_j(y)(d\mathcal{L}_{y^{-1}})^t_y\theta^j_N \,,$$

$$\text{i.e.} \quad (d(\mathcal{L}_{B(y)^{-1}} \circ B)_y)^t\theta^i_N = (d\mathcal{L}_{y^{-1}})^t_y(M^i_j(y)\theta^j_N) \,,$$

$$\text{i.e.} \quad ((d\mathcal{L}_{y^{-1}})^t_y)^{-1}(d(\mathcal{L}_{B(y)^{-1}} \circ B)_y)^t\theta^i_N = M^i_j(y)\theta^j_N \,,$$

$$\text{i.e.} \quad ((d\mathcal{L}_{y^{-1}})^{-1}_y)^t(d(\mathcal{L}_{B(y)^{-1}} \circ B)_y)^t\theta^i_N = M^i_j(y)\theta^j_N \,,$$

$$\text{i.e.} \quad (d\mathcal{L}_y)^t_N(d(\mathcal{L}_{B(y)^{-1}} \circ B)_y)^t\theta^i_N = M^i_j(y)\theta^j_N \,,$$

$$\text{i.e.} \quad (d(\mathcal{L}_{B(y)^{-1}} \circ B \circ \mathcal{L}_y)_N)^t\theta^i_N = M^i_j(y)\theta^j_N \,, \tag{3.27}$$

where here we have used nothing more than simple properties of diffeomorphisms, adjoints and inverses.

To solve now for $M^i_j(y)$ from this equation we need to understand the map $\mathcal{L}_{B(y)^{-1}} \circ B \circ \mathcal{L}_y$, which, as an isometry of S^3 fixing N, lies in I_N. This is best done by using the following lemma which shows, as promised, how nicely the action of B intertwines with the $SU(2)$ group structure on S^3.

Lemma 3.5 *Under the identification 2.20 of S^3 with $SU(2)$, B acts as a homomorphism, i.e.*

$$B(y'y'') = B(y')B(y'') \quad \textit{for all } y', y'' \in SU(2). \tag{3.28}$$

Proof of Lemma: It is possible to check this simply by protracted computation, but the following is a cleaner argument.

As a map $B : SU(2) \to SU(2)$, B fixes the identity, i.e. N, and so its differential at N is a Lie algebra map $dB_N : \underline{su}(2) \to \underline{su}(2)$. Now, since $B \in SO(4)$ is a linear map its differential dB_N is just $\tilde{B}$ as a map on $T_NS^3 \equiv \Re^3$, and since at N the left-invariant vector-fields X_i are just the ambient coordinate vectors $\frac{\partial}{\partial w^i}$, $i = 1, 2, 3$, so moreover we can write down dB_N explicitly as a map with respect to the $\{X_i\}$ basis of $\underline{su}(2)$, namely

$$dB_N(X_i) = \tilde{B}^j_i X_j \quad \textit{for all } i = 1, 2, 3. \tag{3.29}$$

It follows, together with 2.22, that

$$dB_N\left([X_i, X_j]\right) = 2\varepsilon_{ij}{}^{k}\, \tilde{B}_k^m\, X_m\ ,$$

while

$$[dB_N(X_i), dB_N(X_j)] = 2\tilde{B}_i^p\, \tilde{B}_j^q\, \varepsilon_{pq}{}^{m}\, X_m.$$

But now using the fact that $\tilde{B}^t = \tilde{B}^{-1}$ and the well-known formula for the inverse matrix in terms of the transpose of the adjugate, we have the relation

$$\tilde{B}_k^m = \frac{1}{2}\varepsilon_{pq}{}^{m}\, \varepsilon_k{}^{lr}\, \tilde{B}_l^p\, \tilde{B}_r^q\ . \tag{3.30}$$

Substituting this into the preceding equations we deduce that

$$\begin{aligned}
dB_N\left([X_i, X_j]\right) &= \varepsilon_{ij}{}^{k}\, \varepsilon_{pq}{}^{m}\, \varepsilon_k{}^{lr}\, \tilde{B}_l^p\, \tilde{B}_r^q\, X_m \\
&= (\delta_i^l\, \delta_j^r - \delta_i^r\, \delta_j^l)\, \varepsilon_{pq}{}^{m}\, \tilde{B}_l^p\, \tilde{B}_r^q\, X_m \\
&= (\varepsilon_{pq}{}^{m}\, \tilde{B}_i^p\, \tilde{B}_j^q - \varepsilon_{pq}{}^{m}\, \tilde{B}_j^p\, \tilde{B}_i^q)\, X_m \\
&= 2\varepsilon_{pq}{}^{m}\, \tilde{B}_i^p\, \tilde{B}_j^q\, X_m \\
&= [dB_N(X_i), dB_N(X_j)]
\end{aligned}$$

and so the differential of B at the identity, dB_N, is in fact a Lie algebra homomorphism.

It thus follows by standard Lie theory (see e.g. [War], theorem 3.27) that there exists a unique homomorphism, say β, having dB_N as its differential at the identity, and it only remains to prove that in fact $\beta = B$.

But to see this, recall that the same standard Lie theory (see e.g. [War], theorem 3.32) also tells us explicitly what the extending group homomorphism β is, namely the conjugate of the differential at the identity, dB_N, by the exponential map; i.e.

$$\beta = exp \circ dB_N \circ exp^{-1}.$$

Since dB_N acts by $\tilde{B}$ it is easy to see from this that $\beta = B$, provided exp just acts by mapping radial lines from 0 in $\underline{su}(2)$ down onto meridians in S^3, i.e. curves corresponding to the intersection with S^3 of a 2-plane containing the w^4-axis. But this last assertion follows easily from the observation in chapter 2 that the Killing metric on $SU(2)$ is simply a multiple of the standard metric g inherited from $\Re^4$, so that the geodesics through N used in defining exp (which as ordinary matrix exponentials are defined relative to the Killing metric) are the same as the geodesics through N relative to g (which are the meridians just mentioned).

♣

With this lemma we can now quickly complete our proof of Proposition 1. We were trying to understand the isometry $\mathcal{L}_{B(y)^{-1}} \circ B \circ \mathcal{L}_y \in I_N$, and with 3.28 we can see that it is in fact just the rotation B again, independent of y, since for any $y' \in S^3 \equiv SU(2)$ we have

$$(\mathcal{L}_{B(y)^{-1}} \circ B \circ \mathcal{L}_y)(y') = B(y)^{-1}(B(yy')) = B(y)^{-1}(B(y)B(y')) = B(y').$$

In 3.27 it follows that $M^i_j(y)$ is actually independent of y and given by

$$M^i_j\, \theta^j_N = (dB_N)^t \theta^i_N = \tilde{B}^i_j\, \theta^j_N \ ,$$

the latter equality following easily from equation 3.29 regarding the action of the differential dB_N. The proposition then follows immediately from this in light of the definition of M^i_j in 3.26.

♣

Having now proven Proposition 3.4, observe also, as a corollary of it, that

$$B^* \left((\theta \wedge \theta)^{(i)}\right) = \tilde{B}^i_j\, (\theta \wedge \theta)^{(j)} \quad \textit{for all } i = 1, 2, 3\ , \tag{3.31}$$

which follows immediately on invoking relation 3.30.

With 3.25 and 3.31 we can now return and answer the question we were originally interested in, namely how I_N-invariance restricts the form of $G_{1,2}(N, y)$?

Consider the half-meridian which runs from N to S and which, for each value of $\alpha \in (0, \pi]$, intersects the 2-sphere of radius $\sin\alpha$ obtained by slicing S^3 at height $w^4 = \cos\alpha$ at the point $N_\alpha \equiv (0, 0, \sin\alpha, \cos\alpha)$ (the "North pole" of the slice). Then naively we might expect to be able to define each function A_{ij} arbitrarily at each point N_α, $\alpha \in (0, \pi]$, with the values of the A_{ij} at every other point of each slice being then obtained by using the action of I_N, which is clearly transitive on slices, to "push" the resulting 1,2-form at each N_α around the whole slice via 3.23. This would leave us still with nine arbitrary (modulo smoothness and asymptotic conditions yet to be discussed) functions A_{ij}, although each now only a function of the single polar variable $\alpha \in (0, \pi]$.

But, in fact, I_N-invariance restricts the form of the $A_{ij}(y)$ considerably more than this. The reason is that although the action of I_N on slices is transitive, it is not *free*, and, in 3.23, the 1,2-form $G_{1,2}(N, N_\alpha)$ at each N_α must therefore be invariant under the isotropy subgroup, I_{N,N_α}, of N_α inside I_N. Noting that we only need to consider this issue of isotropy invariance at the single point N_α on each slice (since it's easy to see that if $G_{1,2}(N, N_\alpha)$ is invariant under I_{N,N_α} then the full 1,2 form $G_{1,2}(N, \cdot)$ obtained by "pushing" $G_{1,2}(N, N_\alpha)$ around using I_N as described above, is invariant under the isotropy subgroup at each point of the slice) let us see how this isotropy invariance at N_α further restricts the $A_{ij}(y)$. There are two cases:

(i) $\alpha = \pi$. Then we are at the point (N, S) and the isotropy subgroup $I_{N,S}$ is all of I_N.

(ii) $\alpha \in (0, \pi)$. Then clearly the isotropy subgroup I_{N,N_α} is, for all such α, just

$$I_{N,N_\alpha} = \left\{ \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$(\breve{B})$}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) : \breve{B} \in SO(2) \right\}$$

$$= \left\{ \left(\begin{array}{ccc|c} & & & 0 \\ & (\tilde{B}) & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array} \right) : \tilde{B} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \psi \in [0, 2\pi) \right\}. \tag{3.32}$$

Case (i): Here the invariance of $G_{1,2}(N,S) = A_{ij}(S)\theta_N^i \wedge (\theta_S \wedge \theta_S)^{(j)}$ under all of I_N implies immediately, by 3.25 and 3.31, that the $A_{ij}(S)$ must satisfy

$$A_{ij}(S)\,\tilde{B}^i_k\,\tilde{B}^j_m = A_{km}(S) \quad \textit{for all } k, m = 1, 2, 3 \textit{ and for all } \tilde{B} \in SO(3).$$

Writing this as a matrix equation, this means that the matrix $A(S) \in GL(3, \Re)$ must satisfy

$$\tilde{B}^t\,A(S)\,\tilde{B} = A(S) \quad \textit{for all } \tilde{B} \in SO(3).$$

But since $\tilde{B}^t = \tilde{B}^{-1}$ for all $\tilde{B} \in SO(3)$ this implies at once that $A(S)$ must commute with all of $SO(3)$, and it follows at once that $A(S)$ must be just some constant multiple of the identity matrix.

In summary then, at $\alpha = \pi$ the $A_{ij}(S)$ are given, in matrix notation, by

$$A(S) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \quad \textit{for some constant } a \in \Re. \tag{3.33}$$

Case (ii): Here the same reasoning now yields that, for each α, the matrix $A(N_\alpha) \in GL(3, \Re)$, with entries $A_{ij}(N_\alpha)$, must commute with all $\tilde{B} \in SO(3)$ of the restricted form shown in 3.32. Writing this out in full, with the temporary abbreviation of $A_{ij}(N_\alpha)$ as just A_{ij}, this means that for all $\alpha \in (0, \pi)$ and all $\psi \in [0, 2\pi)$ we have

$$\begin{pmatrix} [A_{11}\cos\psi - A_{12}\sin\psi] & [A_{11}\sin\psi + A_{12}\cos\psi] & A_{13} \\ [A_{21}\cos\psi - A_{22}\sin\psi] & [A_{21}\sin\psi + A_{22}\cos\psi] & A_{23} \\ [A_{31}\cos\psi - A_{32}\sin\psi] & [A_{31}\sin\psi + A_{32}\cos\psi] & A_{33} \end{pmatrix} =$$

$$\begin{pmatrix} [A_{11}\cos\psi + A_{21}\sin\psi] & [A_{12}\cos\psi + A_{22}\sin\psi] & [A_{13}\cos\psi + A_{23}\sin\psi] \\ [-A_{11}\sin\psi + A_{21}\cos\psi] & [-A_{12}\sin\psi + A_{22}\cos\psi] & [-A_{13}\sin\psi + A_{23}\cos\psi] \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

from which it follows readily that

$$
\begin{aligned}
&A_{11}(N_\alpha) = A_{22}(N_\alpha) \equiv \bar{f}(\alpha), \ say \\
&and\ A_{12}(N_\alpha) = -A_{21}(N_\alpha) \equiv \bar{g}(\alpha), \ say \\
&and\ A_{13}(N_\alpha) = A_{23}(N_\alpha) = A_{32}(N_\alpha) = A_{31}(N_\alpha) = 0 \\
&and\ A_{33}(N_\alpha) \equiv \bar{h}(\alpha), \ say, \ is\ not\ further\ constrained.
\end{aligned} \tag{3.34}
$$

Combining 3.33 and 3.34 with our earlier prescription for defining $G_{1,2}(N,\cdot)$ on all of S^3 by "pushing it around" off our chosen meridian using the transitive I_N-action, we have now, in principle, applied rotational invariance to the maximum extent in restricting the allowable form of the 1,2-piece of G. It still remains, however, to work out in *concrete* terms what these constraints imply about the general form of $G_{1,2}(N,\cdot)$. Specifically, we want to determine what the explicit closed form of $G_{1,2}(N,y)$ looks like at arbitrary $y \in S^3$.

At $y = S$, 3.33 already gives us the exact form of $G_{1,2}(N,S)$, so we only need to consider $\alpha_y \in (0,\pi)$. We thus now fix some such α_y and try to determine the explicit form of $G_{1,2}(N,y)$ at the arbitrary point $y = (w_y^1, w_y^2, w_y^3, \cos\alpha_y)$ on the $w_y^4 = \cos\alpha_y$ slice of S^3.

Well, adopting the natural notation of representing $G_{1,2}(N,y)$ just by its matrix of coefficient functions, suppressing explicit mention of the forms at N and y (i.e. we write $G_{1,2}(N,y) = A_{ij}(y)\theta_N^i \wedge (\theta_y \wedge \theta_y)^{(j)}$ as just the matrix $A(y)$), we know from 3.34 that at N_{α_y} we have

$$
G_{1,2}(N, N_{\alpha_y}) = A(N_{\alpha_y}) = \begin{pmatrix} \bar{f}(\alpha_y) & \bar{g}(\alpha_y) & 0 \\ -\bar{g}(\alpha_y) & \bar{f}(\alpha_y) & 0 \\ 0 & 0 & \bar{h}(\alpha_y) \end{pmatrix} \tag{3.35}
$$

where the functions $\bar{f}$, $\bar{g}$, and $\bar{h}$ on $(0,\pi)$ must, moreover, be smooth since we know, on general grounds (see Step 4), that $G_{1,2}(N,y)$ is smooth on $S^3 \setminus \{N\}$.

Let $\tilde{B}$ be any element of $SO(3)$ such that B as defined in 3.24 maps N_{α_y} to y. Note that this means that

$$
\tilde{B}_3^i = \frac{w_y^i}{\sin\alpha_y} \quad for\ all\ i = 1,2,3. \tag{3.36}
$$

Then, by our (unambiguous) "pushing around" prescription and 3.25 and 3.31, we have that

$$G_{1,2}(N,y) = A_{ij}(N_{\alpha_y})\,(\tilde{B}^t)^i_k\,(\tilde{B}^t)^j_m\,\theta^k_N \wedge (\theta_y \wedge \theta_y)^{(m)}$$

which, in our matrix notation, is just

$$G_{1,2}(N,y) = A(y) = \tilde{B}\,A(N_{\alpha_y})\,\tilde{B}^t\,. \tag{3.37}$$

[Note that the transposes here arise because our "pushing around" is more precisely a pulling back by the inverse map.]

To compute $\tilde{B}\,A(N_{\alpha_y})\,\tilde{B}^t$ explicitly now, we make the one small simplifying observation that this product can be rewritten as

$$\tilde{B}\,A(N_{\alpha_y})\,\tilde{B}^t = \tilde{B}\,\left\{\left[A(N_{\alpha_y}),\tilde{B}^t\right] + \tilde{B}^t\,A(N_{\alpha_y})\right\} = \tilde{B}\,\left[A(N_{\alpha_y}),\tilde{B}^t\right] + A(N_{\alpha_y})\;, \tag{3.38}$$

and then simply evaluate $\tilde{B}\,\left[A(N_{\alpha_y}),\tilde{B}^t\right]$ by a straightforward, if tedious, calculation. We find that $\left[A(N_{\alpha_y}),\tilde{B}^t\right]$ equals

$$\begin{pmatrix} \bar{g}(\alpha_y)(\tilde{B}^1_2+\tilde{B}^2_1) & \bar{g}(\alpha_y)(\tilde{B}^2_2-\tilde{B}^1_1) & \begin{bmatrix} (\bar{f}(\alpha_y)-\bar{h}(\alpha_y))\tilde{B}^3_1 \\ +\bar{g}(\alpha_y)\tilde{B}^3_2 \end{bmatrix} \\ \bar{g}(\alpha_y)(\tilde{B}^2_2-\tilde{B}^1_1) & -\bar{g}(\alpha_y)(\tilde{B}^1_2-\tilde{B}^2_1) & \begin{bmatrix} (\bar{f}(\alpha_y)-\bar{h}(\alpha_y))\tilde{B}^3_2 \\ -\bar{g}(\alpha_y)\tilde{B}^3_1 \end{bmatrix} \\ \begin{bmatrix} (\bar{h}(\alpha_y)-\bar{f}(\alpha_y))\tilde{B}^1_3 \\ +\bar{g}(\alpha_y)\tilde{B}^2_3 \end{bmatrix} & \begin{bmatrix} (\bar{h}(\alpha_y)-\bar{f}(\alpha_y))\tilde{B}^2_3 \\ -\bar{g}(\alpha_y)\tilde{B}^1_3 \end{bmatrix} & 0 \end{pmatrix}$$

and so, in $\tilde{B}\left[A(N_{\alpha_y}),\tilde{B}^t\right]$, we see, by orthogonality of $\tilde{B}$ and 3.36, that the 1,1-entry is

$$\begin{aligned}(\tilde{B}\,\left[A(N_{\alpha_y}),\tilde{B}^t\right])_{11} &= \left\{\begin{array}{l} \bar{g}(\alpha_y)\left[\tilde{B}^1_1(\tilde{B}^1_2+\tilde{B}^2_1)+\tilde{B}^1_2(\tilde{B}^2_2-\tilde{B}^1_1)\right] \\ +\left[(\bar{h}(\alpha_y)-\bar{f}(\alpha_y))(\tilde{B}^1_3)^2+\bar{g}(\alpha_y)\tilde{B}^1_3\tilde{B}^2_3\right] \end{array}\right\} \\ &= \left[\frac{(\bar{h}(\alpha_y)-\bar{f}(\alpha_y))}{\sin^2\alpha_y}\right](w^1_y)^2\,.\end{aligned}$$

Likewise the 1,2-entry is

$$(\tilde{B}\left[A(N_{\alpha_y}),\tilde{B}^t\right])_{12} = \bar{g}(\alpha_y)\left[\tilde{B}_1^1(\tilde{B}_2^2-\tilde{B}_1^1)-\tilde{B}_2^1(\tilde{B}_2^1+\tilde{B}_1^2)\right]$$

$$+\left[(\bar{h}(\alpha_y)-\bar{f}(\alpha_y))\tilde{B}_3^1\tilde{B}_3^2-\bar{g}(\alpha_y)(\tilde{B}_3^1)^2\right]$$

$$= -\bar{g}(\alpha_y)+\left[\frac{\bar{g}(\alpha_y)}{\sin\alpha_y}\right]w_y^3+\left[\frac{(\bar{h}(\alpha_y)-\bar{f}(\alpha_y))}{\sin^2\alpha_y}\right]w_y^1w_y^2\ ,$$

on invoking 3.30 aswell as orthogonality of $\tilde{B}$ and 3.36 again, and in similar fashion the other seven entries are

$$(\tilde{B}\left[A(N_{\alpha_y}),\tilde{B}^t\right])_{13} = -\left[\frac{\bar{g}(\alpha_y)}{\sin\alpha_y}\right]w_y^2+\left[\frac{(\bar{h}(\alpha_y)-\bar{f}(\alpha_y))}{\sin^2\alpha_y}\right]w_y^1w_y^3\ ,$$

$$(\tilde{B}\left[A(N_{\alpha_y}),\tilde{B}^t\right])_{21} = \bar{g}(\alpha_y)-\left[\frac{\bar{g}(\alpha_y)}{\sin\alpha_y}\right]w_y^3+\left[\frac{(\bar{h}(\alpha_y)-\bar{f}(\alpha_y))}{\sin^2\alpha_y}\right]w_y^2w_y^1\ ,$$

$$(\tilde{B}\left[A(N_{\alpha_y}),\tilde{B}^t\right])_{22} = \left[\frac{(\bar{h}(\alpha_y)-\bar{f}(\alpha_y))}{\sin^2\alpha_y}\right](w_y^2)^2\ ,$$

$$(\tilde{B}\left[A(N_{\alpha_y}),\tilde{B}^t\right])_{23} = \left[\frac{\bar{g}(\alpha_y)}{\sin\alpha_y}\right]w_y^1+\left[\frac{(\bar{h}(\alpha_y)-\bar{f}(\alpha_y))}{\sin^2\alpha_y}\right]w_y^2w_y^3\ ,$$

$$(\tilde{B}\left[A(N_{\alpha_y}),\tilde{B}^t\right])_{31} = \left[\frac{\bar{g}(\alpha_y)}{\sin\alpha_y}\right]w_y^2+\left[\frac{(\bar{h}(\alpha_y)-\bar{f}(\alpha_y))}{\sin^2\alpha_y}\right]w_y^3w_y^1\ ,$$

$$(\tilde{B}\left[A(N_{\alpha_y}),\tilde{B}^t\right])_{32} = -\left[\frac{\bar{g}(\alpha_y)}{\sin\alpha_y}\right]w_y^1+\left[\frac{(\bar{h}(\alpha_y)-\bar{f}(\alpha_y))}{\sin^2\alpha_y}\right]w_y^3w_y^2\ ,\ and$$

$$(\tilde{B}\left[A(N_{\alpha_y}),\tilde{B}^t\right])_{33} = (\bar{f}(\alpha_y)-\bar{h}(\alpha_y))+\left[\frac{(\bar{h}(\alpha_y)-\bar{f}(\alpha_y))}{\sin^2\alpha_y}\right](w_y^3)^2\ .$$

Putting this in 3.38 and 3.37, and defining

$$f(\alpha_y)\equiv\bar{f}(\alpha_y)\ ,\ and$$

$$g(\alpha_y)\equiv\frac{\bar{g}(\alpha_y)}{\sin\alpha_y}\ ,\ and$$

$$h(\alpha_y)\equiv\frac{(\bar{h}(\alpha_y)-\bar{f}(\alpha_y))}{\sin^2\alpha_y}\ ,$$

we then obtain the following final result giving the explicit form of $G_{1,2}(N,y)$ at arbitrary $y\in S^3\setminus\{N,S\}$:

Proposition 3.6 *There are smooth functions f, g, and h on* $(0,\pi)$ *such that in our matrix notation the 1,2 piece of the Green's form,* $G_{1,2}(N,y)=A_{ij}(y)\theta_N^i\wedge(\theta_y\wedge\theta_y)^{(j)}$, *is given explicitly on* $S^3 \setminus \{N, S\}$ *by*

$$G_{1,2}(N,y) = A(y) = f(\alpha_y)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + g(\alpha_y)\begin{pmatrix} 0 & w_y^3 & -w_y^2 \\ -w_y^3 & 0 & w_y^1 \\ w_y^2 & -w_y^1 & 0 \end{pmatrix}$$
$$+ h(\alpha_y)\begin{pmatrix} (w_y^1)^2 & w_y^1 w_y^2 & w_y^1 w_y^3 \\ w_y^2 w_y^1 & (w_y^2)^2 & w_y^2 w_y^3 \\ w_y^3 w_y^1 & w_y^3 w_y^2 & (w_y^3)^2 \end{pmatrix} . \tag{3.39}$$

Together with 3.33 this gives us the explicit closed form of $G_{1,2}(N, y)$ everywhere on $S^3 \setminus \{N\}$.

To conclude this step then there is only one final "house-cleaning" observation to make; namely that we can in fact easily combine 3.39 and 3.33 into a single result giving the explicit form of $G_{1,2}(N, y)$ simultaneously everywhere on $S^3 \setminus \{N\}$, simply by asserting smoothness of f, g and h in 3.39 not just on $(0, \pi)$, but on $(0, \pi]$. This causes the expression in 3.39 to limit smoothly to the form in 3.33 as $y \to S$. Since we know on general grounds that $G_{1,2}(N, y)$ must be smooth at $y = S$ (again, see Step 4), this extended smoothness is in any case *apriori* valid. Indeed, from this viewpoint, the consistency between our forms of $G_{1,2}(N, S)$, as calculated directly in 3.33 and as obtained in the limit $y \to S$ in 3.39 after invoking the extended *apriori* smoothness of f, g and h, actually represents an initial small confirmation of the correctness of our computations thus far.

We have now applied rotational invariance to the maximum extent possible in restricting the form of $G_{1,2}(N, y)$. As promised, we have reduced from the nine unknown functions on all of S^3 that we had at the end of Step 1, to only three unknown functions, f, g and h, depending only on the single polar variable $\alpha_y \in (0, \pi]$. This completes Step 2.

Step 3: Applying Reflection Invariance

In addition to $SO(4)$ invariance there is one other invariance of our problem which can be invoked to further restrict the form of the fundamental solution $G_{1,2}(N,y)$ — reflection invariance. It will allow us to reduce from the three unknown functions in 3.39 to just two.

To be precise, let R_- be the reflection of S^3 given by

$$R_- : S^3 \longrightarrow S^3 : (w^1, w^2, w^3, w^4) \longmapsto (-w^1, -w^2, -w^3, w^4) \tag{3.40}$$

which, as observed in Chapter 2, is the inversion map under our identification of S^3 with $SU(2)$. Note that R_- fixes N.

Then R_- is an isometry and so it is easy to see that $[\Delta, R_-{}^*] = 0$. As R_- is orientation-reversing it follows by theorem 1 that we have

$$G_{1,2}(N, R_-(y)) = -(d(R_- \times R_-)_{(N,R_-(y))})^t G_{1,2}(N,y) \quad \textit{for all} \quad y \in S^3. \tag{3.41}$$

Unfortunately it is not easy to see directly in 3.39 what constraints this imposes on f, g and h because, unlike Step two where 3.25 gave us a very simple formula for the pull-back of the θ^i under rotations, there is no such simple corresponding formula for $R_-{}^*\theta^i$, i.e. the $\{\theta^i\}$ basis of 1-forms is not well adapted to studying reflections.

By contrast, the system of stereographic coordinates $\{v^i\}_{i=1}^3$ on $S^3\backslash\{S\}$ described in Chapter 2 is ideally suited for studying R_-. Indeed a moment's thought reveals that its coordinate 1-forms satisfy the trivial pull-back relation

$$R_-{}^*\, dv^i = -dv^i \quad \textit{for all}\ 1 = 1,2,3. \tag{3.42}$$

We thus should translate 3.39 into stereographic coordinates in order to use 3.42 to investigate the anti-invariance condition 3.41. This involves using the rather messy formula ?? from chapter 2, giving the change of basis matrix relating the θ^i to the coordinate 1-forms dv^i. The computation is expedited by noting that $v^i = \frac{2}{1+\cos\alpha} w^i = \frac{\tilde{r}^2+4}{4} w^i$, and that the matrix in ?? is (of course) orthogonal with respect to the metric in stereographic coordinates as given in 2.15, i.e. its rows (or columns) are perpendicular in this metric and the determinant of the i,jth minor is $(-1)^{i+j}\frac{\tilde{r}^2+4}{4}$, in analogy with 3.30. After a somewhat lengthy computation we find that the translation of 3.39 into stereographic coordinates is

$$G_{1,2}(N,y) = A^{st}_{ij}(y)dv^i_N \wedge (dv_y \wedge dv_y)^{(j)} \quad \textit{where } A^{st} \textit{ is given in matrix form by}$$

$$A^{st} = \left(\tfrac{4}{\tilde{r}^2+4}\right)^4 \left\{ f_1(\alpha_y)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + g_1(\alpha_y)\begin{pmatrix} 0 & v_y^3 & -v_y^2 \\ -v_y^3 & 0 & v_y^1 \\ v_y^2 & -v_y^1 & 0 \end{pmatrix} \right.$$
$$\left. + h_1(\alpha_y)\begin{pmatrix} (v_y^1)^2 & v_y^1 v_y^2 & v_y^1 v_y^3 \\ v_y^2 v_y^1 & (v_y^2)^2 & v_y^2 v_y^3 \\ v_y^3 v_y^1 & v_y^3 v_y^2 & (v_y^3)^2 \end{pmatrix} \right\} \tag{3.43}$$

and

$$f_1(\alpha) = -g(\alpha)\tilde{r}^2 + \left(1 - \tfrac{\tilde{r}^4}{16}\right) f(\alpha)\,,$$
$$\textit{and } g_1(\alpha) = \left(1 + \tfrac{\tilde{r}^2}{4}\right) f(\alpha) + \left(1 - \tfrac{\tilde{r}^2}{4}\right) g(\alpha)\,, \tag{3.44}$$
$$\textit{and } h_1(\alpha) = \tfrac{1}{2}\left(1 + \tfrac{\tilde{r}^2}{4}\right) f(\alpha) + g(\alpha) + h(\alpha)\,.$$

[Note that the structure of $G_{1,2}(N,y)$ is the same in stereographic coordinates as it was in 3.39. This simply reflects the fact that the same general $SO(3)$-invariance constraints apply equally in both settings, i.e. we could have started in stereographic coordinates and then the same reasoning as performed in Step 2 would have led us in identical fashion to the general form given in 3.43.]

We can now apply 3.42 in 3.43 to see how the anti-invariance condition 3.41 further constrains the form of $G_{1,2}(N,y)$. For 3.42 implies at once that

$$(d(R_- \times R_-)_{(N,R_-(y))})^t dv^i_N \wedge (dv_y \wedge dv_y)^{(j)} = -dv^i_N \wedge (dv_{R_-(y)} \wedge dv_{R_-(y)})^{(j)}$$

and so in 3.43 the anti-invariance means that the coefficient functions, A^{st}_{ij}, of these 1,2-forms must be invariant under $R_-{}^*$. Since $\tilde{r}$, and hence also α_y, are clearly invariant under $R_-{}^*$ and equally clearly $R_-{}^* v^i_y = -v^i_y$, it follows at once that this in turn implies the single constraint that

$$g_1 \equiv 0\,. \tag{3.45}$$

In terms of our original functions f, g and h in 3.39 this then means, by 3.44, that we have the relationship

$$f(\alpha_y) = -\left(\frac{4-\tilde{r}^2}{\tilde{r}^2+4}\right) g(\alpha_y) = -g(\alpha_y)\cos\alpha_y\,, \tag{3.46}$$

and this leaves us, as promised, with only really two independent undetermined functions, which without loss of generality we shall take as g and h in what follows, instead of three in our expression for $G_{1,2}(N,y)$ in 3.39. This completes Step 3 applying reflection invariance.

Step 4: Smoothness and Asymptotic Considerations

Between 3.39 (with the small extension discussed immediately thereafter) and 3.46 we have now exhausted all invariances of our problem in reducing the possible form of $G_{1,2}(N,y)$ as much as we can.

At this point there remain just the two independent, unknown smooth functions g and h on $(0,\pi]$, for which we need to solve in order to completely determine $G_{1,2}(N,y)$. This must be done, of course, by returning to the defining equations for the $A_{ij}(y)$ in 3.22 at the end of Step 1, and substituting in the explicit form in 3.39 and 3.46. This will reduce the nine PDE's there to simply a pair of coupled ODE's in g and h, which we then try to solve directly. We shall begin this program in Step 5.

But first we need briefly to consider two qualitative features of $G_{1,2}(N,y)$ — its smoothness on $S^3 \setminus \{N\}$ and its asymptotics as $y \to N$. This is necessary for two reasons; first to explain the rigorous rationale for some important smoothness claims made in Step 2, whose justification we deferred at the time, and secondly to obtain boundary conditions on g and h that we will need in attempting to tackle the above-mentioned pair of ODE's in subsequent sections.

(i) Smoothness: We have already considered the smoothness of G on a number of occasions. In remarks in section 3.2.1 we observed the smoothness of $G_{0,3}(N,y)$ everywhere away from N, and in Step 2 of this section we invoked a claim of *apriori* smoothness of $G_{1,2}(N,y)$ on all of $S^3 \setminus \{N\}$ on two occasions; first in asserting smoothness of $\bar{f}$, $\bar{g}$, and $\bar{h}$ on $(0,\pi)$ in 3.35, and then again in claiming that f, g and h in 3.39 in fact extend to be smooth not just on $(0,\pi)$ but on $(0,\pi]$, which allowed us to subsume our separate calculation of the form of $G_{1,2}(N,S)$ into a single result giving the form of $G_{1,2}(N,y)$ everywhere on $S^3 \setminus \{N\}$.

The general fact underlying all of this is the fundamental theorem in analysis that since Δ^{-1} is a pseudodifferential operator its Schwartz kernel $G(x,y)$ must be smooth everywhere away from the diagonal in $S^3 \times S^3$. This simultaneously explains our observation regarding $G_{0,3}(N,y)$ and supplies the missing justification for the above-mentioned claims in Step 2 of this section regarding $G_{1,2}(N,y)$. In doing the latter it fills in the only small gaps in the rigour of our derivation to this point.

There is, however, still more information which can be gleaned from knowing the smoothness of $G_{1,2}(N,y)$ away from N. Clearly the smoothness of f, g and h on $(0,\pi)$ in 3.39 is both necessary and sufficient to ensure the required smoothness of $G_{1,2}(N,y)$ everywhere away from S in $S^3 \setminus \{N\}$ and, as mentioned, smoothness at S necessitates that f, g and h in fact be smooth on $(0,\pi]$. But this smoothness of f, g and h on all of $(0,\pi]$ is not *sufficient* to guarantee smoothness of $G_{1,2}(N,y)$ at $y = S$. This requires the further condition (in analogy with remark (iii) in section 3.2.1) that

All three functions f, g and h in $C^\infty(0,\pi]$ must have power series expansions around $\alpha = \pi$ which are even in $\tilde{\alpha} \equiv (\pi - \alpha)$. (3.47)

The sufficiency of this condition can easily be seen by changing to good coordinates in a neighbourhood of S in S^3, such as stereographic coordinates projecting from N instead of S.

Noting that this condition is of course compatible with relation 3.46, this then completes our application of smoothness requirements to limit further the form of

$G_{1,2}(N,y)$ in 3.39 — we have obtained one more constraint on our independent unknown functions g and h, in the form of a boundary condition at $\alpha_y = \pi$, as promised.

(ii) Asymptotics: We now consider the singularity of the fundamental solution $G_{1,2}(N,y)$ at N in S^3. From analysing this we shall obtain the second set of boundary conditions that we need, at $\alpha_y = 0$.

The underlying result we draw on in this context is also one we have already discussed in remarks in section 3.2.1, namely that the asymptotic singularity of $G(N,y)$ at N should be the same as that of the fundamental solution centred at 0 for the flat Laplacian on stereographic $\Re^3$. In section 3.2.1 we explained the justification for this claim and verified it for the 0,3-piece of G. In the context of the 1,2-piece here, since the corresponding piece of the flat fundamental solution is just (noting our unusual sign convention again)

$$G^{\Re^3}_{1,2}(N,y) = -\tfrac{1}{4\pi\bar{r}}\left\{dv^i_N \wedge (dv_y \wedge dv_y)^{(i)}\right\} ,$$

and since the θ^i_y and w^i_y converge smoothly to dv^i_y and v^i_y respectively as we approach N, so the same reasoning applied to 3.39 implies at once the following asymptotic conditions on f, g and h as $\alpha_y \to 0$;

$$f(\alpha_y) + h(\alpha_y)(w^i_y)^2 \sim -\tfrac{1}{4\pi\alpha_y} + O(\alpha^0_y) \quad \textit{for each } i = 1,2,3, \textit{ and}$$

$$g(\alpha_y)\varepsilon^{ij}{}_k w^k_y + h(\alpha_y) w^i_y w^j_y \sim 0 + O(\alpha^1_y) \quad \textit{for all } i,j = 1,2,3.$$

Together with 3.46, this immediately yields the desired asymptotic boundary conditions at 0 on our independent unknown functions g and h, namely

$$\begin{aligned} &g(\alpha_y) \sim \tfrac{1}{4\pi\alpha_y} + O(\alpha^0_y) , \quad \textit{and} \\ &h(\alpha_y) \textit{ is at most singular of order } -1 \textit{ as } \alpha_y \to 0 . \end{aligned} \tag{3.48}$$

This completes Step 4.

Step 5: Calculating ODE's Satisfied by g and h

To summarise where we stand at present in our computation of $G_{1,2}(N,y)$, we have reduced the possible form of this fundamental solution to that given in 3.39, which, when coupled with relation 3.46, has left us just needing to identify two independent unknown functions g and h in $C^\infty(0,\pi]$ (noting the remarks immediately following 3.39), whose boundary behaviour at 0 and π is as prescribed in 3.47 and 3.48.

To proceed now, we need in this step to go back, as promised, to the defining equations 3.22 for the $A_{ij}(y)$ and translate them into ODE's in g and h, by substituting in the explicit form in 3.39 together with 3.46. Because it will prove more convenient, we initially retain the function f appearing in 3.39 in our working, and only use 3.46 at the end to remove it and leave just g and h. This approach has the additional advantage of providing an internal check on our computations, since we will initially obtain three coupled ODE's in the three functions f, g and h, and we can then check whether applying 3.46 causes one of these to become redundant as it should. Note in passing, along similar lines, that the fact that we will find the nine equations in 3.22 all reducing to combinations of just the same three independent ODE's in the first place (before applying 3.46) is itself strong evidence of the correctness of our working thus far.

We take the equations in 3.22 in turn.

(i) Substituting 3.39 into the first equation in 3.22, this becomes the following equation in f, g and h;

$$\textstyle\int_{S^3_y}\left[f(\alpha_y)+h(\alpha_y)(w^1_y)^2\right]((\Delta+4)u)vol_{S^3_y}-2\int_{S^3_y}\left[g(\alpha_y)w^3_y+h(\alpha_y)w^1_yw^2_y\right]X_3(u)vol_{S^3_y}$$

$$\textstyle+2\int_{S^3_y}\left[-g(\alpha_y)w^2_y+h(\alpha_y)w^1_yw^3_y\right]X_2(u)vol_{S^3_y} \;=\; -u(N), \quad \textit{for all } u\in C^\infty(S^3)\;.$$

That is,

$$I_1(u)+I_2(u)+I_3(u)+I_4(u)=-u(N),\ \textit{for all } u\in C^\infty(S^3) \tag{3.49}$$

where, using formulae 2.24 and 2.11 from chapter 2,

$$I_1(u)=\textstyle\int_{S^3_y} f(\alpha_y)((\Delta+4)u)vol_{S^3_y}, \quad \textit{and}$$

$$I_2(u)=\textstyle\int_{S^3_y}\left[h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\cos^2\theta_y\right]((\Delta+4)u)vol_{S^3_y}, \quad \textit{and}$$

$$I_3(u) = -2\int_{S^3_y}\left[g(\alpha_y)\sin\alpha_y\cos\phi_y + h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y\right]\times$$

$$\left\{\cos\phi_y\partial_{\alpha_y}u - \cot\alpha_y\sin\phi_y\partial_{\phi_y}u - \partial_{\theta_y}u\right\} vol_{S^3_y}, \quad \textit{and}$$

$$I_4(u) = 2\int_{S^3_y}\left[-g(\alpha_y)\sin\alpha_y\sin\phi_y\sin\theta_y + h(\alpha_y)\sin^2\alpha_y\sin\phi_y\cos\phi_y\cos\theta_y\right]\times$$

$$\left\{\begin{array}{c}\sin\phi_y\sin\theta_y\partial_{\alpha_y}u + (\cot\alpha_y\cos\phi_y\sin\theta_y - \cos\theta_y)\partial_{\phi_y}u \\ +(\cot\phi_y\sin\theta_y + \cot\alpha_y\csc\phi_y\cos\theta_y)\partial_{\theta_y}u\end{array}\right\} vol_{S^3_y}\,. \tag{3.50}$$

We now simplify each $I_i(u)$ in turn with the goal of isolating u so that it only appears in each integrand in undifferentiated form. We explain our working in full detail for $I_1(u)$ but only more briefly for the very similar computations of $I_2(u)$, $I_3(u)$ and $I_4(u)$.

(a) $\underline{I_1(u)}$**:** Since $f(\alpha_y) \sim O(\alpha_y^{-1})$ as $\alpha_y \to 0$, f does not lie in the space $H^2(S^3)$ of L^2-functions whose distributional derivatives of degree less than or equal to two are also lie in L^2. Hence, unfortunately, to achieve our goal we cannot simply apply self-adjointness of $(\Delta+4)$ directly in $I_1(u)$ — we have to perform integration by parts carefully and in full.

The first step in doing this is then to use 2.16 to write

$$I_1(u) = \int_{S^3_y}\left[f(\alpha_y)\sin^2\alpha_y\sin\phi_y\right]((\Delta+4)u)\, d\alpha_y \wedge d\phi_y \wedge d\theta_y\,. \tag{3.51}$$

The second step is to express the Laplacian on $C^\infty(S^3)$ in terms of spherical polars. This is done by the following result, which may easily be proven either by substituting 2.24 into 3.19 or simply by direct computation using 2.17.

Result 3.7 *For any* $u \in C^\infty(S^3)$

$$\Delta u = -\partial^2_\alpha u - 2\cot\alpha\partial_\alpha u - \csc^2\alpha\partial^2_\phi u - \csc^2\alpha\cot\phi\partial_\phi u - \csc^2\alpha\csc^2\phi\partial^2_\theta u\,. \tag{3.52}$$

Applying this in 3.51, our expression for $I_1(u)$ becomes

$$\int_{S^3_y}\left[f(\alpha_y)\sin^2\alpha_y\sin\phi_y\right]\left\{\begin{array}{c}-\partial^2_{\alpha_y}u-2\cot\alpha_y\partial_{\alpha_y}u\\ -\csc^2\alpha_y\partial^2_{\phi_y}u-\csc^2\alpha_y\cot\alpha_y\partial_{\phi_y}u\\ -\csc^2\alpha_y\csc^2\phi_y\partial^2_{\theta_y}u\end{array}\right\}d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$+4\int_{S^3_y}\left[f(\alpha_y)\sin^2\alpha_y\sin\phi_y\right]u\ d\alpha_y\wedge d\phi_y\wedge d\theta_y\ .$$

In this expression we then immediately see that the last term in the first integral yields zero, and we can begin performing integration by parts on the other terms. We obtain that $I_1(u)$ equals

$$\int_{S^2_y}vol_{S^2_y}\left\{\begin{array}{c}-\left[f(\alpha_y)\sin^2\alpha_y\partial_{\alpha_y}u\right]^{\alpha_y=\pi}_{\alpha_y=0}+\int_0^\pi\left\{\begin{array}{c}\partial_{\alpha_y}f(\alpha_y)\sin^2\alpha_y\\ +2\sin\alpha_y\cos\alpha_y f(\alpha_y)\end{array}\right\}\partial_{\alpha_y}u\ d\alpha_y\\ \\ -2[f(\alpha_y)\sin\alpha_y\cos\alpha_y u]^{\alpha_y=\pi}_{\alpha_y=0}+2\int_0^\pi\left\{\begin{array}{c}\partial_{\alpha_y}f(\alpha_y)\sin\alpha_y\cos\alpha_y\\ +\cos 2\alpha_y f(\alpha_y)\end{array}\right\}u\ d\alpha_y\end{array}\right\}$$

$$+\int_0^{2\pi}\int_0^\pi f(\alpha_y)\left\{\ -\left[\sin\phi_y\partial_{\phi_y}u\right]^{\phi_y=\pi}_{\phi_y=0}+\int_0^\pi\cos\phi_y\partial_{\phi_y}u\ d\phi_y\ \right\}\ d\alpha_y\wedge d\theta_y$$

$$-\int_0^{2\pi}\int_0^\pi f(\alpha_y)\left\{\ \int_0^\pi\cos\phi_y\partial_{\phi_y}u\ d\phi_y\ \right\}\ d\alpha_y\wedge d\theta_y$$

$$+4\int_{S^3_y}\left[f(\alpha_y)\sin^2\alpha_y\sin\phi_y\right]u\ d\alpha_y\wedge d\phi_y\wedge d\theta_y\ . \tag{3.53}$$

But this formula then simplifies greatly. Of the eight terms on the right hand side, the sixth and seventh cancel immediately, and the boundary pieces of the first and fifth are zero, causing these terms to vanish.

To see this last fact, consider first the case of $\left[f(\alpha_y)\sin^2\alpha_y\partial_{\alpha_y}u\right]^{\alpha_y=\pi}_{\alpha_y=0}$. Since u is smooth, identical arguments to those used in Step 4, part (i) show that, for any fixed ϕ_y and θ_y, u must have an even power series expansion in α_y near $\alpha_y=0$ and in $\tilde{\alpha_y}=(\pi-\alpha_y)$ near $\alpha_y=\pi$ (cf. result 3.47). Thus $\partial_{\alpha_y}u$ converges at least linearly to 0 at both $\alpha_y=0$ and $\alpha_y=\pi$. As f is smooth at $\alpha_y=\pi$ and only singular of order 1 at $\alpha_y=0$ (by 3.48 and 3.46), so when combined with $\sin^2\alpha_y$ and $\partial_{\alpha_y}u$ the function whose boundary values we are considering turns out to be zero at both limits (converging, indeed, to order 2 at $\alpha_y=0$ and order 3 at $\alpha_y=\pi$!). It thus has vanishing boundary piece, as claimed.

As for the other boundary piece, $\left[\sin\phi_y\partial_{\phi_y}u\right]_{\phi_y=0}^{\phi_y=\pi}$, here we fix $\alpha_y \in (0,\pi)$ and consider u on the copy of S^2 making up the $w^4 = \cos\alpha_y$ slice. Now arguments concerning the "North and South poles" of this copy of S^2, exactly analogous to those in Step 4, part (i) regarding S in S^3, show that for each fixed θ_y on this slice, u has an even power series expansion in ϕ_y near $\phi_y = 0$ and in $\tilde{\phi_y} = (\pi - \phi_y)$ near $\phi_y = \pi$. It thus follows at once, by identical reasoning to that just given above, that both limits vanish to second order and hence this boundary piece also disappears, as asserted.

We are left therefore with four terms in 3.53, the second, third, fourth and eighth.

Of these, the third term, the only remaining of the three terms with boundary pieces, is also calculable immediately. For, if we simply quote the smoothness/asymptotic characteristics of u and f used in treating the first term above (i.e. $\left[f(\alpha_y)\sin^2\alpha_y\partial_{\alpha_y}u\right]_{\alpha_y=0}^{\alpha_y=\pi}$), then we see at once that in $[f(\alpha_y)\sin\alpha_y\cos\alpha_y u]_{\alpha_y=0}^{\alpha_y=\pi}$ the top limit is zero while, in light of 3.48 and 3.46, the bottom limit is $-\frac{1}{4\pi}u(N)$. Substituting this in and performing the remaining trivial integral over S^2 it follows that the contribution of the third term is $-2u(N)$.

The three remaining terms, however, do not reduce further. Since the fourth and the eighth are already in the desired form, with u isolated, we leave them untouched, but we need to apply one further integration by parts in the second term . On doing this expression 3.53 becomes

$$
\begin{aligned}
I_1(u) = \; & 2u(N) + 4\int_{S_y^3}\left[f(\alpha_y)\sin^2\alpha_y\sin\phi_y\right]u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y \\
& + \int_{S_y^2} vol_{S_y^2}\left\{\begin{array}{c}\left[\left(\partial_{\alpha_y}f(\alpha_y)\sin^2\alpha_y + 2\sin\alpha_y\cos\alpha_y f(\alpha_y)\right)u\right]_{\alpha_y=0}^{\alpha_y=\pi} \\ -\int_0^\pi\left\{\begin{array}{c}\partial_{\alpha_y}^2 f(\alpha_y)\sin^2\alpha_y + 4\sin\alpha_y\cos\alpha_y\partial_{\alpha_y}f(\alpha_y) \\ +2\cos 2\alpha_y f(\alpha_y)\end{array}\right\}u\, d\alpha_y\end{array}\right\} \\
& +2\int_{S_y^2} vol_{S_y^2}\left\{\int_0^\pi\left\{\partial_{\alpha_y}f(\alpha_y)\sin\alpha_y\cos\alpha_y + \cos 2\alpha_y f(\alpha_y)\right\}u\, d\alpha_y\right\} .
\end{aligned}
\tag{3.54}
$$

We are nearly there. We just need once again to calculate the term involving the boundary piece $\left[\left(\partial_{\alpha_y} f(\alpha_y)\sin^2\alpha_y + 2\sin\alpha_y\cos\alpha_y f(\alpha_y)\right) u\right]_{\alpha_y=0}^{\alpha_y=\pi}$. But in exactly the same way that we analysed the boundary pieces in the previous expression 3.53, we see at once that the upper limit here is zero and the lower is $(\frac{1}{4\pi} - \frac{2}{4\pi})u(N)$ (noting that 3.48 and 3.46 at once imply $\partial_{\alpha_y} f(\alpha_y) \sim \frac{1}{4\pi}\alpha_y^{-2}$), and so the contribution from this term, after performing the S^2-integration as before, is just $u(N)$.

Cleaning up what remains, by cancelling this contribution against the existing $-2u(N)$ and by trivially cancelling off like terms in the integrals on the second and third lines, we thus reach our final expression for $I_1(u)$, namely

$$I_1(u) = -u(N) + \int_{S_y^3} \left\{-\partial_{\alpha_y}^2 f(\alpha_y) - 2\cot\alpha_y \partial_{\alpha_y} f(\alpha_y) + 4f(\alpha_y)\right\} u \, vol_{S_y^3} \qquad (3.55)$$

(b) $\underline{I_2(u)}$: $I_2(u)$ is easier to simplify than $I_1(u)$ because this time the function $h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\cos^2\theta_y$ *is* in $H^2(S^3)$ and so we can apply self-adjointness of $(\Delta+4)$ directly, together with 3.52, to isolate u immediately. We get

$$I_2(u) = \textstyle\int_{S_y^3} \left\{(\Delta+4)h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\cos^2\theta_y\right\} u \, vol_{S_y^3}$$

$$= \int_{S_y^3} \left\{ \begin{array}{l} -\left(\begin{array}{c} \partial_{\alpha_y}^2 h(\alpha_y)\sin^2\alpha_y + 2\partial_{\alpha_y} h(\alpha_y)\sin 2\alpha_y \\ +2h(\alpha_y)\cos 2\alpha_y \end{array}\right) \sin^2\phi_y\cos^2\theta_y \\ -2\cot\alpha_y \left(\partial_{\alpha_y} h(\alpha_y)\sin^2\alpha_y + h(\alpha_y)\sin 2\alpha_y\right)\sin^2\phi_y\cos^2\theta_y \\ -h(\alpha_y)\left(2\cos 2\phi_y\cos^2\theta_y + \cot\phi_y\sin 2\phi_y\cos^2\theta_y - 2\cos 2\theta_y\right) \end{array} \right\} u \, vol_{S_y^3}$$

$$+4\textstyle\int_{S_y^3} \left\{h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\cos^2\theta_y\right\} u \, vol_{S_y^3}$$

$$= \int_{S_y^3} \left\{ \begin{array}{l} -\partial_{\alpha_y}^2 h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\cos^2\theta_y \\ -6\partial_{\alpha_y} h(\alpha_y)\sin\alpha_y\cos\alpha_y\sin^2\phi_y\cos^2\theta_y \\ +h(\alpha_y)\left(\begin{array}{l} \left(-2\cos 2\alpha_y - 4\cos^2\alpha_y + 4\sin^2\alpha_y\right)\sin^2\phi_y\cos^2\theta_y \\ +\left(-4 + 6\sin^2\phi_y\right)\cos^2\theta_y + 2\left(2\cos^2\theta_y - 1\right) \end{array}\right) \end{array} \right\} u \, vol_{S_y^3},$$

$$i.e. \quad I_2(u) = \int_{S_y^3} \left\{ \begin{array}{l} -\partial_{\alpha_y}^2 h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\cos^2\theta_y \\ -6\partial_{\alpha_y} h(\alpha_y)\sin\alpha_y\cos\alpha_y\sin^2\phi_y\cos^2\theta_y \\ +h(\alpha_y)\left(12\sin^2\alpha_y\sin^2\phi_y\cos^2\theta_y - 2\right) \end{array} \right\} u \, vol_{S_y^3}. \qquad (3.56)$$

(c) $\underline{I_3(u)}$: In this case we just need to apply 2.16 to rewrite $vol_{S^3_y}$ and then perform integration by parts once. In doing this it is easily seen, by arguments along the lines of those in our calculation of $I_1(u)$, that all the boundary pieces arising in the integration by parts vanish, which makes the computation considerably quicker. We find

$$\begin{aligned}
I_3(u) = & \; 2\int_{S^3_y} \left\{\partial_{\alpha_y} g(\alpha_y)\sin^3\alpha_y + 3g(\alpha_y)\sin^2\alpha_y\cos\alpha_y\right\}\sin\phi_y\cos^2\phi_y u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y \\
& +2\int_{S^3_y} \left\{\begin{array}{l} \partial_{\alpha_y} h(\alpha_y)\sin^4\alpha_y + \\ 4h(\alpha_y)\sin^3\alpha_y\cos\alpha_y \end{array}\right\}\sin^3\phi_y\cos\phi_y\sin\theta_y\cos\theta_y u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y \\
& -2\int_{S^3_y} g(\alpha_y)\sin^2\alpha_y\cos\alpha_y\left\{2\sin\phi_y\cos^2\phi_y - \sin^3\phi_y\right\} u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y \\
& -2\int_{S^3_y} h(\alpha_y)\sin^3\alpha_y\cos\alpha_y\left\{4\sin^3\phi_y\cos\phi_y\right\}\sin\theta_y\cos\theta_y u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y \\
& -2\int_{S^3_y} h(\alpha_y)\sin^4\alpha_y\sin^3\phi_y(2\cos^2\theta_y - 1)u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y \\
= & \; 2\int_{S^3_y} \partial_{\alpha_y} g(\alpha_y)\sin^3\alpha_y\sin\phi_y\cos^2\phi_y u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y \\
& +2\int_{S^3_y} \partial_{\alpha_y} h(\alpha_y)\sin^4\alpha_y\sin^3\phi_y\cos\phi_y\sin\theta_y\cos\theta_y u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y \\
& +2\int_{S^3_y} g(\alpha_y)\sin^2\alpha_y\cos\alpha_y\sin\phi_y u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y \\
& -2\int_{S^3_y} h(\alpha_y)\sin^4\alpha_y\sin^3\phi_y(2\cos^2\theta_y - 1)u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y\,,
\end{aligned}$$

$$\text{i.e.}\quad I_3(u) = \int_{S^3_y} \left\{\begin{array}{l} 2\partial_{\alpha_y} g(\alpha_y)\sin\alpha_y\cos^2\phi_y + 2g(\alpha_y)\cos\alpha_y \\ +2\partial_{\alpha_y} h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\cos\phi_y\sin\theta_y\cos\theta_y \\ -2h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y(2\cos^2\theta_y - 1) \end{array}\right\} u\, vol_{S^3_y}\,. \tag{3.57}$$

(d) $\underline{I_4(u)}$: Here we proceed in identical fashion to the case of $I_3(u)$, with boundary pieces again vanishing. We obtain

$$\begin{aligned}
I_4(u) = & \; 2\int_{S^3_y} \left\{\partial_{\alpha_y} g(\alpha_y)\sin^3\alpha_y + 3g(\alpha_y)\sin^2\alpha_y\cos\alpha_y\right\}\sin^3\phi_y\sin^2\theta_y u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y \\
& -2\int_{S^3_y} \left\{\begin{array}{l} \partial_{\alpha_y} h(\alpha_y)\sin^4\alpha_y + \\ 4h(\alpha_y)\sin^3\alpha_y\cos\alpha_y \end{array}\right\}\sin^3\phi_y\cos\phi_y\sin\theta_y\cos\theta_y u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y
\end{aligned}$$

$$+2\int_{S^3_y} g(\alpha_y)\sin^2\alpha_y\cos\alpha_y\left\{2\sin\phi_y\cos^2\phi_y-\sin^3\phi_y\right\}\sin^2\theta_y u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$-2\int_{S^3_y} h(\alpha_y)\sin^3\alpha_y\cos\alpha_y\left\{\begin{array}{l}2\sin\phi_y\cos^3\phi_y\\ -2\sin^3\phi_y\cos\phi_y\end{array}\right\}\sin\theta_y\cos\theta_y u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$-2\int_{S^3_y} g(\alpha_y)\sin^3\alpha_y\left\{2\sin\phi_y\cos\phi_y\right\}\sin\theta_y\cos\theta_y u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$+2\int_{S^3_y} h(\alpha_y)\sin^4\alpha_y\left\{2\sin\phi_y\cos^2\phi_y-\sin^3\phi_y\right\}\cos^2\theta_y u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$+2\int_{S^3_y} g(\alpha_y)\sin^3\alpha_y\sin\phi_y\cos\phi_y\left\{2\sin\theta_y\cos\theta_y\right\}u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$-2\int_{S^3_y} h(\alpha_y)\sin^4\alpha_y\sin\phi_y\cos^2\phi_y(2\cos^2\theta_y-1)u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$+2\int_{S^3_y} g(\alpha_y)\sin^2\alpha_y\cos\alpha_y\sin\phi_y(1-2\sin^2\theta_y)u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$-2\int_{S^3_y} h(\alpha_y)\sin^3\alpha_y\cos\alpha_y\sin\phi_y\cos\phi_y\left\{-2\sin\theta_y\cos\theta_y\right\}u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$=\ 2\int_{S^3_y}\partial_{\alpha_y}g(\alpha_y)\sin^3\alpha_y\sin^3\phi_y\sin^2\theta_y u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$-2\int_{S^3_y}\partial_{\alpha_y}h(\alpha_y)\sin^4\alpha_y\sin^3\phi_y\cos\phi_y\sin\theta_y\cos\theta_y u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$+\int_{S^3_y} g(\alpha_y)\left\{\begin{array}{l}4\sin^2\alpha_y\cos\alpha_y\sin\phi_y\sin^2\theta_y)\\ +2\sin^2\alpha_y\cos\alpha_y\sin\phi_y(1-2\sin^2\theta_y)\end{array}\right\}u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$+\int_{S^3_y} h(\alpha_y)\left\{\begin{array}{l}-4\sin^3\alpha_y\cos\alpha_y\sin\phi_y\cos\phi_y\sin\theta_y\cos\theta_y\\ -2\sin^4\alpha_y\sin^3\phi_y\cos^2\theta_y\\ +2\sin^4\alpha_y\sin\phi_y\cos^2\phi_y\\ +4\sin^3\alpha_y\cos\alpha_y\sin\phi_y\cos\phi_y\sin\theta_y\cos\theta_y\end{array}\right\}u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y\ ,$$

$$i.e.\quad I_4(u)=\int_{S^3_y}\left\{\begin{array}{l}2\partial_{\alpha_y}g(\alpha_y)\sin\alpha_y\sin^2\phi_y\sin^2\theta_y+2g(\alpha_y)\cos\alpha_y\\ -2\partial_{\alpha_y}h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\cos\phi_y\sin\theta_y\cos\theta_y\\ +2h(\alpha_y)\sin^2\alpha_y\left\{\cos^2\phi_y-\sin^2\phi_y\cos^2\theta_y\right\}\end{array}\right\}u\, vol_{S^3_y}\ . \qquad (3.58)$$

With the computations of $I_1(u),\ldots,I_4(u)$ in 3.55 - 3.58 now complete, we are at last in a position to return to 3.49 and finish our task of translating the first defining equation in 3.22 into ODE's in f, g and h. We obtain at first that

$$-u(N) = -u(N) + \int_{S_y^3} \left\{ \begin{array}{l} -\partial^2_{\alpha_y} f(\alpha_y) - 2\cot\alpha_y \partial_{\alpha_y} f(\alpha_y) + 4f(\alpha_y) \\ +2\partial_{\alpha_y} g(\alpha_y)\sin\alpha_y \left\{\cos^2\phi_y + \sin^2\phi_y \sin^2\theta_y\right\} \\ +4g(\alpha_y)\cos\alpha_y - \partial^2_{\alpha_y} h(\alpha_y)\sin^2\alpha_y \sin^2\phi_y \cos^2\theta_y \\ -6\partial_{\alpha_y} h(\alpha_y)\sin\alpha_y\cos\alpha_y \sin^2\phi_y\cos^2\theta_y \\ +h(\alpha_y)\left\{6\sin^2\alpha_y\sin^2\phi_y\cos^2\theta_y - 2 + 2\sin^2\alpha_y\right\} \end{array} \right\} u \; vol_{S_y^3}$$

$$for\ all\ u \in C^\infty(S^3)\ ,$$

and, using relations 2.11 from chapter 2, this equation is in turn easily re-expressed solely in terms of α_y and the single ambient coordinate w_y^1, as follows;

$$\int_{S_y^3} \left\{ \begin{array}{l} \left\{ \begin{array}{l} -\partial^2_{\alpha_y} f(\alpha_y) - 2\cot\alpha_y\partial_{\alpha_y} f(\alpha_y) + 4f(\alpha_y) \\ +2\partial_{\alpha_y} g(\alpha_y)\sin\alpha_y + 4g(\alpha_y)\cos\alpha_y - 2\cos^2\alpha_y h(\alpha_y) \end{array} \right\} \\ + \left\{ \begin{array}{l} -2\csc\alpha_y\partial_{\alpha_y} g(\alpha_y) - \partial^2_{\alpha_y} h(\alpha_y) \\ -6\cot\alpha_y\partial_{\alpha_y} h(\alpha_y) + 6h(\alpha_y) \end{array} \right\} (w_y^1)^2 \end{array} \right\} u \; vol_{S_y^3} \qquad (3.59)$$

$$= 0 \quad for\ all\ u \in C^\infty(S^3).$$

But applying standard theory (i.e. the lemma of Dubois-Raymond) it now follows immediately that the function multiplying u in the integrand of 3.59 must be identically zero. That is

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} -\partial^2_{\alpha_y} f(\alpha_y) - 2\cot\alpha_y\partial_{\alpha_y} f(\alpha_y) + 4f(\alpha_y) \\ +2\partial_{\alpha_y} g(\alpha_y)\sin\alpha_y + 4g(\alpha_y)\cos\alpha_y - 2\cos^2\alpha_y h(\alpha_y) \end{array} \right\} \\ + \left\{-2\csc\alpha_y\partial_{\alpha_y} g(\alpha_y) - \partial^2_{\alpha_y} h(\alpha_y) - 6\cot\alpha_y\partial_{\alpha_y} h(\alpha_y) + 6h(\alpha_y)\right\} (w_y^1)^2 \end{array} \right\} \equiv 0. \qquad (3.60)$$

Since α_y and w_y^1 can be varied independently this therefore implies at once the following two independent, coupled ODE's in f, g and h;

$$\left\{ \begin{array}{l} -\partial^2_{\alpha_y} f(\alpha_y) - 2\cot\alpha_y\partial_{\alpha_y} f(\alpha_y) + 4f(\alpha_y) \\ 2\partial_{\alpha_y} g(\alpha_y)\sin\alpha_y + 4g(\alpha_y)\cos\alpha_y - 2\cos^2\alpha_y h(\alpha_y) \end{array} \right\} = 0 \qquad (3.61)$$

and

$$-2\csc\alpha_y\partial_{\alpha_y} g(\alpha_y) - \partial^2_{\alpha_y} h(\alpha_y) - 6\cot\alpha_y\partial_{\alpha_y} h(\alpha_y) + 6h(\alpha_y) = 0\ . \qquad (3.62)$$

These represent our final reduction of the first defining equation in 3.22. We turn now to the second defining equation.

(ii) We perform the translation of this second defining equation in 3.22 into ODE's in f, g and h in identical fashion to the first, and so we will be more schematic and less detailed in our working.

The equation we obtain in analogy with 3.49, on substituting 3.39 in this case, is

$$I_1(u) + I_2(u) + I_3(u) = 0 \; for\ all\ u \in C^\infty(S^3) \tag{3.63}$$

where

$$I_1(u) = 2\int_{S^3_y}\left\{f(\alpha_y) + h(\alpha_y)\sin^2\alpha_y \sin^2\phi_y \cos^2\theta_y\right\} \times$$
$$\left\{\cos\phi_y \partial_{\alpha_y} u - \cot\alpha_y \sin\phi_y \partial_{\phi_y} u - \partial_{\theta_y} u\right\} vol_{S^3_y}, \quad and$$

$$I_2(u) = \int_{S^3_y}\left\{\begin{array}{c} g(\alpha_y)\sin\alpha_y\cos\phi_y + \\ h(\alpha_y)\sin^2\alpha_y \sin^2\phi_y \sin\theta_y\cos\theta_y \end{array}\right\}((\Delta+4)u)vol_{S^3_y}, \quad and$$

$$I_3(u) = -2\int_{S^3_y}\left\{-g(\alpha_y)\sin\alpha_y\sin\phi_y\sin\theta_y + h(\alpha_y)\sin^2\alpha_y\sin\phi_y\cos\phi_y\cos\theta_y\right\}\times$$
$$\left\{\begin{array}{c} \sin\phi_y\cos\theta_y\partial_{\alpha_y}u + (\cot\alpha_y\cos\phi_y\cos\theta_y + \sin\theta_y)\partial_{\phi_y}u \\ +(\cot\phi_y\cos\theta_y - \cot\alpha_y\csc\phi_y\sin\theta_y)\partial_{\theta_y}u \end{array}\right\} vol_{S^3_y}. \tag{3.64}$$

Again we tackle these $I_i(u)$ in turn.

(a) $\underline{I_1(u)}$: Proceeding exactly as for $I_3(u)$ and $I_4(u)$ in case (i), we find here

$$\begin{aligned} I_1(u) \;=\; & -2\int_{S^3_y}\left\{\begin{array}{c} \partial_{\alpha_y} f(\alpha_y)\sin^2\alpha_y + \\ 2f(\alpha_y)\sin\alpha_y\cos\alpha_y \end{array}\right\}\sin\phi_y\cos\phi_y u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y \\ & -2\int_{S^3_y}\left\{\begin{array}{c} \partial_{\alpha_y} h(\alpha_y)\sin^4\alpha_y \\ +4h(\alpha_y)\sin^3\alpha_y\cos\alpha_y \end{array}\right\}\sin^3\phi_y\cos\phi_y\cos^2\theta_y u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y \\ & +2\int_{S^3_y} f(\alpha_y)\sin\alpha_y\cos\alpha_y\{2\sin\phi_y\cos\phi_y\}u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y \\ & +2\int_{S^3_y} h(\alpha_y)\sin^3\alpha_y\cos\alpha_y\left\{4\sin^3\phi_y\cos\phi_y\right\}\cos^2\theta_y u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y \\ & +2\int_{S^3_y} h(\alpha_y)\sin^4\alpha_y\sin^3\phi_y\{-2\sin\theta_y\cos\theta_y\}u\, d\alpha_y \wedge d\phi_y \wedge d\theta_y \end{aligned}$$

$$= -2\int_{S_y^3} \partial_{\alpha_y} f(\alpha_y)\sin^2\alpha_y \sin\phi_y \cos\phi_y u \, d\alpha_y \wedge d\phi_y \wedge d\theta_y$$

$$-2\int_{S_y^3} \partial_{\alpha_y} h(\alpha_y)\sin^4\alpha_y \sin^3\phi_y \cos\phi_y \cos^2\theta_y u \, d\alpha_y \wedge d\phi_y \wedge d\theta_y$$

$$-4\int_{S_y^3} h(\alpha_y)\sin^4\alpha_y \sin^3\phi_y \sin\theta_y \cos\theta_y u \, d\alpha_y \wedge d\phi_y \wedge d\theta_y ,$$

$$i.e. \quad I_1(u) = \int_{S_y^3} \left\{ \begin{array}{l} -2\partial_{\alpha_y} f(\alpha_y)\cos\phi_y \\ -2\partial_{\alpha_y} h(\alpha_y)\sin^2\alpha_y \sin^2\phi_y \cos\phi_y \cos^2\theta_y \\ -4h(\alpha_y)\sin^2\alpha_y \sin^2\phi_y \sin\theta_y \cos\theta_y \end{array} \right\} u \, vol_{S_y^3} . \qquad (3.65)$$

(b) <u>$I_2(u)$</u>: For $I_2(u)$ we proceed as for $I_2(u)$ in case (i), i.e. by noting that $g(\alpha_y)\sin\alpha_y\cos\phi_y + h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y$ is in $H^2(S^3)$ and using self-adjointness of $(\Delta + 4)$, together with 3.52. We deduce that

$$I_2(u) = \int_{S_y^3} \left\{ (\Delta + 4) \left\{ \begin{array}{c} g(\alpha_y)\sin\alpha_y\cos\phi_y + \\ h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y \end{array} \right\} \right\} u \, vol_{S_y^3}$$

$$= \int_{S_y^3} \left\{ \begin{array}{l} - \left\{ \begin{array}{l} \partial^2_{\alpha_y} g(\alpha_y)\sin\alpha_y\cos\phi_y + 2\partial_{\alpha_y} g(\alpha_y)\cos\alpha_y\cos\phi_y \\ -g(\alpha_y)\sin\alpha_y\cos\phi_y + \partial^2_{\alpha_y} h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y \\ +4\partial_{\alpha_y} h(\alpha_y)\sin\alpha_y\cos\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y \\ +2h(\alpha_y)\left\{1 - 2\sin^2\alpha_y\right\}\sin^2\phi_y\sin\theta_y\cos\theta_y \end{array} \right\} \\ -2\cot\alpha_y \left\{ \begin{array}{l} \partial_{\alpha_y} g(\alpha_y)\sin\alpha_y\cos\phi_y + g(\alpha_y)\cos\alpha_y\cos\phi_y \\ +\partial_{\alpha_y} h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y \\ +2h(\alpha_y)\sin\alpha_y\cos\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y \end{array} \right\} \\ -\csc^2\alpha_y \left\{ \begin{array}{l} -g(\alpha_y)\sin\alpha_y\cos\phi_y \\ +2h(\alpha_y)\sin^2\alpha_y\left\{1 - 2\sin^2\phi_y\right\}\sin\theta_y\cos\theta_y \end{array} \right\} \\ -\csc^2\alpha_y\cot\phi_y \left\{ \begin{array}{l} -g(\alpha_y)\sin\alpha_y\sin\phi_y \\ +2h(\alpha_y)\sin^2\alpha_y\sin\phi_y\cos\phi_y\sin\theta_y\cos\theta_y \end{array} \right\} \\ -\csc^2\alpha_y\csc^2\phi_y\left\{-4h(\alpha_y)\sin\theta_y\cos\theta_y\right\} \\ +4g(\alpha_y)\sin\alpha_y\cos\phi_y + 4h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y \end{array} \right\} u \, vol_{S_y^3}$$

$$
= \int_{S_y^3} \left\{ \begin{array}{l} -\partial^2_{\alpha_y} g(\alpha_y)\sin\alpha_y\cos\phi_y - 4\partial_{\alpha_y} g(\alpha_y)\cos\alpha_y\cos\phi_y \\ +g(\alpha_y)\left\{ \begin{array}{l} 5\sin\alpha_y\cos\phi_y + 2\csc\alpha_y\cos\phi_y \\ -2\cos^2\alpha_y\csc\alpha_y\cos\phi_y \end{array} \right\} \\ -\partial^2_{\alpha_y} h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y \\ -6\partial_{\alpha_y} h(\alpha_y)\sin\alpha_y\cos\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y \\ +h(\alpha_y)\left\{ \begin{array}{l} 8\sin^2\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y - 2\sin^2\phi_y\sin\theta_y\cos\theta_y \\ -4\cos^2\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y + 4\sin^2\phi_y\sin\theta_y\cos\theta_y \\ +2\sin\theta_y\cos\theta_y - 2\cos^2\phi_y\sin\theta_y\cos\theta_y \end{array} \right\} \end{array} \right\} u\, vol_{S_y^3}\,,
$$

$$
i.e. \quad I_2(u) = \int_{S_y^3} \left\{ \begin{array}{l} -\partial^2_{\alpha_y} g(\alpha_y)\sin\alpha_y\cos\phi_y - 4\partial_{\alpha_y} g(\alpha_y)\cos\alpha_y\cos\phi_y \\ +7g(\alpha_y)\sin\alpha_y\cos\phi_y \\ -\partial^2_{\alpha_y} h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y \\ -6\partial_{\alpha_y} h(\alpha_y)\sin\alpha_y\cos\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y \\ +12h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y \end{array} \right\} u\, vol_{S_y^3}\,. \tag{3.66}
$$

(c) $\underline{I_3(u)}$**:** Finally, by identical reasoning to that in $I_1(u)$ above (or $I_3(u)$ and $I_4(u)$ from case (i)) we find that

$$
\begin{aligned}
I_3(u) = \; & -2\int_{S_y^3} \left\{ \begin{array}{l} \partial_{\alpha_y} g(\alpha_y)\sin^3\alpha_y \\ +3g(\alpha_y)\sin^2\alpha_y\cos\alpha_y \end{array} \right\} \sin^3\phi_y\sin\theta_y\cos\theta_y u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y \\
& +2\int_{S_y^3} \left\{ \begin{array}{l} \partial_{\alpha_y} h(\alpha_y)\sin^4\alpha_y \\ +4h(\alpha_y)\sin^3\alpha_y\cos\alpha_y \end{array} \right\} \sin^3\phi_y\cos\phi_y\cos^2\theta_y u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y \\
& -2\int_{S_y^3} g(\alpha_y)\sin^2\alpha_y\cos\alpha_y \left\{ \begin{array}{l} 2\sin\phi_y\cos^2\phi_y \\ -\sin^3\phi_y \end{array} \right\} \sin\theta_y\cos\theta_y u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y \\
& +2\int_{S_y^3} h(\alpha_y)\sin^3\alpha_y\cos\alpha_y \left\{ \begin{array}{l} 2\sin\phi_y\cos^3\phi_y \\ -2\sin^3\phi_y\sin\phi_y \end{array} \right\} \cos^2\theta_y u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y \\
& -2\int_{S_y^3} g(\alpha_y)\sin^3\alpha_y \left\{2\sin\phi_y\cos\phi_y\right\} \sin^2\theta_y u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y
\end{aligned}
$$

$$+2\int_{S_y^3} h(\alpha_y)\sin^4\alpha_y \left\{2\sin\phi_y\cos^2\phi_y - \sin^3\phi_y\right\}\sin\theta_y\cos\theta_y u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$-2\int_{S_y^3} g(\alpha_y)\sin^3\alpha_y\sin\phi_y\cos\phi_y\left\{1-2\sin^2\theta_y\right\}u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$+2\int_{S_y^3} h(\alpha_y)\sin^4\alpha_y\sin\phi_y\cos^2\phi_y\left\{-2\sin\theta_y\cos\theta_y\right\}u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$+2\int_{S_y^3} g(\alpha_y)\sin^2\alpha_y\cos\alpha_y\sin\phi_y\left\{2\sin\theta_y\cos\theta_y\right\}u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$-2\int_{S_y^3} h(\alpha_y)\sin^3\alpha_y\cos\alpha_y\sin\phi_y\cos\phi_y\left\{1-2\sin^2\theta_y\right\}u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$= -2\int_{S_y^3}\partial_{\alpha_y}g(\alpha_y)\sin^3\alpha_y\sin^3\phi_y\sin\theta_y\cos\theta_y u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$-2\int_{S_y^3} g(\alpha_y)\sin^3\alpha_y\sin\phi_y\cos\phi_y u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$+2\int_{S_y^3}\partial_{\alpha_y}h(\alpha_y)\sin^4\alpha_y\sin^3\phi_y\cos\phi_y\cos^2\theta_y u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y$$

$$+\int_{S_y^3} h(\alpha_y)\left\{\begin{array}{l}2\sin^3\alpha_y\cos\alpha_y\sin\phi_y\cos\phi_y\\ -2\sin^4\alpha_y\sin^3\phi_y\sin\theta_y\cos\theta_y\end{array}\right\}u\, d\alpha_y\wedge d\phi_y\wedge d\theta_y\ ,$$

$$i.e.\quad I_3(u) = \int_{S_y^3}\left\{\begin{array}{l}-2\partial_{\alpha_y}g(\alpha_y)\sin\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y\\ -2g(\alpha_y)\sin\alpha_y\cos\phi_y\\ +2\partial_{\alpha_y}h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\cos\phi_y\cos^2\theta_y\\ +2h(\alpha_y)\left\{\begin{array}{l}\sin\alpha_y\cos\alpha_y\cos\phi_y\\ -\sin^2\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y\end{array}\right\}\end{array}\right\}u\ vol_{S_y^3}\ . \qquad (3.67)$$

With our expressions for $I_1(u),\ldots,I_3(u)$ in 3.65-3.67 we now return, as before, and substitute them back into 3.63. The second defining equation then becomes

$$\int_{S_y^3}\left\{\begin{array}{l}-2\partial_{\alpha_y}f(\alpha_y)\cos\phi_y - \partial^2_{\alpha_y}g(\alpha_y)\sin\alpha_y\cos\phi_y\\ -\partial_{\alpha_y}g(\alpha_y)\left\{\begin{array}{l}4\cos\alpha_y\cos\phi_y\\ +2\sin\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y\end{array}\right\}\\ +5g(\alpha_y)\sin\alpha_y\cos\phi_y - \partial^2_{\alpha_y}h(\alpha_y)\sin^2\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y\\ -6\partial_{\alpha_y}h(\alpha_y)\sin\alpha_y\cos\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y\\ +h(\alpha_y)\left\{\begin{array}{l}6\sin^2\alpha_y\sin^2\phi_y\sin\theta_y\cos\theta_y\\ +2\sin\alpha_y\cos\alpha_y\cos\phi_y\end{array}\right\}\end{array}\right\}u\ vol_{S_y^3}$$

$$= 0\ for\ all\ u\in C^\infty(S^3)$$

which, as in case (i), can be re-expressed in terms of the ambient coordinates, this time as the statement that

$$\int_{S^3_y} \left\{ \begin{array}{l} \left\{ \begin{array}{c} -2\csc\alpha_y \partial_{\alpha_y} f(\alpha_y) - \partial^2_{\alpha_y} g(\alpha_y) - 4\cot\alpha_y \partial_{\alpha_y} g(\alpha_y) \\ +5g(\alpha_y) + 2h(\alpha_y)\cos\alpha_y \end{array} \right\} w^3_y \\ + \left\{ \begin{array}{l} -2\csc\alpha_y \partial_{\alpha_y} g(\alpha_y) - \partial^2_{\alpha_y} h(\alpha_y) \\ -6\cot\alpha_y \partial_{\alpha_y} h(\alpha_y) + 6h(\alpha_y) \end{array} \right\} w^1_y w^2_y \end{array} \right\} u\, vol_{S^3_y}$$

$$= 0 \quad for\ all\ u \in C^\infty(S^3).$$

As before, we see that this leads directly to two coupled ODE's in f, g and h, given by requiring the coefficients of w^3_y and $w^1_y w^2_y$ in this expression to be identically zero. But since one of these is identical to an ODE we already have, in 3.62, we thus obtain only one new ODE in f, g and h from the second defining equation in 3.22, namely

$$\left\{ \begin{array}{l} -2\csc\alpha_y \partial_{\alpha_y} f(\alpha_y) - \partial^2_{\alpha_y} g(\alpha_y) - 4\cot\alpha_y \partial_{\alpha_y} g(\alpha_y) \\ +5g(\alpha_y) + 2h(\alpha_y)\cos\alpha_y \end{array} \right\} = 0. \qquad (3.68)$$

(iii) Turning now to the seven remaining defining equations in 3.22, thankfully it can be checked that, as predicted, we in fact get no new ODE's in f, g and h from these. Instead the ODE's that arise are simply combinations of the three we already have in 3.61, 3.62 and 3.68, in the same way that even in the second defining equation above one of the resulting ODE's was simply a repetition of the ODE 3.62 that we had obtained from translating the first defining equation. As remarked at the beginning of this step, this massive redundancy, in which nine coupled PDE's in nine functions collapse to just the three ODE's necessary to determine f, g and h, is strong evidence of the validity of our working so far.

It only remains in this step then to further reduce the three ODE's 3.61, 3.62 and 3.68 in f, g and h to just two ODE's in our two *independent* functions g and h by using relation 3.46 to eliminate f.

Clearly equation 3.62 is unaffected in this process, and on substituting 3.46 into 3.61 and 3.68 it is only a moment's work to see that they both degenerate to the same

ODE in the remaining functions g and h, namely

$$\partial^2_{\alpha_y} g(\alpha_y) + 2\cot\alpha_y \partial_{\alpha_y} g(\alpha_y) - 3g(\alpha_y) - 2h(\alpha_y)\cos\alpha_y = 0 . \qquad (3.69)$$

Note once again that, as discussed at the outset of this step, the fact that both ODE's degenerate consistently here to the same equation on applying our apriori relationship 3.46, represents a further successful test of the internal consistency of our working to this point.

We have now completed the task we set ourselves in this step, of translating the defining equations 3.22 for $G_{1,2}(N,y)$ into a pair of ODE's, 3.62 and 3.69, for g and h. This completes Step 5.

Step 6: The Final Form of $G_{1,2}$

Let us summarise where we are now in this lengthy section. In steps 1-5 we have performed intensive computations to reduce the calculation of the fundamental solution $G_{1,2}(N,y)$ to the determination of just two unknown independent functions g and h in $C^\infty(0,\pi]$, for which we have a pair of coupled second-order ODE's, 3.62 and 3.69, together with the qualitative boundary conditions at 0 and π, in 3.48 and 3.47, needed to ensure a unique solution. If g and h could be determined from this information then the form of $G_{1,2}(N,y)$ would be given explicitly by 3.39 together with the relation 3.46.

Clearly therefore, the next and final step in determining $G_{1,2}(N,y)$ should be simply to solve for g and h exactly from the ODE's 3.62 and 3.69 coupled with the boundary data 3.47 and 3.48.

Unfortunately this is easier said than done. Despite extensive efforts, we have in fact been unable so far to completely solve the above-mentioned ODE's in elementary terms and, at this time, the exact form of the functions g and h is still unknown.

Fortunately, however, this does *not* mean that the situation we find ourselves in is hopeless. The reasons are two-fold.

The first, and main, one is that, after all, for this thesis it is not ultimately the

Green's form G, of $\triangle$, that we are looking for, but rather the Green's form L, of d. Thus our inability to completely determine G need not, *apriori*, be fatal.

The second reason is that, although we have been unable to *completely* solve our ODE's for g and h, this does not mean that we are unable to make *any* progress in dealing with them.

Combining these points, the claim that we make, and which we will verify in Section 3.3, is that although we cannot currently determine g and h, and hence $G_{1,2}$, exactly in simple terms, we *can* solve precisely for the combinations of g, h and their derivatives that arise in the expression for L which we obtain on applying δ in the x-variable to G, as prescribed by equation 3.5. Thus, although G remains out of reach, we are able to make just sufficient progress to solve exactly for L.

To expand briefly on this and make things more precise, we will find once we have made good on this claim, that it allows us to reduce the pair of coupled ODE's in g and h above to a single second-order ODE in g alone. Our stated inability to solve exactly for g and h (and hence for $G_{1,2}$) is thus, in precise terms, the statement that we cannot, as yet, solve this last ODE in elementary form. This inability to carry out the second half of the solution for g and h, however, is immaterial, since it is really only L, and not G, that we need explicitly.

We make one last, more general remark before continuing. This is that, from a moral point of view, the state of affairs just described, in which we just manage to solve for precisely what we need, no more and no less, is not perhaps as miraculously lucky as it seems at first glance. For, if we consider the extraordinary power and success of the Witten/TQFT treatment of Chern-Simons theory and in particular its exact solubility on S^3, it is no longer so surprising that the key ingredient of the same theory from the perturbative viewpoint, namely the "propagator" L, should have a simple closed form, whereas we have no corresponding *apriori* reason for expecting that the Green's form G should be equally simple. From this perspective, indeed, it is a *natural* thing to do, in seeking to solve 3.62 and 3.69 for g and h, to consider straight away the combinations of g, h and derivatives thereof that turn up in computing L, and to try to evaluate them explicitly first.

Having made all these remarks then, and resigned ourselves to being unable to carry out the last step in computing $G_{1,2}(N, y)$, it only remains in this final step of this section to pull together the summary that we gave at the outset of the step of the current state of our calculation of $G_{1,2}(N, y)$, into a single, coherent, complete statement while simultaneously generalising it from a result regarding just the fundamental solution $G_{1,2}(N, y)$ to a result describing the full Green's form $G_{1,2}(x, y)$ for arbitrary x.

This latter generalisation is done in identical spirit to the way it was performed for the $0,3$-piece of G at the end of subsection 3.2.1. However, rather than use an arbitrary isometry mapping x to N as we did there, in this case we need to use precisely the left-translation map $\mathcal{L}_{x^{-1}}$, since here we have to preserve not just the volume-form $vol_{S^3_y}$ but individual left-invariant 1-forms θ^i, in pulling back our expression 3.39 for $G_{1,2}(N, y)$. The final result is easily seen to be the following;

Proposition 3.8 *The 1,2-piece of the Green's form G is given at arbitrary $x, y \in S^3$, $x \neq y$ by $G_{1,2}(x, y) = A_{ij}(x, y)\theta^i_x \wedge (\theta_y \wedge \theta_y)^{(j)}$, where $A_{ij}(x, y)$ is given in standard matrix form by*

$$A(x,y) = -g(\alpha_{x^{-1}y})\cos\alpha_{x^{-1}y}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + g(\alpha_{x^{-1}y})\begin{pmatrix} 0 & w^3_{x^{-1}y} & -w^2_{x^{-1}y} \\ -w^3_{x^{-1}y} & 0 & w^1_{x^{-1}y} \\ w^2_{x^{-1}y} & -w^1_{x^{-1}y} & 0 \end{pmatrix}$$

$$+ h(\alpha_{x^{-1}y})\begin{pmatrix} (w^1_{x^{-1}y})^2 & w^1_{x^{-1}y}w^2_{x^{-1}y} & w^1_{x^{-1}y}w^3_{x^{-1}y} \\ w^2_{x^{-1}y}w^1_{x^{-1}y} & (w^2_{x^{-1}y})^2 & w^2_{x^{-1}y}w^3_{x^{-1}y} \\ w^3_{x^{-1}y}w^1_{x^{-1}y} & w^3_{x^{-1}y}w^2_{x^{-1}y} & (w^3_{x^{-1}y})^2 \end{pmatrix}. \tag{3.70}$$

Here the functions g, $h \in C^\infty(0, \pi]$ are the unique solutions of the coupled ODE's

$$-2\csc\alpha_y\partial_{\alpha_y}g(\alpha_y) - \partial^2_{\alpha_y}h(\alpha_y) - 6\cot\alpha_y\partial_{\alpha_y}h(\alpha_y) + 6h(\alpha_y) = 0 \tag{3.71}$$

and

$$\partial^2_{\alpha_y} g(\alpha_y) + 2\cot\alpha_y \partial_{\alpha_y} g(\alpha_y) - 3g(\alpha_y) - 2h(\alpha_y)\cos\alpha_y = 0 \tag{3.72}$$

on $(0,\pi]$, *with the boundary conditions that, for* α *near* 0

$$g(\alpha) \sim \tfrac{1}{4\pi\alpha} + O(\alpha^0) \tag{3.73}$$

and hence, by 3.71,

$$h(\alpha) \sim -\tfrac{1}{8\pi\alpha} + O(\alpha^0), \tag{3.74}$$

while, for α *near* π, *both* g *and* h *have power series expansions around* $\alpha = \pi$ *which are even in* $\tilde{\alpha} \equiv (\pi - \alpha)$.

This concludes this section analysing $G_{1,2}$.

3.2.3 Computing $G_{2,1}(x,y)$ and $G_{3,0}(x,y)$ on S^3

So far we have now computed the $0,3$ and $1,2$-pieces of the Green's form G on S^3. Fortunately, calculating the remaining two pieces of G is nowhere near as long and arduous as were these initial two. Indeed, all the hard work is already done and we can obtain $G_{2,1}(x,y)$ and $G_{3,0}(x,y)$ very simply from our existing formulae for $G_{1,2}(x,y)$ and $G_{0,3}(x,y)$, by a trick using the trivial observation that $\triangle$ commutes with Hodge star, $*$, as operators on $\Omega^*(S^3)$.

Consider first the $2,1$-piece $G_{2,1}(x,y)$. From 3.7 its defining equation is that

$$\int_{S^3_y} G(x,y) \wedge \Delta\nu(y) \;=\; \nu(x) \quad \textit{for all } x \in S^3 \textit{ and for all } \nu \in \Omega^2(S^3). \tag{3.75}$$

Expanding this equation in components, along the lines of 3.17, we write

$$\begin{aligned} &\nu(y) = \nu_i(y)(\theta_y \wedge \theta_y)^{(i)} \;\; \textit{and} \;\; \Delta\nu(y) = (\Delta\nu)_i(y)(\theta_y \wedge \theta_y)^{(i)} \\ &\textit{and} \;\; G_{2,1}(x,y) = B_{ij}(x,y)(\theta_x \wedge \theta_x)^{(i)} \wedge \theta^j_y\,, \end{aligned} \tag{3.76}$$

and then 3.75 clearly becomes the component equation

$$\begin{aligned} &\textstyle\int_{S^3_y} \left\{ B_{ij}(x,y)(\Delta\nu)_j(y) \right\} vol_{S^3_y} = \nu_i(x) \\ &\textit{for all } x \in S^3,\ \nu \in \Omega^2(S^3) \textit{ and } i = 1,2,3. \end{aligned} \tag{3.77}$$

But now, as a 2-form, ν can be written as the Hodge star of a 1-form $\tilde{\nu} = \tilde{\nu}_i \theta^i$, and we see easily using the relation $*\theta^i = (\theta \wedge \theta)^{(i)}$ that the components of ν and $\tilde{\nu}$ are in fact the same,

$$i.e. \quad \tilde{\nu}_i = \nu_i \quad for\ all\ i = 1,2,3. \tag{3.78}$$

Moreover, since $\triangle$ commutes with $*$ as operators on $\Omega^*(S^3)$, it is equally easy to see that

$$(\triangle\nu)_i(\theta \wedge \theta)^{(i)} = \triangle\nu = \triangle * \tilde{\nu} = *\triangle\tilde{\nu} = *\left\{(\triangle\tilde{\nu})_i \theta^i\right\} = (\triangle\tilde{\nu})_i(\theta \wedge \theta)^{(i)}$$

so that we also have equality of the components of $\triangle\nu$ and $\triangle\tilde{\nu}$,

$$i.e. \quad (\triangle\tilde{\nu})_i = (\triangle\nu)_i \quad for\ all\ i = 1,2,3. \tag{3.79}$$

By substituting 3.78 and 3.79 into 3.77 and noting the bijectivity of $*$ between $\Omega^1(S^3)$ and $\Omega^2(S^3)$, we can thus transform this defining equation for $G_{2,1}$ on 2-forms into an equivalent equation concerning the Laplacian on 1-forms, namely

$$\begin{array}{l} \int_{S^3_y} \left\{B_{ij}(x,y)(\triangle\tilde{\nu})_j(y)\right\} vol_{S^3_y} = \tilde{\nu}_i(x) \\ for\ all\ x \in S^3,\ \tilde{\nu} \in \Omega^1(S^3)\ and\ i = 1,2,3. \end{array} \tag{3.80}$$

But we have already considered the Laplacian on 1-forms in depth in the last section, and equation 3.18 appearing there for $G_{1,2}(x,y) = A_{ij}(x,y)\theta^i_x \wedge (\theta_y \wedge \theta_y)^{(j)}$ indeed translates at once into an equation almost identical to 3.80, namely

$$\begin{array}{l} \int_{S^3_y} \left\{\sum_{j=1}^3 A_{ij}(x,y)(\triangle\tilde{\nu})_j(y)\right\} vol_{S^3_y} = -\tilde{\nu}_i(x) \\ for\ all\ x \in S^3,\ \tilde{\nu} \in \Omega^1(S^3)\ and\ i = 1,2,3. \end{array} \tag{3.81}$$

Comparing 3.80 and 3.81 it follows immediately that we must simply have $B_{ij}(x,y) = -A_{ij}(x,y)$. And combining this with result 3.8 from the last section and 3.76 we thus obtain directly our final result describing the 2,1-piece of G;

Proposition 3.9 *The 2,1-piece of the Green's form G is given at arbitrary $x, y \in S^3$, $x \neq y$ by $G_{2,1}(x,y) = B_{ij}(x,y)(\theta_x \wedge \theta_x)^{(i)} \wedge \theta^j_y$, where the coefficients $B_{ij}(x,y)$ are related to the coefficients $A_{ij}(x,y)$ of $G_{1,2}(x,y)$, described in detail in result 3.8,* *simply by* $B_{ij}(x,y) = -A_{ij}(x,y)$, *i.e.*

$$B(x,y) = g(\alpha_{x^{-1}y})\cos\alpha_{x^{-1}y}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - g(\alpha_{x^{-1}y})\begin{pmatrix} 0 & w^3_{x^{-1}y} & -w^2_{x^{-1}y} \\ -w^3_{x^{-1}y} & 0 & w^1_{x^{-1}y} \\ w^2_{x^{-1}y} & -w^1_{x^{-1}y} & 0 \end{pmatrix}$$

$$- h(\alpha_{x^{-1}y})\begin{pmatrix} (w^1_{x^{-1}y})^2 & w^1_{x^{-1}y}w^2_{x^{-1}y} & w^1_{x^{-1}y}w^3_{x^{-1}y} \\ w^2_{x^{-1}y}w^1_{x^{-1}y} & (w^2_{x^{-1}y})^2 & w^2_{x^{-1}y}w^3_{x^{-1}y} \\ w^3_{x^{-1}y}w^1_{x^{-1}y} & w^3_{x^{-1}y}w^2_{x^{-1}y} & (w^3_{x^{-1}y})^2 \end{pmatrix} \qquad (3.82)$$

where the functions $g,\, h \in C^\infty(0,\pi]$ *are the same as described in result 3.8 (i.e. satisfy the coupled ODE's 3.71 and 3.72 and the associated boundary conditions at* 0 *and* π *given with these).*

In identical fashion we can likewise show that the 3,0-piece of G is very closely related to the 0,3-piece, up to a convention-based minus sign. Omitting the (straightforward) details, the result we obtain for this final piece of G, in light of result 3.2, is that

Proposition 3.10 *For* $x,\, y \in S^3$, $x \neq y$, *the 3,0-piece of* G *is given by*

$$G_{3,0}(x,y) = -\frac{1}{4\pi^2}\left[(\pi - \alpha_{x^{-1}y})\cot\alpha_{x^{-1}y} + 1\right] vol_{S^3_x}\,, \qquad (3.83)$$

with the same smoothness observation as in result 3.2 regarding extension by 0 when $\alpha_{x^{-1}y} = \pi$.

This concludes this section.

3.2.4 Summary

Combining results 3.2, 3.8, 3.9, and 3.10 we have now described as explicitly as we can the full Green's form, $G(x,y)$, of $\triangle$ on S^3. The time has come to turn from G to the object we are ultimately really interested in, namely L, the Green's form of d described at the beginning of the chapter. We need to make good on the claims we made at the end of section 3.2.2 regarding L on S^3. In so doing we will obtain a complete, explicit, closed-form description of L on S^3. We will then, in the final section of this chapter, turn to deriving from this the corresponding complete, closed-form description of L on the lens spaces $L[p]$, which was our whole purpose in this chapter.

3.3 Computing the Green's form $L(x,y)$ of d on S^3

In equation 3.5 and the discussion immediately following it, we outlined how the Green's form L, of d, on S^3 can be deduced from the Green's form G, of Δ, that we have just spent the last section deriving, by "applying δ to it in the x-variable." Let us be more precise about what we mean by this.

By the definition of G as the Schwartz kernel of Δ^{-1}, equation 3.5 means that

$$\nu(x) = \delta_x \left\{ \int_{S^3_y} G(x,y) \wedge d\nu(y) \right\} \qquad for\ all\ x \in S^3\ and\ for\ all\ \nu \in Im\delta \tag{3.84}$$

where we have written the operator δ here as δ_x simply to emphasise the fact that it acts in the x-variable on the form that arises *after* performing the integral over y.

To compare this with the defining equation 3.2 for L we would clearly like to take δ_x inside the integral. The legitimacy of doing this, however, is somewhat subtle since the usual sufficient conditions to permit this involve the existence, for each x, of a neighbourhood U of x and an L^1-function on S^3 which, for all $x' \in U$, bounds first x-derivatives of the coefficient functions of G pointwise a.e. in y, and it is easy to see that in fact such a state of affairs *never* subsists.

Suffice it to say, nevertheless, that these "usual sufficient conditions" are stronger than necessary, and we *can* rigorously justify taking the δ_x inside the integral. The proof of this is modelled closely on that of a very similar result, in [GT, Lemma 4.1], for the Newtonian potential of a smooth function on $\Re^n$. The only new ingredient that has to be supplied beyond straightforward translation of the argument to our setting, is *apriori* estimates on the coefficient functions of G and their first x-derivatives, which we can derive from compactness of S^3 and our knowledge of the asymptotics of G near the diagonal.

To avoid interrupting our derivation of L at this stage with a somewhat messy and technical analytic excursion, however, we omit a detailed proof. The precise result we obtain when we do take the δ_x inside the integral is that

$$\delta_x \left\{ \int_{S^3_y} G(x,y) \wedge d\nu(y) \right\} = - \int_{S^3_y} \{\delta_x G(x,y)\} \wedge d\nu(y) \qquad \text{for all } \nu \in Im\delta \quad (3.85)$$

where here we have adopted the convention on the right hand side that the operator δ_x on mixed forms (i.e. elements of $\Omega^*(S^3_x \times S^3_y)$) is defined as acting by partial differentiation (i.e. keeping y fixed) and as though y-form pieces (i.e. θ^i_y, $(\theta_y \wedge \theta_y)^{(i)}$, *or* $vol_{S^3_y}$) are absent; i.e., $\delta_x G(x,y)$ is defined by expanding $G(x,y)$ in the generators $\left\{\theta^i_x, \theta^j_y\right\}^3_{i,j=1}$ of the algebra $\Omega^*(S^3_x \times S^3_y)$, then ignoring the θ^j_y's and their products, fixing y, and applying ordinary δ_x to the resulting element of $\Omega^*(S^3_x)$. Note that the unexpected minus sign in 3.85 is a consequence of this choice of convention, combined with the unusual sign convention from chapter 1 for mixed integrals which has arisen on several occasions already.

Comparing 3.85 with the defining equation 3.2 for L, we arrive at our final formula giving L in terms of G on S^3, namely

$$L(x,y) = -\delta_x G(x,y) \qquad (3.86)$$

where δ_x is defined as just described.

Using this we now turn to evaluating each of the three pieces of L ($L_{0,2}$, $L_{1,1}$, and $L_{2,0}$) explicitly from our knowledge of G. We start with the 0,2-piece, $L_{0,2}(x,y)$.

3.3.1 Computing $L_{0,2}(x,y)$ on S^3

Since δ_x reduces x-form degree by one, contributions to $L_{0,2}$ in 3.86 come only from the 1,2-piece of G. That is,

$$L_{0,2}(x,y) = -\delta_x G_{1,2}(x,y)$$

which, in light of our definition of δ_x and result 3.8, becomes

$$\begin{aligned} L_{0,2}(x,y) &= *_x d_x *_x \left\{A_{ij}(x,y)\theta_x^i \wedge (\theta_y \wedge \theta_y)^{(j)}\right\} \\ &= \{(X_i)_x (A_{ij}(x,y))\} (\theta_y \wedge \theta_y)^{(j)} . \end{aligned} \tag{3.87}$$

We thus need to compute $(X_i)_x (A_{ij}(x,y))$ explicitly for each $j = 1,2,3$ from our description of the A_{ij} in result 3.8. Clearly this involves first understanding the derivatives $(X_i)_x (\alpha_{x^{-1}y})$ and $(X_i)_x \left(w^k_{x^{-1}y}\right)$, $k = 1,2,3$. This is accomplished by the following lemma, which mirrors result 2.35 from chapter 2.

Lemma 3.11 *For $i, j \in \{1,2,3\}$, $i \neq j$, we have*

$$\begin{gathered} (X_i)_x \left(w^i_{x^{-1}y}\right) = -w^4_{x^{-1}y} \ (\textit{no sum}),\ (X_i)_x \left(w^j_{x^{-1}y}\right) = \varepsilon_i{}^j{}_k w^k_{x^{-1}y} , \\ \textit{and } (X_i)_x \left(w^4_{x^{-1}y}\right) = w^i_{x^{-1}y} \end{gathered} \tag{3.88}$$

and, as a corollary of the last relation,

$$(X_i)_x (\alpha_{x^{-1}y}) = -\csc \alpha_{x^{-1}y}\, w^i_{x^{-1}y} . \tag{3.89}$$

Proof: This lemma not only mirrors result 2.35, it follows directly from it. For, by formulae 2.34 from chapter 2 and our related observations there regarding the inversion map, we have that the ambient coordinates of $x^{-1}y$ are given by

$$\begin{aligned} w^1_{x^{-1}y} &= w^4_x w^1_y + w^3_x w^2_y - w^2_x w^3_y - w^1_x w^4_y , \\ w^2_{x^{-1}y} &= w^4_x w^2_y - w^3_x w^1_y + w^1_x w^3_y - w^2_x w^4_y , \\ w^3_{x^{-1}y} &= w^4_x w^3_y - w^3_x w^4_y - w^1_x w^2_y + w^2_x w^1_y , \textit{ and} \\ w^4_{x^{-1}y} &= w^1_x w^1_y + w^2_x w^2_y + w^3_x w^3_y + w^4_x w^4_y , \end{aligned} \tag{3.90}$$

and the formulae for $(X_i)_x \left(w^1_{x^{-1}y}\right), \ldots, (X_i)_x \left(w^4_{x^{-1}y}\right)$ in 3.88 then follow easily from this by simple calculations using result 2.35. The corollary in turn follows from the last of these formulae and the fact that $w^4_{x^{-1}y} = \cos \alpha_{x^{-1}y}$.

♣

Applying this lemma in 3.70 we can now at once obtain the formulae for the directional derivatives of the A_{ij} that we need to evaluate the quantities $(X_i)_x (A_{ij}(x,y))$

in 3.87. For example, we get

$$\begin{aligned}(X_1)_x(A_{11}(x,y)) = &\ \{g'(\alpha_{x^{-1}y})\cot\alpha_{x^{-1}y} - g(\alpha_{x^{-1}y})\}\, w^1_{x^{-1}y}\\ &-h'(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y}\left(w^1_{x^{-1}y}\right)^3 - 2h(\alpha_{x^{-1}y})\, w^1_{x^{-1}y}w^4_{x^{-1}y}\ , \textit{and}\end{aligned}$$

$$\begin{aligned}(X_2)_x(A_{21}(x,y)) = &\ g'(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y}w^2_{x^{-1}y}w^3_{x^{-1}y} - g(\alpha_{x^{-1}y})\, w^1_{x^{-1}y}\\ &-h'(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y}\left(w^2_{x^{-1}y}\right)^2 w^1_{x^{-1}y}\\ &+h(\alpha_{x^{-1}y})\left\{-w^4_{x^{-1}y}w^1_{x^{-1}y} - w^2_{x^{-1}y}w^3_{x^{-1}y}\right\},\ \textit{and}\end{aligned}$$

$$\begin{aligned}(X_3)_x(A_{31}(x,y)) = &\ -g'(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y}w^3_{x^{-1}y}w^2_{x^{-1}y} - g(\alpha_{x^{-1}y})\, w^1_{x^{-1}y}\\ &-h'(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y}\left(w^3_{x^{-1}y}\right)^2 w^1_{x^{-1}y}\\ &+h(\alpha_{x^{-1}y})\left\{-w^4_{x^{-1}y}w^1_{x^{-1}y} + w^3_{x^{-1}y}w^2_{x^{-1}y}\right\}\end{aligned}$$

so that, adding and noting $w^4_{x^{-1}y} = \cos\alpha_{x^{-1}y}$ and $\sum_{i=1}^3\left(w^i_{x^{-1}y}\right)^2 = \sin^2\alpha_{x^{-1}y}$, we obtain that

$$(X_i)_x(A_{i1}(x,y)) = \left\{\begin{array}{l} g'(\alpha_{x^{-1}y})\cot\alpha_{x^{-1}y} - 3g(\alpha_{x^{-1}y}) \\ -h'(\alpha_{x^{-1}y})\sin\alpha_{x^{-1}y} - 4h(\alpha_{x^{-1}y})\cos\alpha_{x^{-1}y}\end{array}\right\} w^1_{x^{-1}y}\,. \tag{3.91}$$

Similarly,

$$\begin{aligned}(X_1)_x(A_{12}(x,y)) = &\ -g'(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y}w^1_{x^{-1}y}w^3_{x^{-1}y} - g(\alpha_{x^{-1}y})\, w^2_{x^{-1}y}\\ &-h'(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y}\left(w^1_{x^{-1}y}\right)^2 w^2_{x^{-1}y}\\ &+h(\alpha_{x^{-1}y})\left\{-w^4_{x^{-1}y}w^2_{x^{-1}y} + w^1_{x^{-1}y}w^3_{x^{-1}y}\right\},\ \textit{and}\end{aligned}$$

$$\begin{aligned}(X_2)_x(A_{22}(x,y)) = &\ \{g'(\alpha_{x^{-1}y})\cot\alpha_{x^{-1}y} - g(\alpha_{x^{-1}y})\}\, w^2_{x^{-1}y}\\ &-h'(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y}\left(w^2_{x^{-1}y}\right)^3 - 2h(\alpha_{x^{-1}y})\, w^2_{x^{-1}y}w^4_{x^{-1}y},\ \textit{and}\end{aligned}$$

$$\begin{aligned}(X_3)_x(A_{32}(x,y)) = &\ g'(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y}w^3_{x^{-1}y}w^1_{x^{-1}y} - g(\alpha_{x^{-1}y})\, w^2_{x^{-1}y}\\ &-h'(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y}\left(w^3_{x^{-1}y}\right)^2 w^2_{x^{-1}y}\\ &+h(\alpha_{x^{-1}y})\left\{-w^4_{x^{-1}y}w^2_{x^{-1}y} - w^3_{x^{-1}y}w^1_{x^{-1}y}\right\}\end{aligned}$$

so that

$$(X_i)_x(A_{i2}(x,y)) = \left\{\begin{array}{l} g'(\alpha_{x^{-1}y})\cot\alpha_{x^{-1}y} - 3g(\alpha_{x^{-1}y}) \\ -h'(\alpha_{x^{-1}y})\sin\alpha_{x^{-1}y} - 4h(\alpha_{x^{-1}y})\cos\alpha_{x^{-1}y}\end{array}\right\} w^2_{x^{-1}y} \tag{3.92}$$

while finally, in identical fashion, we find that

$$(X_i)_x\,(A_{i3}(x,y)) = \left\{ \begin{array}{l} g'\left(\alpha_{x^{-1}y}\right)\cot\alpha_{x^{-1}y} - 3g\left(\alpha_{x^{-1}y}\right) \\ -h'\left(\alpha_{x^{-1}y}\right)\sin\alpha_{x^{-1}y} - 4h\left(\alpha_{x^{-1}y}\right)\cos\alpha_{x^{-1}y} \end{array} \right\} w^3_{x^{-1}y}\,. \quad (3.93)$$

Substituting 3.91-3.93 now back in 3.87 we arrive at a complete expression for $L_{0,2}(x,y)$ just in terms of our unknown functions g and h, namely

$$L_{0,2}(x,y) = k\left(\alpha_{x^{-1}y}\right) w^i_{x^{-1}y}(\theta_y \wedge \theta_y)^{(i)} \quad (3.94)$$

where $k \in C^\infty(0,\pi]$ is given by

$$k(\alpha) = g'(\alpha)\cot\alpha - 3g(\alpha) - h'(\alpha)\sin\alpha - 4h(\alpha)\cos\alpha\,. \quad (3.95)$$

To determine $L_{0,2}(x,y)$ exactly, therefore, we need to evaluate k explicitly from our (partial) knowledge of g and h in 3.71- 3.74. This is what we do now, and it represents, at least for the 0,2-piece of L, the fulfilment of the claim we made in Step 6 of subsection 3.2.2, namely that even though we cannot determine g and h exactly from 3.71- 3.74 we *can* solve exactly for the combinations of g, h and their derivatives that arise in the expression for L.

Our approach is part systematic and part tinkering. The systematic underlying idea is that since it arises in a natural geometric quantity, L, which we have reason to believe to be nice (see our earlier discussion in subsection 3.2.2, Step 6), the apparently messy function k may nevertheless satisfy a simple ODE. Thus we simply differentiate k, or rather $K(\alpha) \equiv k(\alpha)\sin\alpha$ (which has no denominator in any terms and so is easier to handle), and try tinkering with what turns up using 3.71 and 3.72. We obtain

$$K'(\alpha) = \left\{ \begin{array}{l} g''(\alpha)\cos\alpha - 4g'(\alpha)\sin\alpha - 3g(\alpha)\cos\alpha \\ -h''(\alpha)\sin^2\alpha - 6h'(\alpha)\sin\alpha\cos\alpha - 4h(\alpha)\left\{1 - 2\sin^2\alpha\right\} \end{array} \right\}$$

which, on using 3.71 and 3.72 to remove the second derivatives on the right hand side, becomes

$$K'(\alpha) = \left\{ \begin{array}{l} \{-2g'(\alpha)\cot\alpha + 3g(\alpha) + 2h(\alpha)\cos\alpha\}\cos\alpha \\ -4g'(\alpha)\sin\alpha - 3g(\alpha)\cos\alpha \\ -\{-2g'(\alpha)\csc\alpha - 6h'(\alpha)\cot\alpha + 6h(\alpha)\}\sin^2\alpha \\ -6h'(\alpha)\sin\alpha\cos\alpha - 4h(\alpha)\left\{1 - 2\sin^2\alpha\right\} \end{array} \right\}$$

$$= -2g'(\alpha)\csc\alpha - 2h(\alpha).$$

That is,

$$K'(\alpha) = -2s(\alpha)\csc^2\alpha \tag{3.96}$$

where

$$s(\alpha) \equiv g'(\alpha)\sin\alpha + h(\alpha)\sin^2\alpha\,. \tag{3.97}$$

To proceed, we could try either differentiating K again or differentiating s. It turns out that a wonderful thing happens if we focus on s; namely, using 3.72 again, we get

$$s'(\alpha) = g''(\alpha)\sin\alpha + g'(\alpha)\cos\alpha + h'(\alpha)\sin^2\alpha + 2h(\alpha)\sin\alpha\cos\alpha$$

$$= \left\{ \begin{array}{l} \{-2g'(\alpha)\cot\alpha + 3g(\alpha) + 2h(\alpha)\cos\alpha\}\sin\alpha \\ +g'(\alpha)\cos\alpha + h'(\alpha)\sin^2\alpha + 2h(\alpha)\sin\alpha\cos\alpha \end{array} \right\}$$

$$= -g'(\alpha)\cos\alpha + 3g(\alpha)\sin\alpha + h'(\alpha)\sin^2\alpha + 4h(\alpha)\sin\alpha\cos\alpha\,,$$

i.e.

$$s'(\alpha) = -K(\alpha). \tag{3.98}$$

This is wonderful because 3.98 and 3.96 now combine to yield a single simple ODE in s which turns out to be easily solvable; the ODE is

$$s''(\alpha) = 2\csc^2\alpha\, s(\alpha) \tag{3.99}$$

and, by inspection, its general solution on $(0,\pi]$ is just

$$s(\alpha) = C_1\cot\alpha + C_2\left(\alpha\cot\alpha - 1\right) \tag{3.100}$$

for some $C_1, C_2 \in \Re$.

To determine C_1 and C_2 we use our boundary conditions on g and h from result 3.8. First, since g and h are smooth at π it is easy to see in 3.97 that $s(\pi) = 0$, which, in 3.100, implies that $C_1 = -C_2\pi$ and hence that, more simply, we just have

$$s(\alpha) = C\{(\pi - \alpha)\cot\alpha + 1\} \tag{3.101}$$

for some $C \in \Re$. Applying now also our asymptotic boundary conditions 3.73 and 3.74 for g and h, it follows from 3.97 that $s(\alpha) \sim -\frac{1}{4\pi\alpha}$ for α near 0, and in 3.101, this then implies that $C = -\frac{1}{4\pi^2}$ and so gives us finally

$$s(\alpha) = -\frac{1}{4\pi^2}\{(\pi - \alpha)\cot\alpha + 1\}. \tag{3.102}$$

In 3.98 this now immediately allows us to solve for k as we need. We obtain

$$K(\alpha) \equiv k(\alpha)\sin\alpha = -\frac{1}{4\pi^2}\left\{(\pi - \alpha)\csc^2\alpha + \cot\alpha\right\} ;$$

$$i.e. \quad k(\alpha) = -\frac{1}{4\pi^2}\left\{(\pi - \alpha)\csc^3\alpha + \csc\alpha\cot\alpha\right\}. \tag{3.103}$$

And, in 3.94, this then gives us the final, completely explicit closed-form expression for $L_{0,2}(x,y)$ that was our goal in this section;

Proposition 3.12 *For $x, y \in S^3$, $x \neq y$, the 0,2-piece of L is given by*

$$L_{0,2}(x,y) = -\frac{1}{4\pi^2}\left\{(\pi - \alpha_{x^{-1}y})\csc^3\alpha_{x^{-1}y} + \csc\alpha_{x^{-1}y}\cot\alpha_{x^{-1}y}\right\} w^i_{x^{-1}y}(\theta_y \wedge \theta_y)^{(i)} . \tag{3.104}$$

We turn now to the 1,1-piece, $L_{1,1}(x,y)$.

3.3.2 Computing $L_{1,1}(x,y)$ on S^3

We proceed in similar fashion to the case of the 0,2-piece just considered. Here contributions to $L_{1,1}$ come only from $G_{2,1}$ in 3.86 and so, using result 3.9, we get

$$\begin{aligned}
L_{1,1}(x,y) &= -\delta_x G_{1,2}(x,y) = -*_x d_x *_x \left\{B_{ij}(x,y)(\theta_x \wedge \theta_x)^{(i)} \wedge \theta^j_y\right\} \\
&= -*_x d_x \left\{B_{ij}(x,y)\theta^i_x \wedge \theta^j_y\right\} \\
&= -*_x \left\{(X_k)_x (B_{ij}(x,y))\,\theta^k_x \wedge \theta^i_x \wedge \theta^j_y - 2B_{ij}(x,y)(\theta_x \wedge \theta_x)^{(i)} \wedge \theta^j_y\right\} \\
&= -\left\{\varepsilon^{ki}{}_m (X_k)_x (B_{ij}(x,y))\,\theta^m_x \wedge \theta^j_y - 2B_{mj}(x,y)\theta^m_x \wedge \theta^j_y\right\} .
\end{aligned}$$

That is, rejigging indices,

$$L_{1,1}(x,y) = C_{ij}(x,y)\theta^i_x \wedge \theta^j_y \tag{3.105}$$

where

$$C_{ij}(x,y) = -\left\{\varepsilon^{km}{}_i \left(X_k\right)_x \left(B_{mj}(x,y)\right) - 2B_{ij}(x,y)\right\} . \tag{3.106}$$

To compute the nine entry functions of the matrix $C_{ij}(x,y)$ now, we proceed term by term, using identical techniques to those of the previous section to evaluate the necessary directional derivatives $\left(X_k\right)_x \left(B_{mj}(x,y)\right)$ from result 3.9 and lemma 3.11.

We start with $C_{11}(x,y)$. By 3.106 we have

$$C_{11}(x,y) = -\left\{\left(X_2\right)_x \left(B_{31}(x,y)\right) - \left(X_3\right)_x \left(B_{21}(x,y)\right) - 2B_{11}(x,y)\right\} \tag{3.107}$$

and, by 3.82 and lemma 3.11,

$$\left(X_2\right)_x \left(B_{31}(x,y)\right) = \left\{\begin{array}{l} g'(\alpha_{x^{-1}y}) \csc\alpha_{x^{-1}y}(w^2_{x^{-1}y})^2 + g(\alpha_{x^{-1}y})w^4_{x^{-1}y} \\ +h'(\alpha_{x^{-1}y}) \csc\alpha_{x^{-1}y} w^1_{x^{-1}y} w^2_{x^{-1}y} w^3_{x^{-1}y} \\ -h(\alpha_{x^{-1}y})\left\{(w^1_{x^{-1}y})^2 - (w^3_{x^{-1}y})^2\right\} \end{array}\right\}, \textit{ and}$$

$$\left(X_3\right)_x \left(B_{21}(x,y)\right) = \left\{\begin{array}{l} -g'(\alpha_{x^{-1}y}) \csc\alpha_{x^{-1}y}(w^3_{x^{-1}y})^2 - g(\alpha_{x^{-1}y})w^4_{x^{-1}y} \\ +h'(\alpha_{x^{-1}y}) \csc\alpha_{x^{-1}y} w^1_{x^{-1}y} w^2_{x^{-1}y} w^3_{x^{-1}y} \\ -h(\alpha_{x^{-1}y})\left\{-(w^1_{x^{-1}y})^2 + (w^2_{x^{-1}y})^2\right\} \end{array}\right\},$$

while

$$B_{11}(x,y) = g(\alpha_{x^{-1}y})\cos\alpha_{x^{-1}y} - h(\alpha_{x^{-1}y})(w^1_{x^{-1}y})^2 .$$

Substituting into 3.107 this gives

$$C_{11}(x,y) = -\left\{g'(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y} + h(\alpha_{x^{-1}y})\right\}\left\{\sin^2\alpha_{x^{-1}y} - (w^1_{x^{-1}y})^2\right\} .$$

But, in light of 3.97, we can in fact evaluate the right hand side of this formula immediately to get $C_{11}(x,y)$ exactly. We obtain that

$$\begin{array}{l} C_{11}(x,y) = -s(\alpha_{x^{-1}y}) + s(\alpha_{x^{-1}y})\csc^2\alpha_{x^{-1}y}(w^1_{x^{-1}y})^2 \\ \textit{where the function } s \textit{ is as defined in } 3.102. \end{array} \tag{3.108}$$

Turning now to $C_{12}(x,y)$, in identical fashion this is given by

$$C_{12}(x,y) = -\left\{(X_2)_x\left(B_{32}(x,y)\right) - (X_3)_x\left(B_{22}(x,y)\right) - 2B_{12}(x,y)\right\}$$

where

$$\begin{aligned}
(X_2)_x\left(B_{32}(x,y)\right) &= \left\{\begin{array}{l} -g'(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y}w^2_{x^{-1}y}w^1_{x^{-1}y} - g(\alpha_{x^{-1}y})w^3_{x^{-1}y} \\ +h'(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y}(w^2_{x^{-1}y})^2 w^3_{x^{-1}y} \\ -h(\alpha_{x^{-1}y})\left\{w^1_{x^{-1}y}w^2_{x^{-1}y} - w^3_{x^{-1}y}w^4_{x^{-1}y}\right\} \end{array}\right\}, \textit{ and} \\
(X_3)_x\left(B_{22}(x,y)\right) &= \left\{\begin{array}{l} -g'(\alpha_{x^{-1}y})\cot\alpha_{x^{-1}y}w^3_{x^{-1}y} + g(\alpha_{x^{-1}y})w^3_{x^{-1}y} \\ +h'(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y}(w^2_{x^{-1}y})^2 w^3_{x^{-1}y} \\ +2h(\alpha_{x^{-1}y})w^1_{x^{-1}y}w^2_{x^{-1}y} \end{array}\right\}, \textit{ and} \\
B_{12}(x,y) &= -g(\alpha_{x^{-1}y})w^3_{x^{-1}y} - h(\alpha_{x^{-1}y})w^1_{x^{-1}y}w^2_{x^{-1}y}\,.
\end{aligned}$$

Thus

$$\begin{aligned}
C_{12}(x,y) = {} & \left\{g'(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y} + h(\alpha_{x^{-1}y})\right\} w^1_{x^{-1}y}w^2_{x^{-1}y} \\
& - \left\{g'(\alpha_{x^{-1}y})\cot\alpha_{x^{-1}y} + h(\alpha_{x^{-1}y})\cos\alpha_{x^{-1}y}\right\} w^3_{x^{-1}y}
\end{aligned}$$

and so 3.97 again gives us $C_{12}(x,y)$ explicitly, as

$$C_{12}(x,y) = \left\{\begin{array}{l} -s(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y}\cot\alpha_{x^{-1}y}w^3_{x^{-1}y} \\ +s(\alpha_{x^{-1}y})\csc^2\alpha_{x^{-1}y}w^1_{x^{-1}y}w^2_{x^{-1}y} \end{array}\right\}. \tag{3.109}$$

It is easy to check that the remaining seven entries can likewise be determined exactly, and a pattern soon becomes clear. The general formula is that, for any $i,j \in \{1,2,3\}$,

$$C_{ij}(x,y) = \left\{\begin{array}{l} -s(\alpha_{x^{-1}y})\delta_{ij} - s(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y}\cot\alpha_{x^{-1}y}\varepsilon_{ijk}w^k_{x^{-1}y} \\ +s(\alpha_{x^{-1}y})\csc^2\alpha_{x^{-1}y}w^i_{x^{-1}y}w^j_{x^{-1}y} \end{array}\right\} \tag{3.110}$$

where s *is as defined in* 3.102.

Back in 3.105 this then gives us the explicit, closed-form of the 1,1-piece of L, which we were seeking in this section. Note, in passing, that in obtaining this we also fulfil the remaining half of our claim in Step 6 of subsection 3.2.2 — that even though g and h remain undetermined, we can still evaluate exactly the pieces of L which depend on them.

Our final result is

Proposition 3.13 *The 1,1-piece of the Green's form L is given at arbitrary $x,\ y \in S^3$, $x \neq y$ by $L_{1,1}(x,y) = C_{ij}(x,y)\theta_x^i \wedge \theta_y^j$, where $C_{ij}(x,y)$ is given in standard matrix form by*

$$-s(\alpha_{x^{-1}y})\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - s(\alpha_{x^{-1}y})\csc\alpha_{x^{-1}y}\cot\alpha_{x^{-1}y}\begin{pmatrix} 0 & w^3_{x^{-1}y} & -w^2_{x^{-1}y} \\ -w^3_{x^{-1}y} & 0 & w^1_{x^{-1}y} \\ w^2_{x^{-1}y} & -w^1_{x^{-1}y} & 0 \end{pmatrix}$$

$$+ s(\alpha_{x^{-1}y})\csc^2\alpha_{x^{-1}y}\begin{pmatrix} (w^1_{x^{-1}y})^2 & w^1_{x^{-1}y}w^2_{x^{-1}y} & w^1_{x^{-1}y}w^3_{x^{-1}y} \\ w^2_{x^{-1}y}w^1_{x^{-1}y} & (w^2_{x^{-1}y})^2 & w^2_{x^{-1}y}w^3_{x^{-1}y} \\ w^3_{x^{-1}y}w^1_{x^{-1}y} & w^3_{x^{-1}y}w^2_{x^{-1}y} & (w^3_{x^{-1}y})^2 \end{pmatrix} \qquad (3.111)$$

and $s \in C^\infty(0,\pi]$ is the function $s(\alpha) = -\frac{1}{4\pi^2}\{(\pi-\alpha)\cot\alpha + 1\}$.

Remark: Note that $C_{ij}(x,y)$ has the same basic structure as $A_{ij}(x,y)$ and $B_{ij}(x,y)$, reflecting the fact that L must satisfy the same rotation invariance constraints as we applied to G in Step 2 of subsection 3.2.2.

We now turn to the final, 2,0-piece of L, $L_{2,0}(x,y)$.

3.3.3 Computing $L_{2,0}(x,y)$ on S^3

Again we proceed in the same way as for the other two pieces of L, but things are actually much simpler here because $G_{3,0}$, which provides the only contribution to $L_{2,0}$ in 3.86, is already given *exactly* in result 3.10, i.e. it has no dependence on the unknown functions g and h. Thus, from 3.83, we have

$$\begin{aligned} L_{2,0}(x,y) &= -\delta_x G_{3,0}(x,y) = *_x d_x *_x \left\{-\tfrac{1}{4\pi^2}\left[(\pi-\alpha_{x^{-1}y})\cot(\alpha_{x^{-1}y}) + 1\right] vol_{S^3_x}\right\} \\ &= -\tfrac{1}{4\pi^2}(X_i)_x\{(\pi-\alpha_{x^{-1}y})\cot(\alpha_{x^{-1}y}) + 1\}(\theta_x \wedge \theta_x)^{(i)}, \end{aligned}$$

which yields an expression in very close analogy with result 3.12 as our final formula for $L_{2,0}(x,y)$;

Proposition 3.14 *For $x,\, y \in S^3$, $x \neq y$, the 2,0-piece of L is given by*

$$L_{2,0}(x,y) = -\frac{1}{4\pi^2}\left\{(\pi - \alpha_{x^{-1}y})\csc^3\alpha_{x^{-1}y} + \csc\alpha_{x^{-1}y}\cot\alpha_{x^{-1}y}\right\} w^i_{x^{-1}y}(\theta_x \wedge \theta_x)^{(i)}. \tag{3.112}$$

3.3.4 Summary

In results 3.12 - 3.14 we have now obtained an exact description of all three pieces of the Green's form L on S^3. Various further tests can be performed, moreover, to verify the validity of these formulae; we can check that L satisfies the four defining properties (PL0)-(PL3) used to characterise it in [AS1], we can test whether L satisfies the reflection-invariance property it should (cf. the analogous discussion for G in subsection 3.2.2), and we can verify that the different pieces of L are related as they should be under Hodge star (cf. the relations between $G_{1,2}$ and $G_{2,1}$ on the one hand, and $G_{0,3}$ and $G_{3,0}$ on the other, discussed in section 3.2.3).

These checks are good not only in confirming the correctness of our calculations to this point, but also in providing greater geometrical understanding of L; for example, the Hodge star test just referred to gives us a geometrical explanation of the equality of the coefficient functions of $L_{0,2}$ and $L_{2,0}$.

Nevertheless we choose to omit the details here, in part because this chapter is already lengthy, but also so as to avoid delaying our derivation of L on the lens spaces $L[p]$. Suffice it to say that all our tests confirm the correctness of our formulae in results 3.12 - 3.14.

We conclude this section instead, then, by simply drawing these results together into a single place, giving us the full Green's form, L, of d explicitly on S^3. In doing so, however, there is one trivial notational innovation that we make, namely breaking the different pieces of L explicitly into their coefficient functions and their form-pieces, which will prove useful when we come to describing the corresponding Green's form on the lens spaces $L[p]$ in the next section.

Proposition 3.15 *The Green's form, L, of d on S^3 is given explicitly at arbitrary $x, y \in S^3$, $x \neq y$, by*

$$L(x,y) = L_{0,2}(x,y) + L_{1,1}(x,y) + L_{2,0}(x,y) \tag{3.113}$$

where, if we break the pieces of L into coefficient functions and form-pieces by writing

$$\begin{aligned} L_{0,2}(x,y) &= (\tilde{L}_{0,2})_i(x,y)(\theta_y \wedge \theta_y)^{(i)}, \text{ and} \\ L_{1,1}(x,y) &= (\tilde{L}_{1,1})_{ij}(x,y)\theta_x^i \wedge \theta_y^j, \text{ and} \\ L_{2,0}(x,y) &= (\tilde{L}_{2,0})_i(x,y)(\theta_x \wedge \theta_x)^{(i)}, \end{aligned} \tag{3.114}$$

then we have

$$\begin{aligned}
(\tilde{L}_{0,2})_i(x,y) &= t(\alpha_{x^{-1}y}) w^i_{x^{-1}y}, \text{ and} \\
(\tilde{L}_{1,1})_{ij}(x,y) &= -s(\alpha_{x^{-1}y}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&\quad - s(\alpha_{x^{-1}y}) \csc\alpha_{x^{-1}y} \cot\alpha_{x^{-1}y} \begin{pmatrix} 0 & w^3_{x^{-1}y} & -w^2_{x^{-1}y} \\ -w^3_{x^{-1}y} & 0 & w^1_{x^{-1}y} \\ w^2_{x^{-1}y} & -w^1_{x^{-1}y} & 0 \end{pmatrix} \\
&\quad + s(\alpha_{x^{-1}y}) \csc^2\alpha_{x^{-1}y} \begin{pmatrix} (w^1_{x^{-1}y})^2 & w^1_{x^{-1}y} w^2_{x^{-1}y} & w^1_{x^{-1}y} w^3_{x^{-1}y} \\ w^2_{x^{-1}y} w^1_{x^{-1}y} & (w^2_{x^{-1}y})^2 & w^2_{x^{-1}y} w^3_{x^{-1}y} \\ w^3_{x^{-1}y} w^1_{x^{-1}y} & w^3_{x^{-1}y} w^2_{x^{-1}y} & (w^3_{x^{-1}y})^2 \end{pmatrix}, \text{ and} \\
(\tilde{L}_{2,0})_i(x,y) &= t(\alpha_{x^{-1}y}) w^i_{x^{-1}y}.
\end{aligned} \tag{3.115}$$

Here s and t in $C^\infty(0,\pi]$ are the functions

$$s(\alpha) = -\frac{1}{4\pi^2}\{(\pi - \alpha)\cot\alpha + 1\} \tag{3.116}$$

and

$$t(\alpha) = -\frac{1}{4\pi^2}\left\{(\pi - \alpha)\csc^3\alpha + \csc\alpha\cot\alpha\right\}. \tag{3.117}$$

We now turn to generalising this result to the lens spaces $L[p]$.

3.4 Computing the Green's Form of d on the Lens Spaces $L[p]$

Denote the Green's form of d on $L[p]$ by L^p. Then, as promised, obtaining L^p from our just-completed computation of L on S^3 turns out to be very easy.

We start by giving a somewhat abstract result describing the relationship between these quantities. In fact this indirect description is easily seen to characterise L^p uniquely, and is all that we need when we turn to computation of the graphical pieces of the 2-loop invariants $\tilde{I}_2^{conn}(L[p], A_{triv}, \sigma)$ in the next chapter. But for completeness we will also follow it with a result which translates this characterisation into a more concrete, explicit description of L^p.

Our abstract characterisation is as follows;

Proposition 3.16 *The pull-back of L^p to $\Omega^2(S^3 \times S^3)$ under $\pi_p \times \pi_p$ is related to the Green's form L already lying in $\Omega^2(S^3 \times S^3)$ by*

$$(\pi_p \times \pi_p)^* L^p = \sum_{k=0}^{p-1} (id_{S^3} \times \mathcal{L}_{z_p^k})^* L \tag{3.118}$$

where z_p is the generator of the finite group $\mathbf{Z}_p$, as introduced in chapter 2 in defining $L[p]$.

Proof: Recall the convention from subsection 2.2.3 of denoting points on lens spaces with a "bar" over them to distinguish them from their "unbarred" preimages in S^3 under π_p (i.e. for any $x \in S^3$ we denote $\pi_p(x) \in L[p]$ by $\bar{x}$). Then, in analogy with equation 3.2 the defining equation characterising L^p is that

$$\int_{L[p]_{\bar{y}}} L^p(\bar{x}, \bar{y}) \wedge d\nu(\bar{y}) = \nu(\bar{x}) \quad for\ all\ \bar{x} \in L[p]\ and\ for\ all\ \nu \in Im\delta \subset \Omega^*(L[p]). \tag{3.119}$$

Now recall from chapter 2 (subsection 2.2.3) our definition of the maps ρ and π_p^* and the isomorphism they provide between $\Omega^*(L[p])$ and $\Omega^*_{\mathbf{Z}_p}(S^3)$ (the $\mathbf{Z}_p$-invariant forms on S^3). The discussion there clearly generalises trivially to yield a similar isomorphism between $\Omega^*(L[p] \times L[p])$ and $\Omega^*_{\mathbf{Z}_p \times \mathbf{Z}_p}(S^3 \times S^3)$ (the $\mathbf{Z}_p \times \mathbf{Z}_p$-invariant forms on $S^3 \times S^3$), via the map $(\pi_p \times \pi_p)^*$ and its inverse, which we shall call τ. But, since L is left-invariant (i.e. $(\mathcal{L}_g \times \mathcal{L}_g)^* L = L$ for all $g \in S^3$), it is easy to see that $(\pi_p \times \pi_p)^* L^p$ and $\sum_{k=0}^{p-1}(id_{S^3} \times \mathcal{L}_{z_p^k})^* L$ are both elements of $\Omega^2_{\mathbf{Z}_p \times \mathbf{Z}_p}(S^3 \times S^3)$. Thus equation 3.118 holds if and only if $\sum_{k=0}^{p-1} \tau((id_{S^3} \times \mathcal{L}_{z_p^k})^* L)$ satisfies the defining equation 3.119.

But this is easy to verify. For, given any $\nu \in Im\delta \subset \Omega^*(L[p])$, let $\tilde{\nu} \in \Omega^*(L[p])$ be defined by

$$\tilde{\nu}(\bar{x}) = \sum_{k=0}^{p-1} \int_{L[p]_{\bar{y}}} \left[\tau\left((id_{S^3} \times \mathcal{L}_{z_p^k})^* L\right)\right](\bar{x}, \bar{y}) \wedge d\nu(\bar{y})$$

for all $\bar{x} \in L[p]$, and apply π_p^* to it. Since $\pi_p^* vol_{L[p]} = vol_{S^3}$, in taking π_p^* inside the integral defining $\tilde{\nu}$ we can change it to $(\pi_p \times \pi_p)^*$ and convert the integral into an integral over S^3 with a compensating overall factor of $1/p$; we obtain that, for all $x \in S^3$,

$$\begin{aligned}
(\pi_p^* \tilde{\nu})(x) &= \tfrac{1}{p} \textstyle\sum_{k=0}^{p-1} \int_{S^3_y} \left[(\pi_p \times \pi_p)^* \left(\tau\left((id_{S^3} \times \mathcal{L}_{z_p^k})^* L\right)\right)\right](x, y) \wedge \left(d(\pi_p^* \nu)\right)(y) \\
&= \tfrac{1}{p} \textstyle\sum_{k=0}^{p-1} \int_{S^3_y} \left[(id_{S^3} \times \mathcal{L}_{z_p^k})^* L\right](x, y) \wedge \left(d(\pi_p^* \nu)\right)(y) .
\end{aligned}$$

But $d\pi_p^* \nu$ is in $\Omega^*_{\mathbf{Z}_p}(S^3)$. Thus for each $k \in \{0, 1, \ldots, p-1\}$, we can write $d\pi_p^* \nu = \mathcal{L}_{z_p^k}{}^* d\pi_p^* \nu$ and, using the left-invariance of vol_{S^3} and the defining property 3.2 of L, it follows that, for all $x \in S^3$,

$$\begin{aligned}
(\pi_p^* \tilde{\nu})(x) &= \tfrac{1}{p} \textstyle\sum_{k=0}^{p-1} \int_{S^3_y} \left(id_{S^3} \times \mathcal{L}_{z_p^k}\right)^* \left[L(x, y) \wedge \left(d(\pi_p^* \nu)\right)(y)\right] \\
&= \tfrac{1}{p} \textstyle\sum_{k=0}^{p-1} \int_{S^3_y} L(x, y) \wedge \left(d(\pi_p^* \nu)\right)(y) \\
&= (\pi_p^* \nu)(x) .
\end{aligned}$$

The injectivity of π_p^* then implies at once that $\tilde{\nu} = \nu$, and this, in view of the definition of $\tilde{\nu}$, confirms that $\sum_{k=0}^{p-1} \tau((id_{S^3} \times \mathcal{L}_{z_p^k})^* L)$ does satisfy the defining equation 3.119.

♣

Remark: We have already observed that, even though for our species of lens spaces $L[p]$ we can (and shortly will) translate 3.118 into a much more concrete description of L^p, the somewhat indirect characterisation is actually sufficient for all our future needs in computing the graphical pieces of 2-loop invariants.

It is worth pointing out that this is significant as regards the potential for generalising our computations to arbitrary lens spaces. This is because our capacity to generate a more explicit description of L^p from 3.118 will rely heavily on a feature peculiar to the lens spaces $L[p]$, which their more general counterparts $L(p,q)$ do not possess; namely the existence of a left-invariant framing, $\{\theta^i\}$, of T^*S^3 which descends to $L[p]$. For arbitrary p,q there is no comparable framing compatible with the projection map from S^3 onto $L(p,q)$, and so the best we can hope for is a result along the lines of 3.118 regarding the pull-back of the Green's form on $L(p,q)$ to S^3.

For clarity, therefore, we emphasise the sufficiency of 3.118, and that our reason for restricting to lens spaces of the type $L[p]$ in this thesis is not because we can't obtain the necessary Green's form explicitly enough on the more general lens spaces, but for different reasons that we will discuss in the next chapter.

To conclude this chapter now, however, let us, for the sake of completeness, provide the promised translation of 3.118 into a result giving L^p very explicitly. This translation is easy, relying on the natural descent of the θ^i to $L[p]$ just mentioned, and our separation of L on S^3 into coefficient functions and form-pieces in summary result 3.15. We leave the details to the reader.

Proposition 3.17 *The Green's form, L^p, of d on $L[p]$ is given at arbitrary $\bar{x}$, $\bar{y} \in L[p]$, $\bar{x} \neq \bar{y}$, by*

$$L^p(\bar{x},\bar{y}) = L^p_{0,2}(\bar{x},\bar{y}) + L^p_{1,1}(\bar{x},\bar{y}) + L^p_{2,0}(\bar{x},\bar{y}) \tag{3.120}$$

where, if we break the pieces of L^p into coefficient functions and form-pieces by writing

$$\begin{aligned} L^p_{0,2}(\bar{x},\bar{y}) &= (\tilde{L}^p_{0,2})_i(\bar{x},\bar{y})(\theta_{\bar{y}} \wedge \theta_{\bar{y}})^{(i)}, \textit{ and} \\ L^p_{1,1}(\bar{x},\bar{y}) &= (\tilde{L}^p_{1,1})_{ij}(\bar{x},\bar{y})\theta^i_{\bar{x}} \wedge \theta^j_{\bar{y}}, \textit{ and} \\ L^p_{2,0}(\bar{x},\bar{y}) &= (\tilde{L}^p_{2,0})_i(\bar{x},\bar{y})(\theta_{\bar{x}} \wedge \theta_{\bar{x}})^{(i)}, \end{aligned} \tag{3.121}$$

then the coefficient functions of the pieces are related (in the same way for each) to those of the corresponding pieces of the Green's form, L *, on* S^3 *, by*

$$(\tilde{L}^p_{0,2})_i(\bar{x},\bar{y}) = \sum_{k=0}^{p-1}(\tilde{L}_{0,2})_i(x, z_p^k y), \text{ and}$$
$$(\tilde{L}^p_{1,1})_{ij}(\bar{x},\bar{y}) = \sum_{k=0}^{p-1}(\tilde{L}_{1,1})_{ij}(x, z_p^k y), \text{ and} \tag{3.122}$$
$$(\tilde{L}^p_{2,0})_i(\bar{x},\bar{y}) = \sum_{k=0}^{p-1}(\tilde{L}_{2,0})_i(x, z_p^k y).$$

Here $(x,y) \in S^3 \times S^3$ *is any one of the* p^2 *pre-images of* $(\bar{x},\bar{y})$ *under* $\pi_p \times \pi_p$ *, it being easy to see from the form of* $\tilde{L}_{2,0}, \tilde{L}_{1,1}$ *and* $\tilde{L}_{0,2}$ *in 3.115 - 3.117 that the choice of preimage is irrelevant, as it should be.*

Using these explicit formulae (3.115 - 3.117) for $\tilde{L}_{2,0}, \tilde{L}_{1,1}$ *and* $\tilde{L}_{0,2}$ *this then gives us a fully explicit, closed form expression for the Green's form* L^p *on the lens space* $L[p]$ *.*

Chapter 4

The Graphical Term

Now that we have at our disposal results 3.15 and 3.16 (or 3.17) regarding the Green's form, L^p, of d on the lens spaces $L[p]$, we turn in this chapter to calculating from them the "graphical integral" contributions in expression 2.10 for the 2-loop perturbative invariants $\tilde{I}_2^{conn}(L[p], A_{triv}, \sigma)$.

4.1 Initial Simplifications

Recall from chapter 2, section 2.1.1, that by the "graphical" contribution to formula 2.10 for the 2-loop invariant, $\tilde{I}_2^{conn}(L[p], A_{triv}, \sigma)$, of the lens space $L[p]$ we mean the first term in expression 2.10, namely

$$I_2^{conn}(L[p], A_{triv}, g) = \int_{L[p]_{\bar{x}} \times L[p]_{\bar{y}}} [L^p \wedge L^p \wedge L^p](\bar{x}, \bar{y}) . \qquad (4.1)$$

We want to use our explicit knowledge of L^p from chapter 3 to compute this integral exactly. Before rushing in, however, there are a number of observations that we can make which will greatly simplify the task.

The first is that the only terms in the integrand which contribute to the overall integral are, of course, simply those in the form of some function times $vol_{L[p]_{\bar{x}}} \wedge vol_{L[p]_{\bar{y}}}$; since $(\pi_p \times \pi_p)^*(vol_{L[p]_{\bar{x}}} \wedge vol_{L[p]_{\bar{y}}}) = vol_{S^3_x} \wedge vol_{S^3_y}$, it is therefore easy to see (cf. the earlier proof of result 3.16) that we can pull the integral back up to $S^3 \times S^3$, where it will be much easier to analyse, at the expense of an overall factor of $1/p^2$; i.e.

$$I_2^{conn}(L[p], A_{triv}, g) = \frac{1}{p^2}\int_{S^3_x \times S^3_y} [(\pi_p \times \pi_p)^* L^p \wedge (\pi_p \times \pi_p)^* L^p \wedge (\pi_p \times \pi_p)^* L^p](x, y) \ .$$

But now, by result 3.16, we know that $(\pi_p \times \pi_p)^* L^p = \sum_{k=0}^{p-1}(id_{S^3} \times \mathcal{L}_{z_p^k})^* L$. Thus we can in fact express $I_2^{conn}(L[p], A_{triv}, g)$ just in terms of the Green's form, L, on S^3, namely as

$$\frac{1}{p^2}\int_{S^3_x \times S^3_y} \sum_{k,m,n=0}^{p-1} \left[(id_{S^3} \times \mathcal{L}_{z_p^k})^* L \wedge (id_{S^3} \times \mathcal{L}_{z_p^m})^* L \wedge (id_{S^3} \times \mathcal{L}_{z_p^n})^* L\right](x, y) \ . \quad (4.2)$$

The following result then allows us to simplify this expression even further.

Result 4.1 *For any $k \in \{0, 1, \ldots, p-1\}$, the 2-form $(id_{S^3} \times \mathcal{L}_{z_p^k})^* L$ on $S^3 \times S^3$ is right-invariant; i.e.*

$$(\mathcal{R}_g \times \mathcal{R}_g)^*(id_{S^3} \times \mathcal{L}_{z_p^k})^* L = (id_{S^3} \times \mathcal{L}_{z_p^k})^* L \quad \textit{for all } g \in S^3 \ . \quad (4.3)$$

Proof: This is a trivial consequence of the fact that $SU(2)_R$ commutes with $\mathbf{Z}_p \subset SU(2)_L$ inside SO(4), so that $(\mathcal{R}_g \times \mathcal{R}_g)^*(id_{S^3} \times \mathcal{L}_{z_p^k})^* = (id_{S^3} \times \mathcal{L}_{z_p^k})^*(\mathcal{R}_g \times \mathcal{R}_g)^*$, together with the fact that L is *apriori* invariant under all of SO(4) (cf. the remark after result 3.13, chapter 3) and hence under $SU(2)_R$ in particular (it is only a matter of convention that in 3.15 we have chosen to represent it in manifestly left-invariant form using the θ^i-basis).

♣

Remark: Note, in passing, that $(id_{S^3} \times \mathcal{L}_{z_p^k})^* L$ is *not* left-invariant. The key to right-invariance was that $SU(2)_R$ lies in the centraliser of $\mathbf{Z}_p$ in SO(4). We shall return to this observation in a little while when we discuss the possibility of generalising our computations to arbitrary lens spaces $L(p, q)$.

Applying result 4.1 in 4.2 now and noting the right-invariance of vol_{S^3} (again because right-translation is an isometry), it is then easy to see that for any fixed x we get the *same* multiple of the volume form at x if we integrate out the y-variable in 4.2. It follows that we can greatly simplify affairs by reducing 4.2 to just an integral over y, introducing a compensating factor of $2\pi^2$ (= volume of S^3) and leaving x fixed at N in the remaining integrand; i.e. $I_2^{conn}(L[p], A_{triv}, g)$ is given by

$$\frac{2\pi^2}{p^2} *_N \int_{S^3_y} \sum_{k,m,n=0}^{p-1} \left[(id_{S^3} \times \mathcal{L}_{z_p^k})^* L \wedge (id_{S^3} \times \mathcal{L}_{z_p^m})^* L \wedge (id_{S^3} \times \mathcal{L}_{z_p^n})^* L\right] (N, y) \,. \quad (4.4)$$

(Note that the use of the Hodge star operator at N here simply reflects the notational difficulty of expressing the fact that the x-form-pieces should no longer appear in the integrand; instead, we have therefore chosen to leave vol_N in the integrand and remove it *after* integrating, using the Hodge star.)

To reach the final form of our expression for $I_2^{conn}(L[p], A_{triv}, g)$, from which we shall then attempt to compute $I_2^{conn}(L[p], A_{triv}, g)$ directly using the explicit formula for L in result 3.15, it now only remains to write the integrand in 4.4 explicitly in the form of a function times $vol_{S^3_y}$, dropping the terms which do not contribute to the overall integral.

In doing this the first thing to note is that, since the 1-forms θ^i are left-invariant, our expression for L in result 3.15 allows us to understand the terms of the form $(id_{S^3} \times \mathcal{L}_{z_p^k})^* L$ in 4.4 very concretely; the pull-back affects only the coefficient functions of L and we get

$$\begin{aligned}\left[(id_{S^3} \times \mathcal{L}_{z_p^k})^* L\right] (N, y) = \; & (\tilde{L}_{2,0})_i(N, z_p^k y)(\theta_N \wedge \theta_N)^{(i)} + (\tilde{L}_{1,1})_{ij}(N, z_p^k y)\theta_N^i \wedge \theta_y^j \\ & + (\tilde{L}_{0,2})_i(N, z_p^k y)(\theta_y \wedge \theta_y)^{(i)} \,.\end{aligned} \quad (4.5)$$

In 4.4 therefore, the three terms of the type analysed in 4.5 combine to generate a total of twenty-seven expressions, arising from combinations of $\tilde{L}_{2,0}, \tilde{L}_{1,1}$ and $\tilde{L}_{0,2}$ at the different points $(N, z_p^k y), (N, z_p^m y)$ and $(N, z_p^n y)$. Of these, however, only seven arise from terms which yield the necessary overall $vol_N \wedge vol_{S^3_y}$ form-piece, with six coming from combinations in which one term contributes a 2,0-coefficient, one a 1,1-coefficient, and the remaining term a 0,2-coefficient, while the seventh represents the case in which all three terms contribute a 1,1-coefficient.

Among the six similar expressions, moreover, it is clear that even though for any given values of k, m and n the resulting integrals are not all equal (varying according to how the 2,0-, 1,1-, and 0,2-coefficients are paired with the points $(N, z_p^k y), (N, z_p^m y)$ and $(N, z_p^n y)$), nonetheless this inequality disappears upon summing over all possible values of k, m and n. Hence, without loss of generality, we may take out a factor of six and decree that the 2,0-coefficient arises with argument $(N, z_p^k y)$, the 1,1-coefficient with argument $(N, z_p^m y)$, and the 0,2-coefficient with argument $(N, z_p^n y)$.

Overall, therefore, after spending a few moments to check how the form-indices on the various coefficient functions must fit together in the expressions we have just outlined in order to yield the claimed $vol_N \wedge vol_{S_y^3}$ combination, we arrive at our final result expressing the graphical term $I_2^{conn}(L[p], A_{triv}, g)$ in fully simplified form;

Proposition 4.2 *The graphical contribution,* $I_2^{conn}(L[p], A_{triv}, g)$, *to the 2-loop invariant of the lens space* $L[p]$ *is given by*

$$I_2^{conn}(L[p], A_{triv}, g) = \frac{2\pi^2}{p^2} \int_{S_y^3} \sum_{k,m,n=0}^{p-1} \{J_1(k,m,n) + J_2(k,m,n)\}\, vol_{S_y^3} \tag{4.6}$$

where

$$J_1(k,m,n) = 6\left[(\tilde{L}_{2,0})_i(N, z_p^k y)\right]\left[(\tilde{L}_{1,1})_{ij}(N, z_p^m y)\right]\left[(\tilde{L}_{0,2})_j(N, z_p^n y)\right] \tag{4.7}$$

and

$$J_2(k,m,n) = -\varepsilon^{iab}\varepsilon^{jcd}\left[(\tilde{L}_{1,1})_{ij}(N, z_p^k y)\right]\left[(\tilde{L}_{1,1})_{ac}(N, z_p^m y)\right]\left[(\tilde{L}_{1,1})_{bd}(N, z_p^n y)\right]. \tag{4.8}$$

Remarks: No further general reductions are possible in this expression for $I_2^{conn}(L[p], A_{triv}, g)$. We have to turn instead to analysing the two integral quantities $J_1(k,m,n)$ and $J_2(k,m,n)$ using our exact formulae for the coefficient functions $\tilde{L}_{2,0}, \tilde{L}_{1,1}$ and $\tilde{L}_{0,2}$ in 3.115-3.117 and our knowledge from chapter 2 of the explicit form of z_p and of the group structure and geometry of S^3. We will undertake this in the next two sections.

But before turning to this, we conclude this section on initial simplifications with a few remarks, promised at the end of chapter 3, on the feasibility of generalising our calculations in this thesis from just the $L[p]$ species of lens spaces to arbitrary $L(p,q)$.

We observed, in discussing result 3.16, that any difficulties in generalising to arbitrary $L(p,q)$ in our calculations are not due to problems in describing the Green's form of d on $L(p,q)$ explicitly. Indeed, already in expression 4.2 we have made good on our claim there that the indirect description in result 3.16 suffices for our calculations, and it is easy to see that the same reduction of I_2^{conn} to an expression just involving the S^3 Green's form can be carried through on any $L(p,q)$.

The greater difficulties in treating arbitrary $L(p,q)$ are twofold and arise after reaching 4.2 in our simplifications.

The first turns up, in fact, in the immediately ensuing result 4.1. This result was crucial in allowing us, in 4.4, to reduce our expression for I_2^{conn} from an integral over the product space $S_x^3 \times S_y^3$, to just an integral over S_y^3. The key point in the reduction was that, for $L[p]$, the centraliser of $\mathbf{Z}_p$ in SO(4), which we noted there is the subgroup for which the invariance condition 4.3 can be obtained, is precisely large enough to act freely transitively on S^3. Unfortunately, the same is not true for arbitrary $L(p,q)$, where the centraliser of the relevant $\mathbf{Z}_p$ in SO(4) turns out to be just a 2-torus inside SO(4) (corresponding to arbitrary left and right multiplication by diagonal elements in SU(2)) and so cannot act transitively on S^3. Thus no correspondingly simple reduction of our product domain of integration can be achieved.

The second difficulty arises at the stage of equation 4.5. This allowed us to greatly simplify the integrand in 4.4 and to write it with the $vol_{S_y^3}$ explicitly separated from the different combinations of coefficient functions which turn up. The essential feature which facilitated this easy simplification and splitting was the invariance of the basis 1-forms θ^i under the elements $\mathcal{L}_{z_p^k}$ of $\mathbf{Z}_p$ appearing in 4.4. When we go to arbitrary $L(p,q)$, however, 4.5 must be adapted not just to replace z_p by the generator of the relevant new $\mathbf{Z}_p$, but also to include extra matrix factors which arise from pulling back the θ_y^i form-pieces by elements of the new $\mathbf{Z}_p$. These matrix factors actually turn out to have a very simple form (depending only on p, q and the power of the generator involved in the pulling back, and independent of y!), but they still make the simplification process enormously much more complicated, and lead to integrands which are considerably harder to handle even than those in $J_1(k,m,n)$ and $J_2(k,m,n)$.

This explains, then, our reasons for restricting our attention to the lens spaces $L[p]$ in this thesis. We hope that the obstacles to generalising to arbitrary $L(p,q)$ just described will not ultimately prove insurmountable (and there does seem some reason to hope that they simply represent an increase in the labour required to compute I_2^{conn} rather than a genuine theoretical barrier to such computations), but we do not pursue these speculations further here. The work involved in computing I_2^{conn} just for the $L[p]$ spaces from result 4.2 will already prove quite substantial, and so we proceed with this without further ado. We start with $J_1(k,m,n)$.

4.2 Evaluation of $J_1(k,m,n)$

Suppose at first that $m=0$. Then, substituting our formulae for $\tilde{L}_{2,0}, \tilde{L}_{1,1}$ and $\tilde{L}_{0,2}$ in 3.115-3.117 into the definition in 4.7, we have that $J_1(k,0,n)$ is given explicitly by

$$J_1(k,0,n) = 6t(\alpha_{z_p^k y})t(\alpha_{z_p^n y})s(\alpha_y)\csc^2\alpha_y\left\{-\sin^2\alpha_y\delta_{ij} - \varepsilon_{ijl}w_y^l w_y^4 + w_y^i w_y^j\right\} w_{z_p^k y}^i w_{z_p^n y}^j \,. \tag{4.9}$$

Here the ambient coordinates of $z_p^k y$ and $z_p^n y$ are related to those of y by the following basic result, which follows immediately from equation 2.38 in chapter 2;

Result 4.3 *For any $q \in \{0,1,\ldots,p-1\}$ we have*

$$\begin{aligned} w_{z_p^q y}^1 &= w_y^1 c_{q,p} - w_y^2 s_{q,p}\,, \\ w_{z_p^q y}^2 &= w_y^1 s_{q,p} + w_y^2 c_{q,p}\,, \\ w_{z_p^q y}^3 &= w_y^3 c_{q,p} + w_y^4 s_{q,p}\,, \text{and} \\ w_{z_p^q y}^4 &= -w_y^3 s_{q,p} + w_y^4 c_{q,p}\,, \end{aligned} \tag{4.10}$$

where $c_{q,p} = \cos(\frac{2\pi q}{p})$ *and* $s_{q,p} = \sin(\frac{2\pi q}{p})$.

To simplify 4.9 now we see that we need to understand quantities of the form $w^i_{z^q_p y} w^i_{z^r_p y}$ and $\varepsilon_{ijl} w^l_y w^i_{z^q_p y} w^j_{z^r_p y}$ using this result. We do this with the following lemma;

Lemma 4.4 *For any* $q, r \in \{0, 1, \ldots, p-1\}$ *we have that, first,*

$$w^i_{z^q_p y} w^i_{z^r_p y} = \sin^2\alpha_y c_{q-r,p} + \left\{(w^4_y)^2 - (w^3_y)^2\right\} s_{q,p} s_{r,p} + w^3_y w^4_y s_{q+r,p} \,, \tag{4.11}$$

and secondly,

$$\varepsilon_{ijl} w^l_y w^i_{z^q_p y} w^j_{z^r_p y} = 0 \,. \tag{4.12}$$

Proof: By 4.10, we have that

$$\begin{aligned} w^i_{z^q_p y} w^i_{z^r_p y} &= \left[w^1_y c_{q,p} - w^2_y s_{q,p}\right]\left[w^1_y c_{r,p} - w^2_y s_{r,p}\right] + \left[w^1_y s_{q,p} + w^2_y c_{q,p}\right]\left[w^1_y s_{r,p} + w^2_y c_{r,p}\right] \\ &\quad + \left[w^3_y c_{q,p} + w^4_y s_{q,p}\right]\left[w^3_y c_{r,p} + w^4_y s_{r,p}\right] \\ &= \sin^2\alpha_y c_{q,p} c_{r,p} + \left\{\sin^2\alpha_y - (w^3_y)^2 + (w^4_y)^2\right\} s_{q,p} s_{r,p} + w^3_y w^4_y \begin{pmatrix} s_{q,p} c_{r,p} \\ + c_{q,p} s_{r,p} \end{pmatrix} \\ &= \sin^2\alpha_y c_{q-r,p} + \left\{(w^4_y)^2 - (w^3_y)^2\right\} s_{q,p} s_{r,p} + w^3_y w^4_y s_{q+r,p} \,, \end{aligned}$$

which proves the first formula, while also

$$\begin{aligned} \varepsilon_{ijl} w^l_y w^i_{z^q_p y} w^j_{z^r_p y} &= w^1_y \left\{ \begin{array}{l} \left[w^1_y s_{q,p} + w^2_y c_{q,p}\right]\left[w^3_y c_{r,p} + w^4_y s_{r,p}\right] \\ -\left[w^3_y c_{q,p} + w^4_y s_{q,p}\right]\left[w^1_y s_{r,p} + w^2_y c_{r,p}\right] \end{array} \right\} \\ &\quad + w^2_y \left\{ \begin{array}{l} \left[w^3_y c_{q,p} + w^4_y s_{q,p}\right]\left[w^1_y c_{r,p} - w^2_y s_{r,p}\right] \\ -\left[w^1_y c_{q,p} - w^2_y s_{q,p}\right]\left[w^3_y c_{r,p} + w^4_y s_{r,p}\right] \end{array} \right\} \\ &\quad + w^3_y \left\{ \begin{array}{l} \left[w^1_y c_{q,p} - w^2_y s_{q,p}\right]\left[w^1_y s_{r,p} + w^2_y c_{r,p}\right] \\ -\left[w^1_y s_{q,p} + w^2_y c_{q,p}\right]\left[w^1_y c_{r,p} - w^2_y s_{r,p}\right] \end{array} \right\} \\ &= w^1_y \left\{ \begin{array}{c} w^2_y w^3_y \left(c_{q,p} c_{r,p} - c_{q,p} c_{r,p}\right) + w^1_y w^4_y \left(s_{q,p} s_{r,p} - s_{q,p} s_{r,p}\right) \\ + w^2_y w^4_y s_{q-r,p} - w^1_y w^3_y s_{q-r,p} \end{array} \right\} \\ &\quad + w^2_y \left\{ -w^2_y w^3_y s_{q-r,p} - w^1_y w^4_y s_{q-r,p} \right\} + w^3_y \left\{ (w^1_y)^2 + (w^2_y)^2 \right\} s_{q-r,p} \\ &= 0 \,, \end{aligned}$$

which proves the second.

♣

With this lemma we can now immediately simplify our expression for $J_1(k,0,n)$, as claimed. For in 4.9 we see that

$$\left\{\begin{array}{c} -\sin^2\alpha_y\delta_{ij} - \\ \varepsilon_{ijl}w_y^l w_y^4 + w_y^i w_y^j \end{array}\right\} w^i_{z_p^k y} w^j_{z_p^n y} = -\sin^2\alpha_y \left\{\begin{array}{c} \sin^2\alpha_y c_{k-n,p} \\ +\left\{(w_y^4)^2 - (w_y^3)^2\right\} s_{k,p}s_{n,p} \\ +w_y^3 w_y^4 s_{k+n,p} \end{array}\right\}$$

$$+\left\{\begin{array}{c} \sin^2\alpha_y c_{k,p} \\ +w_y^3 w_y^4 s_{k,p} \end{array}\right\}\left\{\begin{array}{c} \sin^2\alpha_y c_{n,p} \\ +w_y^3 w_y^4 s_{n,p} \end{array}\right\}$$

$$= \left\{\begin{array}{c} -\sin^4\alpha_y - \sin^2\alpha_y (w_y^4)^2 \\ +\sin^2\alpha_y (w_y^3)^2 + (w_y^4)^2 (w_y^3)^2 \end{array}\right\} s_{k,p}s_{n,p}$$

$$= -\left\{\sin^2\alpha_y - (w_y^3)^2\right\} s_{k,p}s_{n,p}$$

$$= -\left\{(w_y^1)^2 + (w_y^2)^2\right\} s_{k,p}s_{n,p} ,$$

and so our final expression for $J_1(k,0,n)$ becomes simply

$$J_1(k,0,n) = -6t(\alpha_{z_p^k y})t(\alpha_{z_p^n y})s(\alpha_y)\csc^2\alpha_y \left\{(w_y^1)^2 + (w_y^2)^2\right\} s_{k,p}s_{n,p} . \qquad (4.13)$$

To get $J_1(k,m,n)$ for arbitrary m from this is then simply a matter of writing $\tilde{y} = z_p^m y$, quoting 4.13, and finally translating the answer back into an expression in y. In doing this, the only additional simplification we make is to observe trivially from 4.10 that

For any $m \in \{0,1,\ldots,p-1\}$ *we have* $(w^1_{z_p^m y})^2 + (w^2_{z_p^m y})^2 = (w_y^1)^2 + (w_y^2)^2$. (4.14)

Our final result for $J_1(k,m,n)$ in 4.7 becomes

Result 4.5 *For any* $k,m,n \in \{0,1,\ldots,p-1\}$ *we have*

$$J_1(k,m,n) = -6t(\alpha_{z_p^k y})t(\alpha_{z_p^n y})s(\alpha_{z_p^m y})\csc^2\alpha_{z_p^m y} \left\{(w_y^1)^2 + (w_y^2)^2\right\} s_{k-m,p}s_{n-m,p} . \qquad (4.15)$$

We now turn to $J_2(k,m,n)$.

4.3 Evaluation of $J_2(k,m,n)$

$J_2(k,m,n)$ proves to be a great deal more complicated to evaluate than was $J_1(k,m,n)$. Again we begin with the case $m=0$. In 4.8 we thus consider

$$J_2(k,0,n) = -\left[(\tilde{L}_{1,1})_{ij}(N, z_p^k y)\right] A^{ij}(N,y,n) \tag{4.16}$$

where

$$A^{ij}(N,y,n) \equiv \varepsilon^{iab}\varepsilon^{jcd}\left\{\left[(\tilde{L}_{1,1})_{ac}(N,y)\right]\left[(\tilde{L}_{1,1})_{bd}(N,z_p^n y)\right]\right\}, \tag{4.17}$$

and we begin by focusing on $A^{ij}(N,y,n)$. The following lemma gives a surprisingly simple formula for this quantity;

Lemma 4.6 *For any* $n \in \{0,1,\ldots,p-1\}$ *and any* $i,j \in \{1,2,3\}$ *we have*

$$A^{ij}(N,y,n) = s(\alpha_y)\csc^2\alpha_y s(\alpha_{z_p^n y})\csc^2(\alpha_{z_p^n y})\left\{w_y^i w_y^j + w_{z_p^n y}^i w_{z_p^n y}^j - w_{z_p^n}^i w_{y^{-1}z_p^n y}^j\right\}. \tag{4.18}$$

Remarks: (i) Note that, since $w_{z_p^n}^1 = w_{z_p^n}^2 = 0$, the third term in the bracket on the right hand side in 4.18 is non-zero only when $i=3$.

This seems a strange state of affairs — the terms in 4.18 with $i=3$ seem qualitatively different from those with $i=1,2$. It can, however, be understood better if we consider the more general quantity $A^{ij}(x,y,n)$, defined by 4.17 but with arbitrary x in place of N, which would have arisen had we not managed to integrate out the x-dependence of $I_2^{conn}(L[p], A_{triv}, g)$ in section 4.1. For we find that the formula for this quantity is more transparently symmetric than 4.18, namely

$$A^{ij}(x,y,n) = \left\{\begin{array}{c} s(\alpha_{x^{-1}y})\csc^2(\alpha_{x^{-1}y})\times \\ s(\alpha_{x^{-1}z_p^n y})\csc^2(\alpha_{x^{-1}z_p^n y}) \end{array}\right\}\left\{\begin{array}{c} -w_{x^{-1}y}^i w_{x^{-1}y}^j - w_{x^{-1}z_p^n y}^i w_{x^{-1}z_p^n y}^j \\ +w_{x^{-1}z_p^n x}^i w_{y^{-1}z_p^n y}^j \end{array}\right\}, \tag{4.19}$$

and so we see that the anomalous differences between $i=1,2$ and $i=3$ which exist when $x=N$, disappear in this more general context.

Note, in passing, that the general formula 4.19 is itself intrinsically interesting as a property of the Green's form L.

(ii) To understand the term $w^j_{y^{-1}z^n_p y}$ in 4.18, we use equations ?? from chapter 2, giving the product on S^3 in terms of ambient coordinates, together with our observation there that $w^j_{y^{-1}} = -w^j_y$ for all $j = 1,2,3$; we obtain that, for any $n \in \{0,1,\ldots,p-1\}$,

$$\begin{aligned} w^1_{y^{-1}z^n_p y} &= 2\left[w^1_y w^3_y - w^2_y w^4_y\right] s_{n,p} \,, \\ w^2_{y^{-1}z^n_p y} &= 2\left[w^2_y w^3_y + w^1_y w^4_y\right] s_{n,p} \,, and \\ w^3_{y^{-1}z^n_p y} &= \left[(w^4_y)^2 + (w^3_y)^2 - (w^2_y)^2 - (w^1_y)^2\right] s_{n,p} \,. \end{aligned} \tag{4.20}$$

Proof of Lemma: Substituting formula 3.116 for $\tilde{L}_{1,1}$ into 4.18 we obtain that

$$\begin{aligned} A^{ij}(N,y,n) = \; & s(\alpha_y)\mathrm{csc}^2\alpha_y s(\alpha_{z^n_p y})\mathrm{csc}^2\alpha_{z^n_p y}\varepsilon^{iab}\varepsilon^{jcd}\left\{-\sin^2\alpha_y\delta_{ac} - \varepsilon_{acq}w^q_y w^4_y + w^a_y w^c_y\right\} \\ & \times\left\{-\sin^2\alpha_{z^n_p y}\delta_{bd} - \varepsilon_{bdl}w^l_{z^n_p y}w^4_{z^n_p y} + w^b_{z^n_p y}w^d_{z^n_p y}\right\}. \end{aligned} \tag{4.21}$$

Ignoring the common factor of $s(\alpha_y)\mathrm{csc}^2(\alpha_y)s(\alpha_{z^n_p y})\mathrm{csc}^2(\alpha_{z^n_p y})$, this leads to nine terms from the product in 4.21. Evaluating these now one at a time, we obtain

$$\begin{aligned} \sin^2\alpha_y\sin^2\alpha_{z^n_p y}\varepsilon^{iab}\varepsilon^{jab} &= \sin^2\alpha_y\sin^2\alpha_{z^n_p y}\left(\delta_{ij}\delta_{aa} - \delta_{ia}\delta_{ja}\right) \\ &= 2\sin^2\alpha_y\sin^2\alpha_{z^n_p y}\delta_{ij} \,, \end{aligned}$$

and

$$\begin{aligned} \sin^2\alpha_y\varepsilon^{iab}\varepsilon^{jad}\varepsilon_{lbd}w^l_{z^n_p y}w^4_{z^n_p y} &= \sin^2\alpha_y\left(\delta_{jl}\delta_{ab} - \delta_{jb}\delta_{al}\right)\varepsilon^{iab}w^l_{z^n_p y}w^4_{z^n_p y} \\ &= -\sin^2\alpha_y\varepsilon^{ilj}w^l_{z^n_p y}w^4_{z^n_p y} \\ &= \sin^2\alpha_y\varepsilon^{ijl}w^l_{z^n_p y}w^4_{z^n_p y} \,, \end{aligned}$$

and

$$\begin{aligned} -\sin^2\alpha_y\varepsilon^{iab}\varepsilon^{jad}w^b_{z^n_p y}w^d_{z^n_p y} &= -\sin^2\alpha_y\left(\delta_{ij}\delta_{bd} - \delta_{bj}\delta_{id}\right)w^b_{z^n_p y}w^d_{z^n_p y} \\ &= -\sin^2\alpha_y\sin^2\alpha_{z^n_p y}\delta_{ij} + \sin^2\alpha_y w^i_{z^n_p y}w^j_{z^n_p y} \,, \end{aligned}$$

and

$$\begin{aligned} \varepsilon^{iab}\varepsilon^{jcb}\varepsilon^{acq}\sin^2\alpha_{z^n_p y}w^q_y w^4_y &= \left(\delta_{ij}\delta_{ac} - \delta_{ic}\delta_{ja}\right)\varepsilon^{acq}\sin^2\alpha_{z^n_p y}w^q_y w^4_y \\ &= \sin^2\alpha_{z^n_p y}\varepsilon^{ijl}w^l_y w^4_y \,, \end{aligned}$$

and

$$\begin{aligned}
\varepsilon^{iab}\varepsilon^{jcd}\varepsilon_{qac}\varepsilon_{lbd}w^q_y w^4_y w^l_{z^n_p y} w^4_{z^n_p y} &= (\delta_{iq}\delta_{bc} - \delta_{ic}\delta_{bq})(\delta_{jl}\delta_{bc} - \delta_{jb}\delta_{lc})\, w^q_y w^4_y w^l_{z^n_p y} w^4_{z^n_p y} \\
&= \left(3w^i_y w^j_{z^n_p y} - w^i_y w^j_{z^n_p y} - w^i_y w^j_{z^n_p y} + w^j_y w^i_{z^n_p y}\right) w^4_y w^4_{z^n_p y} \\
&= \left(w^i_y w^j_{z^n_p y} + w^j_y w^i_{z^n_p y}\right) w^4_y w^4_{z^n_p y} \;,
\end{aligned}$$

and, in light of 4.12,

$$\begin{aligned}
-\varepsilon^{iab}\varepsilon^{jcd}\varepsilon_{qac}w^q_y w^4_y w^b_{z^n_p y} w^d_{z^n_p y} &= -(\delta_{iq}\delta_{bc} - \delta_{ic}\delta_{bq})\,\varepsilon^{jcd} w^q_y w^4_y w^b_{z^n_p y} w^d_{z^n_p y} \\
&= -0 + \varepsilon^{jid}\left[w^b_y w^b_{z^n_p y}\right] w^4_y w^d_{z^n_p y} \\
&= -\left\{\sin^2\alpha_y c_{n,p} + w^3_y w^4_y s_{n,p}\right\} \varepsilon^{ijl} w^4_y w^l_{z^n_p y} \;,
\end{aligned}$$

and

$$\begin{aligned}
-\varepsilon^{iab}\varepsilon^{jcb}\sin^2\alpha_{z^n_p y} w^a_y w^c_y &= -(\delta_{ij}\delta_{ac} - \delta_{ic}\delta_{ja})\sin^2\alpha_{z^n_p y} w^a_y w^c_y \\
&= -\sin^2\alpha_y \sin^2\alpha_{z^n_p y}\delta_{ij} + \sin^2\alpha_{z^n_p y} w^i_y w^j_y \;,
\end{aligned}$$

and

$$\begin{aligned}
-\varepsilon^{iab}\varepsilon^{jcd}\varepsilon_{dlb}w^l_{z^n_p y} w^4_{z^n_p y} w^a_y w^c_y &= -(\delta_{id}\delta_{al} - \delta_{il}\delta_{ad})\,\varepsilon^{jcd} w^l_{z^n_p y} w^4_{z^n_p y} w^a_y w^c_y \\
&= -\varepsilon^{ijc}\left[w^a_y w^a_{z^n_p y}\right] w^4_{z^n_p y} w^c_y + 0 \\
&= -\left\{\sin^2\alpha_y c_{n,p} + w^3_y w^4_y s_{n,p}\right\} \varepsilon^{ijl} w^4_{z^n_p y} w^l_y \;,
\end{aligned}$$

and, finally,

$$\varepsilon^{iab}\varepsilon^{jcd} w^a_y w^c_y w^b_{z^n_p y} w^d_{z^n_p y} = B_i(y,n) B_j(y,n)$$

where, for any $i = 1, 2, 3$, we define

$$B_i(y,n) \equiv \varepsilon^{iab} w^a_y w^b_{z^n_p y} \;, \tag{4.22}$$

which, in concrete terms, means that

$$\begin{aligned}
B_1(y,n) &= -\left[w^1_y w^3_y - w^2_y w^4_y\right] s_{n,p} \;, \\
B_2(y,n) &= -\left[w^1_y w^4_y + w^2_y w^3_y\right] s_{n,p} \;, \textit{and} \\
B_3(y,n) &= \left[(w^1_y)^2 + (w^2_y)^2\right] s_{n,p} \;.
\end{aligned} \tag{4.23}$$

Adding these contributions together, we see that

$$A^{ij}(N,y,n) = \left\{ \begin{array}{l} s(\alpha_y)\csc^2\alpha_y \times \\ s(\alpha_{z_p^n y})\csc^2\alpha_{z_p^n y} \end{array} \right\} \left\{ \begin{array}{l} \sin^2\alpha_y \left(\varepsilon^{ijl} w^l_{z_p^n y} w^4_{z_p^n y} + w^i_{z_p^n y} w^j_{z_p^n y}\right) \\ +\sin^2\alpha_{z_p^n y}\left(\varepsilon^{ijl} w^l_y w^4_y + w^i_y w^j_y\right) \\ + \left(w^i_y w^j_{z_p^n y} + w^j_y w^i_{z_p^n y}\right) w^4_y w^4_{z_p^n y} \\ - \left\{\sin^2\alpha_y c_{n,p} + w^3_y w^4_y s_{n,p}\right\} \varepsilon^{ijl} w^4_y w^l_{z_p^n y} \\ - \left\{\sin^2\alpha_y c_{n,p} + w^3_y w^4_y s_{n,p}\right\} \varepsilon^{ijl} w^4_{z_p^n y} w^l_y \\ +B_i(y,n)B_j(y,n) \end{array} \right\}. \tag{4.24}$$

To finally get from this expression to that in 4.18 is then simply a (very) tedious matter of taking each of the nine possibilities for i and j in turn, expanding all the terms involving $z_p^n y$ using 4.10, and simplifying while bearing in mind equations 4.20 and 4.23 and the fact that $w^3_{z_p^n} = s_{n,p}$. The details are unenlightening and we leave them to the reader.

♣

Now that we have proven lemma 4.6, we can use it, together with 3.116 again, to proceed with our evaluation of $J_2(k,0,n)$. In 4.16 we obtain that

$$J_2(k,0,n) = -\left\{ \begin{array}{c} s(\alpha_y)\csc^2\alpha_y s(\alpha_{z_p^k y})\csc^2\alpha_{z_p^k y} \\ \times s(\alpha_{z_p^n y})\csc^2\alpha_{z_p^n y} \end{array} \right\} C(y,k,n) \tag{4.25}$$

where

$$C(y,k,n) = \left\{-\sin^2\alpha_{z_p^k y}\delta_{ij} - \varepsilon_{ijl} w^l_{z_p^k y} w^4_{z_p^k y} + w^i_{z_p^k y} w^j_{z_p^k y}\right\} \left\{ \begin{array}{c} w^i_y w^j_y + w^i_{z_p^n y} w^j_{z_p^n y} \\ -w^i_{z_p^n} w^j_{y^{-1} z_p^n y} \end{array} \right\}. \tag{4.26}$$

Clearly it only remains to understand $C(y,k,n)$. Once again nine terms arise from the expression for this in 4.26, but this time it turns out to be best to consider them three at a time, rather than examining each individually. The first set of three we look at is

$$\left\{-\sin^2\alpha_{z_p^k y}\delta_{ij} - \varepsilon_{ijl}w^l_{z_p^k y}w^4_{z_p^k y} + w^i_{z_p^k y}w^j_{z_p^k y}\right\} w^i_y w^j_y$$

$$= -\sin^2\alpha_y \sin^2\alpha_{z_p^k y} - 0 + \left\{\sin^2\alpha_y c_{k,p} + w^3_y w^4_y s_{k,p}\right\}^2$$

$$= \quad -\sin^2\alpha_y \left\{1 - \left(w^4_y c_{k,p} - w^3_y s_{k,p}\right)^2\right\} + \sin^4\alpha_y {c_{k,p}}^2$$

$$+2\sin^2\alpha_y w^3_y w^4_y s_{k,p} c_{k,p} + (w^3_y)^2 (w^4_y)^2 {s_{k,p}}^2$$

$$= -\sin^2\alpha_y s^2_{k,p} + (w^3_y)^2 s^2_{k,p} \; ,$$

$$i.e. \;\; \left\{-\sin^2\alpha_{z_p^k y}\delta_{ij} - \varepsilon_{ijl}w^l_{z_p^k y}w^4_{z_p^k y} + w^i_{z_p^k y}w^j_{z_p^k y}\right\} w^i_y w^j_y = -\left\{(w^1_y)^2 + (w^2_y)^2\right\} s^2_{k,p} \; . \tag{4.27}$$

The second set is $\left\{-\sin^2\alpha_{z_p^k y}\delta_{ij} - \varepsilon_{ijl}w^l_{z_p^k y}w^4_{z_p^k y} + w^i_{z_p^k y}w^j_{z_p^k y}\right\} w^i_{z_p^n y} w^j_{z_p^n y}$. But this can be evaluated immediately; for, letting $\tilde{y} = z_p^n y$, we can simply quote the result we just derived, obtaining an answer of $-\left\{(w^1_{\tilde{y}})^2 + (w^2_{\tilde{y}})^2\right\} s^2_{k-n,p}$, and, in light of 4.14, we thus obtain at once that

$$\left\{-\sin^2\alpha_{z_p^k y}\delta_{ij} - \varepsilon_{ijl}w^l_{z_p^k y}w^4_{z_p^k y} + w^i_{z_p^k y}w^j_{z_p^k y}\right\} w^i_{z_p^n y} w^j_{z_p^n y} = -\left\{(w^1_y)^2 + (w^2_y)^2\right\} s^2_{n-k,p} \; . \tag{4.28}$$

Finally, the last set of three is $\left\{\sin^2\alpha_{z_p^k y}\delta_{ij} + \varepsilon_{ijl}w^l_{z_p^k y}w^4_{z_p^k y} - w^i_{z_p^k y}w^j_{z_p^k y}\right\} w^i_{z_p^n} w^j_{y^{-1}z_p^n y}$. In simplifying this it is again profitable to make a change of variables; this time we write $\tilde{y} = z_p^k y$ and, recalling that $w^1_{z_p^n} = w^2_{z_p^n} = 0$ and $w^3_{z_p^n} = s_{n,p}$ and noting that $y^{-1}z_p^n y = \tilde{y}^{-1}z_p^n\tilde{y}$, we then obtain that

$$\left\{\sin^2\alpha_{z_p^k y}\delta_{ij} + \varepsilon_{ijl}w^l_{z_p^k y}w^4_{z_p^k y} - w^i_{z_p^k y}w^j_{z_p^k y}\right\} w^i_{z_p^n} w^j_{y^{-1}z_p^n y}$$

$$= \sin^2\alpha_{\tilde{y}} w^3_{\tilde{y}^{-1}z_p^n\tilde{y}} s_{n,p} + \left\{w^1_{\tilde{y}^{-1}z_p^n\tilde{y}} w^2_{\tilde{y}} - w^2_{\tilde{y}^{-1}z_p^n\tilde{y}} w^1_{\tilde{y}}\right\} w^4_{\tilde{y}} s_{n,p} - \left\{w^4_{\tilde{y}^{-1}z_p^n\tilde{y}} w^4_{\tilde{y}} - w^4_{z_p^n\tilde{y}}\right\} w^3_{\tilde{y}} s_{n,p}$$

$$= \sin^2\alpha_{\tilde{y}} w^3_{\tilde{y}^{-1}z_p^n\tilde{y}} s_{n,p} - \left\{w^4_{\tilde{y}^{-1}z_p^n\tilde{y}} w^3_{\tilde{y}} - w^1_{\tilde{y}^{-1}z_p^n\tilde{y}} w^2_{\tilde{y}} + w^2_{\tilde{y}^{-1}z_p^n\tilde{y}} w^1_{\tilde{y}}\right\} w^4_{\tilde{y}} s_{n,p} + w^4_{z_p^n\tilde{y}} w^3_{\tilde{y}} s_{n,p}$$

$$= \sin^2\alpha_{\tilde{y}} w^3_{\tilde{y}^{-1}z_p^n\tilde{y}} s_{n,p} - \left\{w^3_{z_p^n\tilde{y}} - w^3_{\tilde{y}^{-1}z_p^n\tilde{y}} w^4_{\tilde{y}}\right\} w^4_{\tilde{y}} s_{n,p} + w^4_{z_p^n\tilde{y}} w^3_{\tilde{y}} s_{n,p}$$

$$= w^3_{\tilde{y}^{-1}z_p^n\tilde{y}} s_{n,p} + \left\{w^4_{z_p^n\tilde{y}} w^3_{\tilde{y}} - w^3_{z_p^n\tilde{y}} w^4_{\tilde{y}}\right\} s_{n,p}$$

$$= \left[(w_{\bar{y}}^4)^2 + (w_{\bar{y}}^3)^2 - (w_{\bar{y}}^2)^2 - (w_{\bar{y}}^1)^2\right] s_{n,p}^2 + \left\{ \begin{array}{c} w_{\bar{y}}^3 \left[w_{\bar{y}}^4 c_{n,p} - w_{\bar{y}}^3 s_{n,p}\right] \\ -w_{\bar{y}}^4 \left[w_{\bar{y}}^3 c_{n,p} + w_{\bar{y}}^4 s_{n,p}\right] \end{array} \right\} s_{n,p}$$

$$= -\left\{(w_{\bar{y}}^1)^2 + (w_{\bar{y}}^2)^2\right\} s_{n,p}^2 ,$$

where here we have twice used equations 2.34 from chapter 2 to reduce cumbersome expressions. Quoting 4.14 again we thus see that this last set of three terms in $C(y,k,n)$ is actually independent of k, namely

$$\left\{-\sin^2\alpha_{z_p^k y}\delta_{ij} - \varepsilon_{ijl} w_{z_p^k y}^l w_{z_p^k y}^4 + w_{z_p^k y}^i w_{z_p^k y}^j\right\} w_{z_p^n}^i w_{y^{-1} z_p^n y}^j = -\left\{(w_y^1)^2 + (w_y^2)^2\right\} s_{n,p}^2 . \tag{4.29}$$

Bringing the three similar expressions in 4.27, 4.28, and 4.29 together now, we finally obtain our desired simplification of $C(y,k,n)$,

$$C(y,k,n) = -\left\{(w_y^1)^2 + (w_y^2)^2\right\} \left[s_{k,p}^2 + s_{n-k,p}^2 + s_{n,p}^2\right] , \tag{4.30}$$

which, in 4.25, then gives us, as promised, a fully simplified expression for $J_2(k,0,n)$;

$$J_2(k,0,n) = \left\{ \begin{array}{c} s(\alpha_y)\csc^2\alpha_y s(\alpha_{z_p^k y})\csc^2\alpha_{z_p^k y} \\ \times s(\alpha_{z_p^n y})\csc^2\alpha_{z_p^n y} \end{array} \right\} \left\{(w_y^1)^2 + (w_y^2)^2\right\} \left[s_{k,p}^2 + s_{n-k,p}^2 + s_{n,p}^2\right] . \tag{4.31}$$

We can then finally go from this to the general expression for $J_2(k,m,n)$, m arbitrary, in exactly the same way as we did for $J1$. Our ultimate result is;

Result 4.7 *For any* $k,m,n \in \{0,1,\ldots,p-1\}$, $J_2(k,m,n)$ *is given by*

$$\left\{ \begin{array}{c} s(\alpha_{z_p^k y})\csc^2\alpha_{z_p^k y} s(\alpha_{z_p^m y})\csc^2\alpha_{z_p^m y} \\ \times s(\alpha_{z_p^n y})\csc^2\alpha_{z_p^n y} \end{array} \right\} \left\{(w_y^1)^2 + (w_y^2)^2\right\} \left[s_{k-m,p}^2 + s_{n-k,p}^2 + s_{n-m,p}^2\right] . \tag{4.32}$$

4.4 Singularity Structure and Regularisation

Now that we have Results 4.5 and 4.7 we can return to 4.6 and obtain an explicit expression for the graphical contribution to the 2-loop invariant of $L[p]$.

Proposition 4.8 *We have*

$$I_2^{conn}(L[p], A_{triv}, g) = -\frac{2\pi^2}{p^2}\int_{S_y^3}\sum_{k,m,n=0}^{p-1} J(k,m,n) vol_{S_y^3} \tag{4.33}$$

where

$$J(k,m,n) = \left\{ \begin{array}{c} 6t(\alpha_{z_p^k y})t(\alpha_{z_p^n y})s(\alpha_{z_p^m y})\csc^2\alpha_{z_p^m y}\left\{(w_y^1)^2+(w_y^2)^2\right\} s_{k-m,p}s_{n-m,p} \\ -\left\{ \begin{array}{c} \left\{ \begin{array}{c} s(\alpha_{z_p^k y})\csc^2\alpha_{z_p^k y}s(\alpha_{z_p^m y})\csc^2\alpha_{z_p^m y} \\ \times s(\alpha_{z_p^n y})\csc^2\alpha_{z_p^n y} \end{array} \right\}\times \\ \left\{(w_y^1)^2+(w_y^2)^2\right\}\left[s_{k-m,p}^2+s_{n-k,p}^2+s_{n-m,p}^2\right] \end{array} \right\} \end{array} \right\}. \tag{4.34}$$

We want to evaluate $I_2^{conn}(L[p], A_{triv}, g)$ exactly from this expression. Before attempting this, however, we devote the rest of this section to investigating the singularity structure of the integrand in 4.33 and 4.34 in order to verify its integrability.

After all, we know from 3.116 and 3.117 that the functions $s(\alpha)\csc^2\alpha$ and $t(\alpha)$ are both singular of order $1/\alpha^3$ as $\alpha \to 0$ and so, superficially, this integrand looks as though it might blow up in a highly non-integrable way at any of the p points, $\left\{z_p^r\right\}_{r=0}^{p-1}$, on S^3. In order to test this we must, of course, begin by taking the different terms in the sum over k, m and n in 4.33 and grouping together those with the same individual degrees of singularity at the same points. Obviously this involves dividing them according to the degree of coincidence among the three points $z_p^k y, z_p^m y$ and $z_p^n y$ which appear. We thus break the sum into three cases.

Case (i): $z_p^k y, z_p^m y$ and $z_p^n y$ all coincide. This occurs only if $k = m = n$, in which case we are considering $\sum_{r=0}^{p-1} J(r,r,r)$. But, since $s_{0,p} = 0$, it is easy to see that each expression $J(r,r,r)$ in fact vanishes identically. Thus

$$\sum_{k,m,n \text{ all equal}} J(k,m,n) = 0. \tag{4.35}$$

Remark: Note that if we were investigating just the 2-loop invariant of S^3, then $J(0,0,0)$ would be the only term that would turn up in expression 4.8 for

$I_2^{conn}(S^3, A_{triv}, g)$. Thus 4.35 shows that $I_2^{conn}(S^3, A_{triv}, g)$ is zero not just for symmetry reasons, as noted in [AS1], but because its integrand in 2.10 actually vanishes identically.

Case (ii): Two of $z_p^k y, z_p^m y$ and $z_p^n y$ coincide and the third is different. Here we are clearly considering $\sum_{r,s=0,r\neq s}^{p-1} \{J(r,r,s) + J(r,s,r) + J(s,r,r)\}$. In this case individual terms in the sum not only do not disappear, they *do* in fact have non-integrable singularities. For example $J(0,0,1)$ can readily be seen to be singular of order -4 near N , making it a non-integrable function on the 3-dimensional manifold S^3 . Fortunately, however, we only have to perform integration over S^3 *after summing* all these individual terms, and we claim that the non-integrable singularities in fact all cancel off, leaving only integrable singularities behind at the points $\left\{z_p^r\right\}_{r=0}^{p-1}$ on S^3 .

To see this, we need to examine $\sum_{r,s=0,r\neq s}^{p-1} \{J(r,r,s) + J(r,s,r) + J(s,r,r)\}$ in more detail. From 4.34 we see readily that this equals

$$\sum_{r,s=0,r\neq s}^{p-1} 6\left\{(t(\alpha_{z_p^r y}))^2 - (s(\alpha_{z_p^r y}))^2 \csc^4\alpha_{z_p^r y}\right\} s(\alpha_{z_p^s y})\csc^2\alpha_{z_p^s y}\left\{(w_y^1)^2 + (w_y^2)^2\right\} s_{r-s,p}^2 ,$$

and, invoking 3.116 and 3.117, this in turn reduces to

$$\sum_{r,s=0,r\neq s}^{p-1} \frac{3}{8\pi^4}\left\{(\pi - \alpha_{z_p^r y})^2 \csc^4\alpha_{z_p^r y} - \csc^2\alpha_{z_p^r y}\right\} s(\alpha_{z_p^s y})\csc^2\alpha_{z_p^s y}\left\{(w_y^1)^2 + (w_y^2)^2\right\} s_{r-s,p}^2 . \tag{4.36}$$

Consider now a general term in this reduced expression, i.e.

$$\frac{3}{8\pi^4}\left\{(\pi - \alpha_{z_p^r y})^2 \csc^4\alpha_{z_p^r y} - \csc^2\alpha_{z_p^r y}\right\} s(\alpha_{z_p^s y})\csc^2\alpha_{z_p^s y}\left\{(w_y^1)^2 + (w_y^2)^2\right\} s_{r-s,p}^2$$

for some fixed $r \neq s$. This has singularities at the two points $y = z_p^{-r}$ and $y = z_p^{-s}$. But since, by 4.14,

$$(w_y^1)^2 + (w_y^2)^2 = \sin^2\alpha_{z_p^r y}\sin^2\phi_{z_p^r y} = \sin^2\alpha_{z_p^s y}\sin^2\phi_{z_p^s y} , \tag{4.37}$$

we see that these singularities are only of orders -2 and -1 respectively, and so integrable. Thus, overall, the expression in 4.36 involves only terms with integrable singularities, the order -4 singularities in individual terms having cancelled each other out as claimed.

Case (iii): All of $z_p^k y, z_p^m y$ and $z_p^n y$ are distinct. In this case, arguments along the lines of the one just given in case (ii), invoking 4.37, show that each individual term $J(k,m,n)$ (k,m,n all different) has only integrable, order -1 singularities (at the three points z_p^{-k}, z_p^{-m} and z_p^{-n}) and so the sum under consideration, namely

$$\sum_{k,m,n \text{ all distinct}} J(k,m,n) \tag{4.38}$$

certainly presents no integrability problems.

Combining cases (i)-(iii), we have now completely investigated the singularity structure of the integrand in 4.33. We see that it *is* in fact integrable, despite appearances, as promised.

There is, moreover, an auxiliary benefit arising from the analysis we have just performed. For it provides us with a simplified expression for $I_2^{conn}(L[p], A_{triv}, g)$, given by the reduction of the sum $\sum_{k,m,n=0}^{p-1} J(k,m,n)$ in 4.33 to just the two smaller sums in 4.36 and 4.38. Although not apparently a dramatic simplification, this actually represents a significant advance over 4.33 and 4.34 in that all of the *individual* terms in these smaller sums are independently integrable.

Thus, in rewriting our expression for $I_2^{conn}(L[p], A_{triv}, g)$, we can now interchange the order of summation and integration in 4.36 and 4.38, and this allows us to perform independent isometric changes of variables on each of the individual integral terms in the resulting sums. It is easy to see that we can thereby reduce the sum over r and s in 4.36 to just a sum over a single index, say $q \in \{1, \ldots, p-1\}$, and reduce the sum over k, m and n in 4.38 to just a double sum over k and n, in both cases at the expense of introducing an overall factor of p. Our final, fully simplified expression for $I_2^{conn}(L[p], A_{triv}, g)$ is thus;

Proposition 4.9 *The graphical contribution to the 2-loop perturbative invariant of* $L[p]$ *is*

$$I_2^{conn}(L[p], A_{triv}, g) = \frac{-3}{4\pi^2 p} \sum_{q=1}^{p-1} I_A(q,p) - \frac{2\pi^2}{p} \sum_{k,n=1, k\neq n}^{p-1} I_B(k,n,p) \, , \tag{4.39}$$

where

$$I_A(q,p) = \int_{S^3_y} \left\{ \begin{array}{c} (\pi-\alpha_y)^2\csc^4(\alpha_y) \\ -\csc^2(\alpha_y) \end{array} \right\} s(\alpha_{z^q_p y})\csc^2\alpha_{z^q_p y} \left\{ (w^1_y)^2 + (w^2_y)^2 \right\} s^2_{q,p} vol_{S^3_y} \tag{4.40}$$

and

$$I_B(k,n,p) = \int_{S^3_y} \left\{ \begin{array}{c} \left\{ \begin{array}{c} 6t(\alpha_{z^k_p y})t(\alpha_{z^n_p y})s_{k,p}s_{n,p} \quad - \\ s(\alpha_{z^k_p y})\csc^2\alpha_{z^k_p y} s(\alpha_{z^n_p y})\csc^2\alpha_{z^n_p y} \left[s^2_{k,p} + s^2_{n-k,p} + s^2_{n,p} \right] \end{array} \right\} \\ \times \; s(\alpha_y)\csc^2\alpha_y \left\{ (w^1_y)^2 + (w^2_y)^2 \right\} \end{array} \right\} vol_{S^3_y}. \tag{4.41}$$

There is now no further simplification of $I_2^{conn}(L[p], A_{triv}, g)$ that we can perform. The time has at last come to face up and try to explicitly numerically evaluate it for arbitrary p by computing $I_A(q,p)$ and $I_B(k,n,p)$. We undertake this in the next two sections.

But it is worth deferring this task for just a moment longer, in order to conclude this section with a few brief remarks explaining the singularity analysis just performed in the broader context of regularisation in perturbative quantum field theory.

At first glance, the disappearance of the non-integrable singularities that we just saw in treating cases (i) and (ii) seems almost miraculous. In case (i) the potentially most singular terms of the form $J(r,r,r)$ were seen to vanish identically, while in case (ii) the non-integrable order -4 singularities that existed in individual terms all combined exactly so as to cancel off and leave behind only harmless lower order singularities.

Of course, however, these cancellations did not occur by chance — properly understood, they are a consequence of the regularisation scheme adopted by Axelrod and Singer in [AS1] as a familiar ingredient in the analysis of Feynman graphs in the perturbation theory framework.

To be more precise, all the singularities that we discussed are really arising down on $L[p]$ from singularities along the diagonal of the propagator in the theory (which is essentially just L^p — see chapter 2). But, as in the perturbative treatment of any quantum field theory, these diagonal singularities have had to be "regularised" in some way (e.g. momentum cutoff, dimensional regularisation) precisely so as to guarantee finiteness of the Feynman amplitudes of graphs arising in the loop expansion of the partition function. In our case this regularisation was done by Axelrod and Singer in [AS1], where it consisted of a "point-splitting" scheme involving antisymmetrising the group theoretic indices (defined relative to parallel transport) on the propagator. In the same paper and its sequel, [AS2], Axelrod and Singer verified that this yielded the required finiteness of the amplitudes of all 2-loop graphs in the theory.

Thus in working, from chapter 2 onwards, using their definition of I_2^{conn}, it has in fact been *apriori* evident that all our integral expressions for this quantity must turn out to be well-defined and finite, since, as described in section 2.1, they all just represent the Feynman amplitude of the sunset graph at the trivial connection. The integrability observed in this section was thus not miraculous, but rather simply a reflection of the regularisation that had already occurred in [AS1] in the formula defining I_2^{conn}, and a special case of the 2-loop finiteness result proven there.

We now turn to the explicit computations of $I_A(q,p)$ and $I_B(k,n,p)$ foreshadowed just prior to these last remarks, starting with the easier of the two, $I_A(q,p)$.

4.5 An explicit formula for $I_A(q,p)$

Take any p and consider $I_A(q,p)$ for arbitrary fixed $q \in \{1,\ldots,p-1\}$, as defined in 4.40. Writing it out as an iterated integral over the polar variables α_y, ϕ_y and θ_y, using 2.16 and 2.11, we see at once that θ_y integrates out trivially, and so our task becomes to compute

$$I_A(q,p) = 2\pi \int_0^\pi \left\{(\pi-\alpha_y)^2 - \sin^2(\alpha_y)\right\} K_{q,p}(\alpha_y)\, s_{q,p}^2\, d\alpha_y\,, \tag{4.42}$$

where

$$K_{q,p}(\alpha_y) = \int_0^\pi s(\alpha_{z_p^q y}) \csc^2\alpha_{z_p^q y} \sin^3(\phi_y)\, d\phi_y \tag{4.43}$$

and where, of course, from result 4.3, $\alpha_{z^q_p y}$ is given implicitly as a function of α_y and ϕ_y by

$$\cos\alpha_{z^q_p y} = \cos\alpha_y c_{q,p} - \sin\alpha_y \cos\phi_y s_{q,p} \,. \tag{4.44}$$

Clearly we must concentrate first on understanding $K_{q,p}(\alpha_y)$. By definition of s, this is given explicitly by

$$K_{q,p}(\alpha_y) = \frac{-1}{4\pi^2}\int_0^\pi \left\{\left(\pi - \alpha_{z^q_p y}\right)\, \csc^2\alpha_{z^q_p y}\, \cot\alpha_{z^q_p y} + \csc^2\alpha_{z^q_p y}\right\} \sin^3(\phi_y)\, d\phi_y \,. \tag{4.45}$$

But note that, since the integration variable here is ϕ_y, so of course α_y should be treated as a *constant* for the duration of the computation of this integral. This means in 4.44 that we have

$$\frac{d\alpha_{z^q_p y}}{d\phi_y} = -\csc\alpha_{z^q_p y}\, \sin\alpha_y\, \sin\phi_y\, s_{q,p} \,, \tag{4.46}$$

and from this it is then easy to see that 4.45 can be rewritten as

$$K_{q,p}(\alpha_y) = \frac{-1}{4\pi^2 \sin\alpha_y\, s_{q,p}}\int_0^\pi \left\{\frac{d}{d\phi_y}\left[\left(\pi - \alpha_{z^q_p y}\right)\, \csc\alpha_{z^q_p y}\right]\right\} \sin^2(\phi_y)\, d\phi_y \,. \tag{4.47}$$

Integrating by parts and noting that the boundary term vanishes for any α_y (since, for any α_y, $\csc\alpha_{z^q_p y}$ is at most ever singular of order -1 as ϕ_y approaches 0 or π) this in turn then becomes

$$K_{q,p}(\alpha_y) = \frac{1}{2\pi^2 \sin\alpha_y\, s_{q,p}}\int_0^\pi \left(\pi - \alpha_{z^q_p y}\right)\, \csc\alpha_{z^q_p y} \sin\phi_y\, \cos\phi_y\, d\phi_y \,. \tag{4.48}$$

To proceed from here now, the key is to observe, either from 4.46 or directly from 4.44, that $\alpha_{z^q_p y}$ varies monotonically with ϕ_y, and so we can in fact change integration variables to $u = \alpha_{z^q_p y}$ itself. In doing this, however, we need to take some care with our new limits of integration to ensure that they always remain in the required range $[0, \pi]$. At $\phi_y = \pi$ we have $\cos u = \cos(\alpha_y - \frac{2\pi q}{p})$, and at $\phi_y = 0$ we have $\cos u = \cos(\alpha_y + \frac{2\pi q}{p})$, but to determine u from these identities we need to know more about the sizes of $\frac{2\pi q}{p}$ and α_y. We thus have to break our calculation of 4.48 into four cases, according to whether $0 < \frac{2\pi q}{p} \leq \pi/2$, $\pi/2 < \frac{2\pi q}{p} \leq \pi$, $\pi < \frac{2\pi q}{p} \leq 3\pi/2$, or $3\pi/2 < \frac{2\pi q}{p} < 2\pi$, and within each case into three subcases according to the value of α_y.

We find, however, that the working in all these cases is essentially identical, and moreover that, even though the form of $K_{q,p}(\alpha_y)$ varies slightly from case to case, nonetheless the eventual formula we obtain for $I_A(q,p)$, which is ultimately all that we are interested in, turns out to be the *same* each time. Thus, in the remainder of this section, we shall, without loss of generality, consider only case (i), where $0 < \frac{2\pi q}{p} \leq \pi/2$. The reader who is concerned to check our assertion that the formula we obtain for $I_A(q,p)$ *is* in fact the same in the other cases, may do so very quickly once we have laid out our working for this case, since the amendments required in these other scenarios are so minimal.

So, assuming as outlined that $0 < \frac{2\pi q}{p} \leq \pi/2$, we now proceed in turn with the three subcases into which we mentioned we would need to split the computation of 4.48.

Subcase (i) $0 \leq \alpha_y < \frac{2\pi q}{p}$; Then 4.48 becomes

$$K_{q,p}(\alpha_y) = \frac{1}{2\pi^2 \sin^3\alpha_y\, s_{q,p}^3} \int_{\frac{2\pi q}{p}-\alpha_y}^{\frac{2\pi q}{p}+\alpha_y} (\pi - u)\left[\cos\alpha_y c_{q,p} - \cos u\right] du \,. \tag{4.49}$$

Now

$$\int_{\frac{2\pi q}{p}-\alpha_y}^{\frac{2\pi q}{p}+\alpha_y} (\pi - u)\, du = \frac{-1}{2}\left[(\pi - u)^2\right]_{\frac{2\pi q}{p}-\alpha_y}^{\frac{2\pi q}{p}+\alpha_y} = 2(\pi - \frac{2\pi q}{p})\alpha_y \,,$$

and

$$\begin{aligned} -\int_{\frac{2\pi q}{p}-\alpha_y}^{\frac{2\pi q}{p}+\alpha_y} (\pi - u)\cos u\, du &= -[(\pi - u)\,\sin u]_{\frac{2\pi q}{p}-\alpha_y}^{\frac{2\pi q}{p}+\alpha_y} - \int_{\frac{2\pi q}{p}-\alpha_y}^{\frac{2\pi q}{p}+\alpha_y} \sin u\, du \\ &= -(\pi - \tfrac{2\pi q}{p} - \alpha_y)(s_{q,p}\cos\alpha_y + c_{q,p}\sin\alpha_y) \\ &\quad +(\pi - \tfrac{2\pi q}{p} + \alpha_y)(s_{q,p}\cos\alpha_y - c_{q,p}\sin\alpha_y) \\ &\quad + \{c_{q,p}\cos\alpha_y - s_{q,p}\sin\alpha_y - c_{q,p}\cos\alpha_y - s_{q,p}\sin\alpha_y\} \\ &= -2(\pi - \tfrac{2\pi q}{p})\sin\alpha_y c_{q,p} + 2\alpha_y\,\cos\alpha_y s_{q,p} - 2\sin\alpha_y s_{q,p} \,. \end{aligned}$$

Substituting these formulae into 4.49, we therefore find that

$$\begin{aligned} K_{q,p}(\alpha_y) &= \frac{1}{\pi^2 \sin^3\alpha_y\, s_{q,p}^3} \left\{ \begin{array}{c} (\pi - \frac{2\pi q}{p})\left[\alpha_y\,\cos\alpha_y c_{q,p} - \sin\alpha_y c_{q,p}\right] \\ +\left[\alpha_y\,\cos\alpha_y s_{q,p} - \sin\alpha_y s_{q,p}\right] \end{array} \right\} \\ &= \frac{1}{\pi^2 \sin^3\alpha_y\, s_{q,p}^3} \left\{(\pi - \tfrac{2\pi q}{p})c_{q,p} + s_{q,p}\right\}\left[\alpha_y\,\cos\alpha_y - \sin\alpha_y\right] \,, \end{aligned}$$

i.e. for this subcase,

$$K_{q,p}(\alpha_y) = \frac{1}{\pi^2 \sin^2\alpha_y\, s_{q,p}^2} \left\{ (\pi - \frac{2\pi q}{p}) \cot \frac{2\pi q}{p} + 1 \right\} \{\alpha_y \cot \alpha_y - 1\} \ . \tag{4.50}$$

Subcase (ii) $\frac{2\pi q}{p} \leq \alpha_y < \pi - \frac{2\pi q}{p}$; In this setting 4.48 becomes

$$K_{q,p}(\alpha_y) = \frac{1}{2\pi^2 \sin^3\alpha_y\, s_{q,p}^3} \int_{\alpha_y - \frac{2\pi q}{p}}^{\frac{2\pi q}{p} + \alpha_y} (\pi - u) \left[\cos \alpha_y c_{q,p} - \cos u\right] du \tag{4.51}$$

and, in identical fashion to the computations just performed for subcase (i), we quickly deduce that, for this subcase,

$$K_{q,p}(\alpha_y) = \frac{1}{\pi^2 \sin^2\alpha_y\, s_{q,p}^2} \left\{ \frac{2\pi q}{p} \cot \frac{2\pi q}{p} - 1 \right\} \{(\pi - \alpha_y) \cot \alpha_y + 1\} \ . \tag{4.52}$$

Subcase (iii) $\pi - \frac{2\pi q}{p} \leq \alpha_y \leq \pi$; Finally, here 4.48 becomes

$$K_{q,p}(\alpha_y) = \frac{1}{2\pi^2 \sin^3\alpha_y\, s_{q,p}^3} \int_{\alpha_y - \frac{2\pi q}{p}}^{2\pi - \alpha_y - \frac{2\pi q}{p}} (\pi - u) \left[\cos \alpha_y c_{q,p} - \cos u\right] du \tag{4.53}$$

and again, by identical arguments, we deduce that, for this subcase,

$$K_{q,p}(\alpha_y) = \frac{1}{\pi^2 \sin^2\alpha_y\, s_{q,p}^2} \left\{ \frac{2\pi q}{p} \cot \frac{2\pi q}{p} - 1 \right\} \{(\pi - \alpha_y) \cot \alpha_y + 1\} \ , \tag{4.54}$$

which is the same formula as for subcase (ii).

Combining the results of these three subcases now, we obtain at last the closed-form expression for $K_{q,p}(\alpha_y)$ which we were seeking;

$$K_{q,p}(\alpha_y) = \begin{cases} \frac{1}{\pi^2 \sin^2\alpha_y\, s_{q,p}^2} \left\{ (\pi - \frac{2\pi q}{p}) \cot \frac{2\pi q}{p} + 1 \right\} \{\alpha_y \cot \alpha_y - 1\} \ , & 0 \leq \alpha_y < \frac{2\pi q}{p} \\ \frac{1}{\pi^2 \sin^2\alpha_y\, s_{q,p}^2} \left\{ \frac{2\pi q}{p} \cot \frac{2\pi q}{p} - 1 \right\} \{(\pi - \alpha_y) \cot \alpha_y + 1\} \ , & \frac{2\pi q}{p} \leq \alpha_y \leq \pi \ . \end{cases} \tag{4.55}$$

Remark: Note that this formula is itself an intrinsically interesting result in the following respect. In 4.43 we see that $K_{q,p}(\alpha_y)$ is, in some sense, a deformation of the function s on S^3, in which its new value at angle α_y is some sort of weighted average over the slice-variable ϕ_y, related also to the action of $\mathbf{Z}_p$ through the presence of $\alpha_{z_p^q y}$. We are familiar with the function s, of course, from chapter 3, where it first arose as the Green's function of $\triangle$ on S^3, satisfying the ODE $\partial_\alpha{}^2 s(\alpha) = 2\csc^2\alpha s(\alpha)$.

Well, a moment's reflection on 4.55 reveals that $K_{q,p}(\alpha_y)$ in fact remains remarkably closely related to s. Not only is it, of course, a function only of α_y, it actually still satisfies the same ODE, $\partial_\alpha{}^2 K_{q,p}(\alpha) = 2\csc^2\alpha K_{q,p}(\alpha)$. (Note: it thus satisfies $\Delta K_{q,p} = 0$, but this does not represent a contradiction of the fact that harmonic functions on S^3 must be constant, since it is not smooth, only continuous, at the join angle $\alpha_y = \frac{2\pi q}{p}$.) Indeed, the only way $K_{q,p}$ differs materially from s is that its singularity at N has been removed, so that it approaches 0 to $O(\alpha_y^2)$ at N aswell as S.

We thus see that $K_{q,p}$ can be understood as a sort of *modulated* version of the Green's function s, having the same differential properties but with its singularity tamed by the weighted averaging process and $\mathbf{Z}_p$-intertwining that produced it.

Returning from these remarks now, with 4.55 in hand, we can go back to 4.42 and complete our computation of $I_A(q,p)$. We obtain that

$$I_A(q,p) = C_1 K_1 + C_2 K_2 \tag{4.56}$$

where

$$C_1 = \frac{2}{\pi}\left\{(\pi - \frac{2\pi q}{p})\cot\frac{2\pi q}{p} + 1\right\} \text{ and } K_1 = \int_0^{\frac{2\pi q}{p}} \left\{ \begin{array}{c} \{(\pi-\alpha_y)^2\csc^2\alpha_y - 1\}\times \\ \{\alpha_y\cot\alpha_y - 1\} \end{array} \right\} d\alpha_y \tag{4.57}$$

and

$$C_2 = \frac{2}{\pi}\left\{\frac{2\pi q}{p}\cot\frac{2\pi q}{p} - 1\right\} \text{ and } K_2 = \int_{\frac{2\pi q}{p}}^{\pi} \left\{ \begin{array}{c} \{(\pi-\alpha_y)^2\csc^2\alpha_y - 1\}\times \\ \{(\pi-\alpha_y)\cot\alpha_y + 1\} \end{array} \right\} d\alpha_y\,. \tag{4.58}$$

To complete our evaluation of $I_A(q,p)$ we thus just have to compute K_1 and K_2. But, using integration by parts and boundary-term analysis of the type used many times already, we have that

$$\int_0^{\frac{2\pi q}{p}} (\pi-\alpha_y)^2\csc^2\alpha_y\,\{\alpha_y\cot\alpha_y - 1\}\,d\alpha_y = \frac{1}{2}\int_0^{\frac{2\pi q}{p}} (\pi-\alpha_y)^2\partial^2_{\alpha_y}\{\alpha_y\cot\alpha_y - 1\}\,d\alpha_y$$

$$= \tfrac{1}{2}[(\pi-\alpha_y)^2\{-\alpha_y \csc^2\alpha_y + \cot\alpha_y\}]_0^{\frac{2\pi q}{p}} + \int_0^{\frac{2\pi q}{p}} (\pi-\alpha_y)\partial_{\alpha_y}\{\alpha_y \cot\alpha_y - 1\}\, d\alpha_y$$

$$= \tfrac{1}{2}\left[(\pi-\tfrac{2\pi q}{p})^2\left\{-\tfrac{2\pi q}{p s_{q,p}^2}+\tfrac{c_{q,p}}{s_{q,p}}\right\}\right] + [(\pi-\alpha_y)\{\alpha_y \cot\alpha_y - 1\}]_0^{\frac{2\pi q}{p}}$$

$$+\int_0^{\frac{2\pi q}{p}} \{\alpha_y \cot\alpha_y - 1\}\, d\alpha_y\ ,$$

and hence, in 4.57 it follows at once that

$$K_1 = \frac{1}{2}\left(\pi-\frac{2\pi q}{p}\right)\left\{(\pi-\frac{2\pi q}{p})\left\{-\frac{2\pi q}{p s_{q,p}^2}+\frac{c_{q,p}}{s_{q,p}}\right\}+2(\frac{2\pi q}{p})\frac{c_{q,p}}{s_{q,p}}-2\right\}\ . \tag{4.59}$$

Likewise, K_2 can be computed in near identical fashion (after making the simplifying substitution $\tilde{\alpha_y} = \pi - \alpha_y$), giving

$$K_2 = \frac{1}{2}\left(\pi-\frac{2\pi q}{p}\right)\left\{(\pi-\frac{2\pi q}{p})\left\{(\pi-\frac{2\pi q}{p})\frac{1}{s_{q,p}^2}+\frac{c_{q,p}}{s_{q,p}}\right\}-2(\pi-\frac{2\pi q}{p})\frac{c_{q,p}}{s_{q,p}}-2\right\}\ . \tag{4.60}$$

Substituting 4.59 and 4.60 into 4.56 we therefore finally obtain, after elementary cancellations, the following very simple expression for $I_A(q,p)$;

Result 4.10 *For any* $q \in \{1,\ldots,p-1\}$ *we have*

$$I_A(q,p) = -\left(1-\frac{2q}{p}\right)^2\pi^2\ . \tag{4.61}$$

This completes the task we set ourselves in this section. We turn now to $I_B(k,n,p)$.

4.6 An explicit formula for $I_B(k,n,p)$

Unfortunately, as might perhaps be expected from comparing formulae 4.40 and 4.41, $I_B(k,n,p)$ is much harder to evaluate than was $I_A(q,p)$. Indeed, to this date, we have been *unable* to deduce a closed form expression for $I_B(k,n,p)$ along the lines of 4.61.

The principal cause of this intractability is the presence of *two* different implicit variables, $\alpha_{z_p^k y}$ and $\alpha_{z_p^n y}$, in 4.41, in contrast to the case of $I_A(q,p)$ in 4.40 where only one quantity of this type, $\alpha_{z_p^q y}$, arises. This has prevented us from successfully mimicking our approach in the previous section, where the key was really our ability to isolate the term $(\pi - \alpha_{z_p^q y})$ and (eventually) understand the remainder of the integrand as a total differential in ϕ_y.

In an effort to sidestep this problem, we decided to run extensive numerical computations, using Mathematica numerical integration software, evaluating $I_B(k,n,p)$ for all p between 1 and 20, and all possible values of k and n for each p. Our hope in doing this was that, perhaps, even though individual terms $I_B(k,n,p)$ had proved intractable, we might find combinations of these terms for different k and n for which the numerical results suggested simple rational formulae; such combinations would then clearly be the natural thing to atempt to evaluate, and might prove calculable where we had failed with individual $I_B(k,n,p)$.

The results of these numerical tests, together with the programs that generated them and brief explanatory remarks, are collected in Appendix 1 of this thesis. As far as the success of our goal in performing them is concerned, however, the outcome can only be described as mixed.

The successful part was that they *did* reveal, as hoped, that (i) individual $I_B(k,n,p)$ don't appear to be expressible in easy rational terms, but that (ii) there do exist natural combinations of such terms, which we shall call "cyclic triples" (we shall state precisely what we mean by this in a moment), which *do* add up to simple rational expressions in k, n and p. They thus suggested that we should not really be surprised at our failure to evaluate individual $I_B(k,n,p)$, and indicated strongly that we should instead be attempting to compute the integrals given by sums of these cyclic triples.

The fact preventing this numerical excursion from being a complete success, however, is that thus far we have, unfortunately, had no more luck in obtaining by exact computations the formula suggested by the numerics for these cyclic triples, than we had earlier in attempting to find a closed-form expression for the individual $I_B(k,n,p)$.

We thus have to admit openly that we are at present unable to complete the computation in exact, theoretical terms of the second sum in expression 4.39 for

$I_2^{conn}(L[p], A_{triv}, g)$. This represents the one gap which currently exists in our work in this thesis. The best we can do, instead, is to present now as a *conjecture* the result mentioned above regarding sums of cyclic triples, which we deduced from our numerical calculations.

We will then use this in the next section, in conjunction with 4.61, to finally produce at least a "conjectural" evaluation of $I_2^{conn}(L[p], A_{triv}, g)$. We are obliged to point out, however, that when we come to using this in turn in our final evaluations of the 2-loop invariants $\tilde{I}_2^{conn}(L[p], A_{triv}, \sigma)$ in chapter 5, these evaluations will thus, unfortunately, only be valid modulo the truth of this conjecture, i.e. equivalently, they will only as yet have been verified numerically for $1 \leq p \leq 20$. In this context it is, perhaps, only a small consolation to remark that we find the numerical evidence in Appendix 1 thoroughly convincing as regards the truth of the conjecture.

Let us now state this conjecture precisely. By cyclic triples we mean triples of terms consisting of $I_B(k, n, p)$, $I_B(n-k, p-k, p)$, and $I_B(p-n, p-n+k, p)$ for any arbitrary $k < n$. These are "cyclic" collections in the sense that, if we had not used changes of variable on each individual term in 4.38 to remove m-dependence and allow us to write I_B as a function only of k and n in 4.39, then these would represent the transparently cyclic triple of integrands $J(k', m', n')$, $J(n', k', m')$, and $J(m', n', k')$, with $k = k' - m'$ and $n = n' - m'$. With this terminology now explained, our conjecture is then that

Conjecture 4.11 *For any* $k, n \in \{1, \ldots, p-1\}, k < n$ *we have*

$$I_B(k, n, p) + I_B(n-k, p-k, p) + I_B(p-n, p-n+k, p) = -\frac{3}{8\pi^2 p^2} \left\{ \begin{array}{c} (p-2n)^2 \\ -4k(n-k) \end{array} \right\}. \tag{4.62}$$

We now turn to combining this with result 4.10 to calculate at last the full graphical piece $I_2^{conn}(L[p], A_{triv}, g)$.

4.7 The Final Evaluation of $I_2^{conn}(L[p], A_{triv}, g)$

Using expression 4.61 for $I_A(q,p)$ and the conjecture we just stated for $I_B(k,n,p)$ we can now return to 4.39 and finally evaluate $I_2^{conn}(L[p], A_{triv}, g)$, at least modulo the truth of our conjecture.

The first sum in 4.39 becomes $\frac{3}{4p^3}\sum_{q=1}^{p-1}(P-2q)^2$ and it takes only a moment to evaluate this and obtain

$$\frac{-3}{4\pi^2 p}\sum_{q=1}^{p-1} I_A(q,p) = \frac{(p-1)(p-2)}{4p^2}\,. \tag{4.63}$$

To evaluate the *other* sum in 4.39 on the basis of 4.62 we must first rewrite it so that we only sum over $k < n$, which is easily done by introducing a factor of two, and then so that the remaining terms are broken up into their disjoint cyclic triples. There is one subtlety in this last decomposition, however, namely that if p is a multiple of three then the three terms in the cyclic triple with $k = p/3$ and $n = 2p/3$ are all identical, and only one of them, not all three, should appear in our overall sum. Taking this into account, we separate our treatment of this second sum into two cases depending on whether p is a multiple of three or not. Letting $\lfloor l \rfloor$ denote the greatest integer less than or equal to l, then it is a simple matter to see that this rewriting ends up taking the form

$$-\frac{2\pi^2}{p}\sum_{k,n=1,k\neq n}^{p-1} I_B(k,n,p) = -\frac{4\pi^2}{p}\sum_{k=1}^{\lfloor p/3\rfloor}\sum_{n=2k}^{p-k-1}\left\{\begin{array}{l} I_B(k,n,p) \\ +I_B(n-k,p-k,p) \\ +I_B(p-n,p-n+k,p) \end{array}\right\}$$

if p is not a multiple of three, and

$$-\frac{2\pi^2}{p}\sum_{k,n=1,k\neq n}^{p-1} I_B(k,n,p) = \left\{\begin{array}{c} -\frac{4\pi^2}{p}\sum_{k=1}^{p/3-1}\sum_{n=2k}^{p-k-1}\left\{\begin{array}{l} I_B(k,n,p) \\ +I_B(n-k,p-k,p) \\ +I_B(p-n,p-n+k,p) \end{array}\right\} \\ -\frac{4\pi^2}{p} I_B(p/3,2p/3,p) \end{array}\right\}$$

if p is a multiple of three. i.e.

$$-\frac{2\pi^2}{p}\sum_{k,n=1,k\neq n}^{p-1} I_B(k,n,p) = \frac{3}{2p^3}\sum_{k=1}^{\lfloor p/3\rfloor}\sum_{n=2k}^{p-k-1}\left\{\begin{array}{l} (p-2n)^2 \\ -4k(n-k) \end{array}\right\}$$

if p is not a multiple of three, and

$$-\frac{2\pi^2}{p}\sum_{k,n=1,k\neq n}^{p-1} I_B(k,n,p) = \left\{ \begin{array}{c} \frac{3}{2p^3}\sum_{k=1}^{p/3-1}\sum_{n=2k}^{p-k-1}\left\{ \begin{array}{c} (p-2n)^2 \\ -4k(n-k) \end{array} \right\} \\ +\frac{1}{2p^3}\left(-\frac{p^2}{3}\right) \end{array} \right\}$$

if p is a multiple of three. These expressions can, however, be substantially further simplified. The first step obviously is to perform the common inner sum over n. After a few lines of elementary algebra we obtain that

$$\sum_{n=2k}^{p-k-1}\left\{(p-2n)^2-4k(n-k)\right\} = \frac{1}{3}\left\{(p^3+2p)-(15p^2+6)k+(54p)k^2-54k^3\right\} . \tag{4.64}$$

To then simplify the expressions we obtain on substituting this above, the easiest thing to do is just to consider the three cases $p=3r$, $p=3r+1$, and $p=3r+2$ in turn, and use the same elementary algebra to perform the sum over k in each case. We omit the tedious details, but the outcome is interesting in that, on retranslating our final answer in terms of p rather than r, we find that we get the *same* expression in all three cases, namely just

$$-\frac{2\pi^2}{p}\sum_{k,n=1,k\neq n}^{p-1} I_B(k,n,p) = -\frac{(p-1)(p-2)}{4p^2} . \tag{4.65}$$

This is our final expression for the second sum in 4.39.

Adding 4.63 and 4.65 now in 4.39, we then at last reach the end of the long journey we have been undertaking in this chapter and the last, namely the explicit computation of the graphical contributions, $I_2^{conn}(L[p], A_{triv}, g)$, to the 2-loop invariants, $\tilde{I}_2^{conn}(L[p], A_{triv}, \sigma)$, of the lens spaces $L[p]$. The final result we obtain, which of course, as mentioned, is valid only modulo the truth of the numerically based conjecture 4.11, could not be simpler. It is just that

Proposition 4.12 *For any $p \geq 1$ we have*

$$I_2^{conn}(L[p], A_{triv}, g) = 0 . \tag{4.66}$$

This now concludes this chapter.

Chapter 5

The Counterterm, the Full 2-loop Invariants, and Comparison with the Exact TQFT Solution

In chapter 4 we finished computing the graphical contributions to the 2-loop invariants of the lens spaces $L[p]$, $p \geq 1$, and found them all to be zero. In expression 2.10 defining $\tilde{I}_2^{conn}$ the only other term which appears is the counterterm $\frac{1}{8}CS_{grav}(g,\sigma)$, which we described in section 2.1.2. Computation of this counterterm for the $L[p]$ spaces therefore represents the final step in the evaluation of the 2-loop invariants $\tilde{I}_2^{conn}(L[p], A_{triv}, \sigma)$, which is our whole goal in this thesis. We now turn to this task. Fortunately it turns out to be simply a matter of pasting together results already in the literature, and therefore much easier than was the arduous treatment of the graphical piece in chapters 3 and 4.

5.1 Evaluation of $CS_{grav}(g,\sigma)$ on $L[p]$

To this date, all other papers (e.g. [FG1], [J1] and [R]) dealing with lens space Chern-Simons-Witten invariants have been conducted in the TQFT setting. The natural definition of the lens spaces in this context is their surgery definition. In this framework the authors of these papers were able to derive formulae describing explicitly how the biframing naturally inherited via surgery on the lens space from the initial canonical biframing on S^3, relates to the actual canonical biframing on the lens space. This then allowed them to handle completely the presence of the canonical biframing, σ, in the definition of their invariants.

Unfortunately, however, in the more analytic/geometric setting of the Axelrod-Singer theory, and in particular for our current problem of evaluating $CS_{grav}(g,\sigma)$ on $L[p]$, this surgery-theoretic understanding of σ is not *directly* useful. This is because we cannot obtain from it an expression for σ (or, alternatively, the biframing inherited from S^3 via surgery, whose known relationship to σ would permit deduction of $CS_{grav}(g,\sigma)$ from a knowledge of the gravitational Chern-Simons invariant in this biframing) in the very concrete geometric terms that we would need in order to calculate the integral defining the gravitational Chern-Simons invariant.

We thus need to work indirectly. Our starting point is the observation in [A] that $CS_{grav}(g,\sigma)$ is simply a multiple of another well-known metric invariant, the eta-invariant of Atiyah-Patodi-Singer; i.e.

$$CS_{grav}(g,\sigma) = 6\eta_g \, . \tag{5.1}$$

Thus we just need to know the value of η_g for lens spaces. But this computation has already been done — indeed it can be found in the second of the original three papers of Atiyah-Patodi-Singer introducing the eta -invariant, [APS II]. There, it follows from Proposition 2.12 together with our expression 2.38 for the generator of $\mathbf{Z}_p$, that on $L[p]$ (with the standard metric inherited from S^3) we have

$$\eta_g = \frac{1}{p}\sum_{k=1}^{p-1} \cot^2(\frac{\pi k}{p}) \, . \tag{5.2}$$

Note that we see here the appearance of a Dedekind sum, a familiar feature of the lens space computations in [FG1], [J1] and [R].

We can, moreover, simplify 5.2 further by quoting standard theory (e.g. [RG]), which gives us a surprisingly simple quadratic formula for this Dedekind sum;

$$\sum_{k=1}^{p-1} \cot^2(\frac{\pi k}{p}) = \frac{1}{3}(p-1)(p-2) \,. \tag{5.3}$$

In 5.2 we thus obtain just

$$\eta_g = \frac{(p-1)(p-2)}{3p} \,, \tag{5.4}$$

and in 5.1 this in turn then gives us a very simple final formula for the counterterm on $L[p]$;

Proposition 5.1 *On the lens space $L[p]$, with the standard metric, g, inherited from S^3, the gravitational Chern-Simons invariant in the canonical biframing, σ, is given by*

$$CS_{grav}(g, \sigma) = 2\frac{(p-1)(p-2)}{p} \,. \tag{5.5}$$

In the definition of the 2-loop invariants $\tilde{I}_2^{conn}(L[p], A_{triv}, \sigma)$ in 2.10, the full counterterm is, of course, 1/8 of this quantity.

This result concludes our discussion of the 2-loop counterterm in this section.

5.2 Final Computation of $\tilde{I}_2^{conn}(L[p], A_{triv}, \sigma)$ and Comparison with the Exact TQFT Solution

The moment of truth has now arrived. In this final section we at last accomplish our whole goal in this thesis, of comparing our calculation, based on the Axelrod-Singer theory, of the perturbative 2-loop invariants, $\tilde{I}_2^{conn}(L[p], A_{triv}, \sigma)$, with those predicted from the sub-leading asymptotics of the trivial-connection contribution to the exact TQFT solution. The first step in this is, of course, to complete *our* calculation of the $\tilde{I}_2^{conn}(L[p], A_{triv}, \sigma)$, $p \geq 1$. But this is simply a trivial matter of adding our results 4.66 and 5.1 for the graphical piece and the counterterm in formula 2.10. We obtain at once that

Proposition 5.2 *For any $p \geq 1$, the 2-loop perturbative invariant, $\tilde{I}_2^{conn}(L[p], A_{triv}, \sigma)$, of the lens space $L[p]$ is given by*

$$\tilde{I}_2^{conn}(L[p], A_{triv}, \sigma) = -\frac{(p-1)(p-2)}{4p} . \tag{5.6}$$

Recall, of course, that at this stage, as discussed in sections 4.6 and 4.7, this result is true for *all* p only "conjecturally" modulo the truth of our conjecture 4.11, which was based on numerical computations for $1 \leq p \leq 20$.

As for the value *predicted* from the exact TQFT solution for Z_k on $L[p]$, this can be obtained either from [J2], using Proposition 2.14, pp 81, which represents a breakdown of the full solution for Z_k (given in any of [J1], [J2], [FG1] or [R]) via Fourier resummation into its contributions from the different flat connections, or more directly and easily from [R], where Rozansky goes even further in extracting the loop coefficients from the trivial-connection series.

We shall work from [R]; there the predicted value, which we shall denote $\tilde{I}_2^{conn,TQFT}(L[p], A_{triv}, \sigma)$, of the 2-loop trivial-connection invariant, representing the sub-leading coefficient in the trivial-connection higher loop series, is in fact given explicitly by equation 3.12 on pp 15. We just need to understand the various symbols arising in this formula.

Well, recalling from chapter 2 that $L[p]$ corresponds to $L(p, p-1)$ in standard notation, and that, in Rozansky's language, a lens space $L(p', q')$ is obtained from a $U(p', -q')$ surgery on the unknot in S^3, we see that, to begin with, we have $M' = L[p], M = S^3$, and $q = 1$. As for ν, by equation 2.21 in [R] this is given by $\nu = -\frac{m_1}{m_2}$, where m_1 and m_2 are defined in terms of cycles on the complement of the unknot in S^3 (see equation 2.13 and the discussion of C_1 and C_2 on pp 1); but it is trivial to see that for the unknot $m_1 = 0$ and $m_2 = 1$, and so $\nu = 0$. Finally, $D_{2,2}$ is given by equation 2.25 in [R], and since the second derivative of the Alexander polynomial of the unknot in S^3 is obviously zero and d, which is also defined by equation 2.13, is equally easily seen to be 1, this simplifies at once to $D_{2,2} = -\frac{\pi^2}{6}$.

In equation 3.12 therefore, we obtain at once, after noting $s(p,1) \equiv 0$ (by the definition of the Dedekind sum in equation 1.8 in [R]) and performing trivial algebraic simplifications, that finally

Proposition 5.3 *For any $p \geq 1$, the predicted value of the 2-loop perturbative invariant of $L[p]$, on the basis of the sub-leading term in the series expansion of the trivial-connection contribution to the exact TQFT solution for Z_k, is given by*

$$\tilde{I}_2^{conn,TQFT}(L[p], A_{triv}, \sigma) = -\frac{(p-1)(p-2)}{4p} . \tag{5.7}$$

Our final conclusion is therefore, that our values in result 5.2 for the 2-loop perturbative invariants of the $L[p]$ class of lens spaces, calculated on the basis of the Axelrod-Singer perturbation theory, *agree* with the results expected from the exact Witten-TQFT solution.

As discussed in chapter 1, this represents further strong experimental evidence, extending the exclusively semi-classical such evidence which currently exists, of the validity of Witten's functional integral heuristics in his treatment of the Chern-Simons quantum field theory. In so doing, this in turn provides support both for the validity of such "exact" quantum-field-theoretic manipulations in general, and for their general internal consistency with alternative perturbative treatments of the same theories.

Appendix 1

This appendix contains the numerical computations referred to in chapter 4, section 4.6 and motivating the important conjecture 4.11 therein, on the strength of which we ultimately based our explicit computation of the values of the graphical terms $I_2^{conn}(L[p], A_{triv}, g)$, $p \geq 1$, in proposition 4.12.

We include here the programs generating these computations aswell as the output files containing the results. These are Mathematica programs and use nothing more than standard Mathematica numerical integration software packages. As programs they are completely self-explanatory (when read in conjunction with the definitions of $I_A(q,p)$ and $I_B(k,n,p)$ in chapter 4) and the resulting output files are equally trivial to interpret. For clarity we treat the computations for each lens space, $L[p]$, separately, giving first the program generating all the $I_A(q,p)$ and $I_B(k,n,p)$ for this value of p (filename IAIBp.m), and then the resulting output file (filename outIAIBp).

Although we have performed these computations for all $L[p]$ with $3 \leq p \leq 20$ (the results for $p = 1, 2$ are trivial), for reasons of brevity we only include here the cases $3 \leq p \leq 8$. These are easily sufficient, nonetheless, to allow the reader to both confirm our general calculation of $I_A(q,p)$ in result 4.10, and see the strength of the numerical evidence for conjecture 4.11 regarding the $I_B(k,n,p)$.

IAIB3.m

$t[x] := -\frac{1}{4\pi^2}((\pi - x)\csc^3 x + \csc x \cot x)$

$s[x] := -\frac{1}{4\pi^2}((\pi - x)\cot x + 1)$

$u[x] := s[x]\csc^2 x$

$a[n] := \cos[(2n\pi)/3]$

$b[n] := \sin[(2n\pi)/3]$

$c[x, y, n] := a[n]\cos x - b[n]\sin x \cos y$

$d[x, y, n] := \arccos(c[x, y, n])$

$fa[x, y, n] := ((\pi - x)^2\csc^4 x - \csc^2 x)\, u[d[x, y, n]]\sin^4 x \sin^3 y\, b[n]^2$

$fb[x,y,k,n] := (6t[d[x,y,k]]\,t[d[x,y,n]]\,b[k]\,b[n] - u[d[x,y,k]]\,u[d[x,y,n]]\,(b[k]^2 + b[n]^2 + b[n-k]^2))\ s[x] \sin^2 x \sin^3 y$

$ja[k] := NIntegrate[fa[x,y,k], \{x,0,\pi\}, \{y,0,\pi\}, WorkingPrecision \to 7, AccuracyGoal \to 5]$

$jb[k,n] := NIntegrate[fb[x,y,k,n], \{x,0,\pi\}, \{y,0,\pi\}, WorkingPrecision \to 7, AccuracyGoal \to 5]$

$IA[k] := 2\pi ja[k]$

$IB[k,n] := 2\pi jb[k,n]$

$IA[1], IA[2]$

$IB[1,2]$

outIAIB3

−1.09661, −1.09661

0.00415954

IAIB4.m

$t[x] := -\frac{1}{4\pi^2}((\pi - x)\csc^3 x + \csc x \cot x)$

$s[x] := -\frac{1}{4\pi^2}((\pi - x)\cot x + 1)$

$u[x] := s[x]\csc^2 x$

$a[n] := \cos[(2n\pi)/4]$

$b[n] := \sin[(2n\pi)/4]$

$c[x,y,n] := a[n]\cos x - b[n]\sin x \cos y$

$d[x,y,n] := \arccos(c[x,y,n])$

$fa[x,y,n] := ((\pi - x)^2\csc^4 x - \csc^2 x)\ u[d[x,y,n]] \sin^4 x \sin^3 y\ b[n]^2$

$fb[x,y,k,n] := (6t[d[x,y,k]]\,t[d[x,y,n]]\,b[k]\,b[n] - u[d[x,y,k]]\,u[d[x,y,n]]\,(b[k]^2 + b[n]^2 + b[n-k]^2))\ s[x] \sin^2 x \sin^3 y$

$ja[k] := NIntegrate[fa[x,y,k], \{x,0,\pi\}, \{y,0,\pi\}, WorkingPrecision \to 7,$

$AccuracyGoal \to 5]$

$jb[k,n] := NIntegrate[fb[x,y,k,n], \{x,0,\pi\}, \{y,0,\pi\}, WorkingPrecision \to 7,$

$AccuracyGoal \to 5]$

$IA[k] := 2\pi ja[k]$

$IB[k,n] := 2\pi jb[k,n]$

$IA[1], IA[2], IA[3]$

$IB[1,2], IB[1,3], IB[2,3]$

outIAIB4

−2.46741, 0, −2.46741

0.000925203, 0.00765263, 0.000925203

IAIB5.m

$t[x] := -\frac{1}{4\pi^2}((\pi - x)\csc^3 x + \csc x \cot x)$

$s[x] := -\frac{1}{4\pi^2}((\pi - x)\cot x + 1)$

$u[x] := s[x]\csc^2 x$

$a[n] := \cos[(2n\pi)/5]$

$b[n] := \sin[(2n\pi)/5]$

$c[x,y,n] := a[n]\cos x - b[n]\sin x \cos y$

$d[x,y,n] := \arccos(c[x,y,n])$

$fa[x,y,n] := ((\pi - x)^2\csc^4 x - \csc^2 x)\, u[d[x,y,n]] \sin^4 x \sin^3 y\, b[n]^2$

$fb[x,y,k,n] := (6t[d[x,y,k]]\,t[d[x,y,n]]\,b[k]\,b[n] - u[d[x,y,k]]\,u[d[x,y,n]]\,(b[k]^2 + b[n]^2 +$

$b[n-k]^2))\, s[x] \sin^2 x \sin^3 y$

$ja[k] := NIntegrate[fa[x,y,k], \{x,0,\pi\}, \{y,0,\pi\}, WorkingPrecision \to 8,$

$AccuracyGoal \to 6]$

$jb[k,n] := NIntegrate[fb[x,y,k,n], \{x,0,\pi\}, \{y,0,\pi\}, WorkingPrecision \to 8,$

$AccuracyGoal \to 6]$

$IA[k] := 2\pi ja[k]$

$IB[k, n] := 2\pi jb[k, n]$

$IA[1], IA[2], IA[3], IA[4]$

$IB[1, 2], IB[1, 3], IB[1, 4], IB[2, 3], IB[2, 4], IB[3, 4]$

outIAIB5

−3.55306, −0.394783, −0.394783, −3.55306

− 0.00360071, 0.0040842, 0.011762, 0.00247065, 0.0040842, −0.00360071

IAIB6.m

$t[x] := -\frac{1}{4\pi^2}((\pi - x)\csc^3 x + \csc x \cot x)$

$s[x] := -\frac{1}{4\pi^2}((\pi - x)\cot x + 1)$

$u[x] := s[x]\csc^2 x$

$a[n] := \cos[(2n\pi)/6]$

$b[n] := \sin[(2n\pi)/6]$

$c[x, y, n] := a[n]\cos x - b[n]\sin x \cos y$

$d[x, y, n] := \arccos(c[x, y, n])$

$fa[x, y, n] := ((\pi - x)^2\csc^4 x - \csc^2 x)\ u[d[x, y, n]]\sin^4 x \sin^3 y\ b[n]^2$

$fb[x, y, k, n] := (6t[d[x, y, k]]t[d[x, y, n]]b[k]b[n] - u[d[x, y, k]]u[d[x, y, n]](b[k]^2 + b[n]^2 +$
$b[n - k]^2))\ s[x]\sin^2 x \sin^3 y$

$ja[k] := NIntegrate[fa[x, y, k], \{x, 0, \pi\}, \{y, 0, \pi\}, WorkingPrecision \to 7,$
$AccuracyGoal \to 5]$

$jb[k, n] := NIntegrate[fb[x, y, k, n], \{x, 0, \pi\}, \{y, 0, \pi\}, WorkingPrecision \to 7,$
$AccuracyGoal \to 5]$

$IA[k] := 2\pi ja[k]$

$IB[k, n] := 2\pi jb[k, n]$

$IA[1], IA[2], IA[3], IA[4], IA[5]$

$IB[1,2], IB[1,3], IB[1,4], IB[1,5], IB[2,3], IB[2,4], IB[2,5], IB[3,4], IB[3,5]$

$IB[4,5]$

outIAIB6

−4.3865, −1.09661, 0, −1.09661, −4.3865

− 0.00840206, 0.00080826, 0.00673106, 0.0168618, 0.00080826, 0.00415954

0.00673106, 0.00080826, 0.00080826, −0.00840206

IAIB7.m

$t[x] := -\frac{1}{4\pi^2}((\pi - x)\csc^3 x + \csc x \cot x)$

$s[x] := -\frac{1}{4\pi^2}((\pi - x)\cot x + 1)$

$u[x] := s[x]\csc^2 x$

$a[n] := \cos[(2n\pi)/7]$

$b[n] := \sin[(2n\pi)/7]$

$c[x, y, n] := a[n]\cos x - b[n]\sin x \cos y$

$d[x, y, n] := \arccos(c[x, y, n])$

$fa[x, y, n] := ((\pi - x)^2\csc^4 x - \csc^2 x)\, u[d[x, y, n]] \sin^4 x \sin^3 y\, b[n]^2$

$fb[x, y, k, n] := (6t[d[x, y, k]]t[d[x, y, n]]b[k]b[n] - u[d[x, y, k]]u[d[x, y, n]]\,(b[k]^2 + b[n]^2 +$

$b[n-k]^2))\, s[x]\sin^2 x \sin^3 y$

$ja[k] := NIntegrate[fa[x, y, k], \{x, 0, \pi\}, \{y, 0, \pi\}, WorkingPrecision \to 7,$

$AccuracyGoal \to 5]$

$jb[k, n] := NIntegrate[fb[x, y, k, n], \{x, 0, \pi\}, \{y, 0, \pi\}, WorkingPrecision \to 7,$

$AccuracyGoal \to 5]$

$IA[k] := 2\pi ja[k]$

$IB[k, n] := 2\pi jb[k, n]$

$IA[1], IA[2], IA[3], IA[4], IA[5], IA[6]$

$IB[1,2], IB[1,3], IB[1,4], IB[1,5], IB[1,6], IB[2,3], IB[2,4], IB[2,5], IB[2,6]$

$IB[3,4], IB[3,5], IB[3,6], IB[4,5], IB[4,6], IB[5,6]$

outIAIB7

−5.03552, −1.81279, −0.206116, −0.206116, −1.81279, −5.03552

− 0.0134471, −0.00207463, 0.00334714, 0.00961433, 0.0228712, −0.00225343

0.00284302, 0.00585409, 0.00961433, 0.00174967, 0.00284302, 0.00334714

− 0.00225343, −0.00207463, −0.0134471

IAIB8.m

$t[x] := -\frac{1}{4\pi^2}((\pi - x)\csc^3 x + \csc x \cot x)$

$s[x] := -\frac{1}{4\pi^2}((\pi - x)\cot x + 1)$

$u[x] := s[x]\csc^2 x$

$a[n] := \cos[(2n\pi)/8]$

$b[n] := \sin[(2n\pi)/8]$

$c[x, y, n] := a[n]\cos x - b[n]\sin x \cos y$

$d[x, y, n] := \arccos(c[x, y, n])$

$fa[x, y, n] := ((\pi - x)^2\csc^4 x - \csc^2 x)\, u[d[x, y, n]]\sin^4 x \sin^3 y\, b[n]^2$

$fb[x, y, k, n] := (6t[d[x, y, k]]\,t[d[x, y, n]]\,b[k]\,b[n] - u[d[x, y, k]]\,u[d[x, y, n]]\,(b[k]^2 + b[n]^2 +$
$b[n-k]^2))\, s[x]\sin^2 x \sin^3 y$

$ja[k] := NIntegrate[fa[x, y, k], \{x, 0, \pi\}, \{y, 0, \pi\}, WorkingPrecision \to 7,$
$AccuracyGoal \to 5]$

$jb[k, n] := NIntegrate[fb[x, y, k, n], \{x, 0, \pi\}, \{y, 0, \pi\}, WorkingPrecision \to 7,$
$AccuracyGoal \to 5]$

$IA[k] := 2\pi ja[k]$

$IB[k, n] := 2\pi jb[k, n]$

$IA[1], IA[2], IA[3], IA[4], IA[5], IA[6], IA[7]$

$IB[1,2], IB[1,3], IB[1,4], IB[1,5], IB[1,6], IB[1,7], IB[2,3], IB[2,4], IB[2,5],$
$IB[2,6], IB[2,7], IB[3,4], IB[3,5], IB[3,6], IB[3,7], IB[4,5], IB[4,6], IB[4,7],$
$IB[5,6], IB[5,7], IB[6,7]$

outIAIB8

−5.55166, −2.46741, −0.61686, 0, −0.61686, −2.46741, −5.55166

− 0.0186155, −0.00465577, 0.000646365, 0.0056727, 0.0129729, 0.0301646

− 0.0058397, 0.000925203, 0.00444217, 0.00765263, 0.0129729, 0.000646365

0.00309504, 0.00444217, 0.0056727, 0.000646365, 0.000925203, 0.000646365

− 0.0058397, −0.00465577, −0.0186155

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