# DEVELOPMENTS OF CHARTS FOR DETERMINING SHEAR, MOMENT AND NORMAL STRESSES IN CIRCULAR, RECTANGULAR AND ELLIPTICAL RINGS, SUCH AS ARE USED IN TRANSVERSE FRAMES OF FUSELAGES.

by

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F.L.

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# DEVELOPMENT OF CHARTS FOR DETERMINING SHEAR, MOMENT AND NORMAL STRESSES IN CIRCULAR, RECTANGULAR AND ELLIPTICAL RINGS, SUCH AS ARE USED IN TRANSVERSE FRAMES OF FUSELAGES.


## Summary.

Charts for shear, moment and normal stresses in rings were developed, by using the so-called "Symmetric and Anti-Symmetric Loadings".The rings are subjected to a single concentrated load and the stress due to shear is distributed .according to the ordinary theories of bending and torsion.

The formulas for the circular rings have already been developed For rectangular rings, the development of formulas for shear, moment and normal stresses is presented. No formulas were derived for the elliptical case , due to the impossibility of setting up a relation between the deflection and the moments.

Curves giving the coefficients for the determination of the bending, shear and axial loads at any point in a circular ring are presented. For the rectangular case, the curves give the coefficients for the bending, shear and normal loads at one point on the ring, and that for various positions of the load.No curve was drawn for elliptical rings .

## INTRODUCTION.

This method substitutes two loading systems for the original one. In one system, the forces are symmetrically disposed about the plane of symmetry of the structure, in the other the forces are arranged anti-symmetrically. There exist I2 relations connecting forces, moments, slopes, and deflections.

The basic assumptions made are:

1.- The concentrated loads acting on the rings are held in equilibrium under the distributed shear reaction in the fuselage skin.

2.- The fuselage skin is equally effective in supporting shear loads at all points.

3.- The rings have a fairly large diameter as compared to depth.

4.- The rings have constant EI.

## Development of formulas.

### Circular Rings.

When a force of magnitude 2P is acting perpendicular to the plane of the section, the formulas were derived for the moment and the loads at point A.[1] (fig.1)

$$M_{\alpha} = \frac{(PR)}{\pi}(\cos\alpha - 1)(\pi - \alpha - \sin\alpha) \qquad (I)$$

$$N_{\alpha} = \frac{(P)}{\pi}(\pi - \alpha - \sin\alpha\cos\alpha) \qquad (2)$$

$$S_{\alpha} = \frac{(P)}{\pi}\left[\cos\alpha(1+\cos\alpha) - \frac{1}{2}\right] \qquad (3)$$

When the force is acting parallel with the plane of symmetry and anti-symmetry, the expressions for moment, normal force and shear force at point A become: (fig.11)

$$M_\alpha = \frac{(PR)}{\pi}\left[\frac{1}{2} + \cos\alpha - (\pi-\alpha)\sin\alpha + \sin^2\alpha\right] \quad (4)$$

$$N_\alpha = \frac{(P)}{\pi}\left(3/2 - \sin^2\alpha\right) \quad (5)$$

$$S_\circ = \left(\frac{P}{\pi}\right)\left[\left(\pi - \alpha\right) - \sin\alpha - \frac{1}{2}\sin 2\alpha\right] \quad (6)$$

In these formulas, the variable is $\alpha$, P and R for a given problem being fixed.

Equations (1), (2), (3), (4), (5), (6), allow us to determine the moment, normal and shear forces on one section of a circular ring, for loads acting at various angles with the plane of a section. The loads are resolved into components acting parallel and perpendicular to the plane , and the results are added up .

Rectangular Rings:

Load acting perpendicular to the plane of section. (fig.3)

The following formulas were derived for the symmetrical loading :

$$\int_0^\pi \frac{M}{EI}.ds = 0 \quad (7)$$

$$\int_0^\pi \frac{M.(R\cos\varphi)\,ds}{EI} = 0 \quad (8)$$

Eq. (7) is similar to the expression for the change in slope obtained from the moment-area relations on a straight beam.

Eq. (8) is a moment-area relation for the tangential deflection at B taken with respect to the tangent to the elastic curve at A.

Eq. (7) and (8) enable us to solve for the two unknowns $M_\alpha$ and $N_\alpha$, by setting up expressions for the moments at any section , in terms of these two unknowns and the known quantities.

From figure (3b);

for $0 < \varphi < \varphi_1$ $\quad M_1 = M_\alpha$

$\varphi_1 < \varphi < \pi - \varphi_1$ $\quad M_\varphi = M_\alpha + N_\alpha\left(\frac{b}{2} - \frac{a}{2}\cot\varphi\right)$

for $\alpha < \phi < (\pi-\phi_1)$ $\quad M_3 = M_2 - P.\frac{a}{2}(\cot\alpha - \cot\phi)$

$(\pi-\phi_1) < \phi < \pi$ $\quad M_4 = M_a + b.N_a - P\left(\frac{a}{2}\cot\alpha + \frac{b}{2}\right)$

Also:

for $0 < \phi < \phi_1$ $\quad ds_1 = \frac{b}{2}\sec^2\phi.d\phi \qquad X_1 = \frac{b}{2}$

$\phi_1 < \phi < \pi-\phi_1$ $\quad ds_2 = \frac{a}{2}\operatorname{cosec}^2\phi.d\phi \qquad X_2 = \frac{a}{2}.\cot\phi$

$(\pi-\phi_1) < \phi < \pi$ $\quad ds_3 = \frac{b}{2}\sec^2\phi.d\phi \qquad X_3 = \left(-\frac{b}{2}\right)$

where X is the distance from the section to the axis AB. and is similar to ($R\cos\phi$) in the case of the circular ring.

Substituting these quantities in Eq. (7) and (8) ( EI being constant)

$$\int_0^\pi M.ds = 0 = \int_0^{\phi_1}(M_a)\left(\frac{b}{2}\right)\sec^2\phi\,d\phi + \int_{\phi_1}^{(\pi-\phi_1)}(M_2)\left(\frac{a}{2}\right)(\operatorname{cosec}^2\phi.d\phi) - \int_\alpha^{(\pi-\phi_1)} P\left(\frac{a}{2}\right)(\cot\alpha - \cot\phi)\left(\frac{a}{2}\operatorname{cosec}^2\phi.d\phi\right) + \int_{(\pi-\phi_1)}^{\pi}(M_4)\left(\frac{b}{2}\sec^2\phi\,d\phi\right)$$

$$\int_0^\pi M(R\cos\phi)ds = \int_0^{\phi_1}(M_a)\left(\frac{b}{2}\right)\left(\frac{b}{2}\sec^2\phi.d\phi\right) + \int_{\phi_1}^{(\pi-\phi_1)}(M_2)\left(\frac{a}{2}\cot\phi\right)\left(\frac{a}{2}\operatorname{cosec}^2\phi.d\phi\right) - \int_\alpha^{(\pi-\phi_1)} P\left(\frac{a}{2}\right)(\cot\alpha - \cot\phi)\left(\frac{a}{2}\cot\phi\right)\left(\frac{a}{2}\operatorname{cosec}^2\phi.d\phi\right) + \int_{\pi-\phi_1}^{\pi}(M_4)\left(-\frac{b}{2}\right)\left(\frac{b}{2}\sec^2\phi.d\phi\right) -$$

Performing the indicated integration and simplifying the equations:

$$M_a = \frac{P\{a^3(a+b)\cot^3\alpha + a^2b(3a+b)\cot^2\alpha - ab^2(a+b).\cot\alpha - b^3(3a+b)\}}{8b(3a+b)(a+b)} \qquad (9)$$

$$N_a = \frac{P\{-a^3\cot^3\alpha + 3ab(2a+b)\cot\alpha + 2b^2(3a+b)\}}{4b^2(3a+b)} \qquad (10)$$

Eq. (9) and (10) suffice for the determination of the moment and the normal force at point A, in the case of the symmetrically loaded structure.

In the anti-symmetrically loaded structure (fig.3c), the concentrated load P produces twisting and shearing in the skin, and the ring acts as a transverse stiffener.

The twisting moment due to the load P is $(2P)(\frac{a}{2}\cot\alpha)$ and is resisted by the torsional couple provided by the four sides of the ring. For the side CD, this couple is (taking moments about the center of the figure):

$$\int_{-b/2}^{+b/2} q_{\circ}\sqrt{x^2+\frac{a^2}{4}}\,dx = \left\{\frac{x}{2}\sqrt{x^2+\frac{a^2}{4}} + \frac{a^2}{8}\ln_e\left(x+\sqrt{x^2+\frac{a^2}{4}}\right)\right\}_{-b/2}^{+b/2}$$

$$= q_{\circ}\left\{\frac{b}{2}\sqrt{\frac{a^2}{4}+\frac{b^2}{4}} + \left[\frac{a^2}{8}\ln_e\left(\frac{b}{2}+\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}\right) - \frac{a^2}{8}\ln_e\left(-\frac{b}{2}+\sqrt{\frac{b^2}{4}+\frac{a^2}{4}}\right)\right]\right\}$$

and for the four sides of the ring:

$$(2P)\left(\frac{a}{2}\cot\alpha\right) = \frac{q_{\circ}}{8}\left\{2(a+b)\sqrt{a^2+b^2} + a^2\ln\frac{b+\sqrt{a^2+b^2}}{-b+\sqrt{a^2+b^2}} + b^2\ln\frac{a+\sqrt{a^2+b^2}}{-a+\sqrt{a^2+b^2}}\right\}$$

or $\qquad P.a\cot\alpha = M_T(q_{\circ})$

$$\text{where } M_T = \frac{1}{8}\left\{2(a+b)\sqrt{a^2+b^2} + a^2\ln\frac{b+\sqrt{a^2+b^2}}{-b+\sqrt{a^2+b^2}} + b^2\ln\frac{a+\sqrt{a^2+b^2}}{-a+\sqrt{a^2+b^2}}\right\} \tag{11}$$

$$\text{so} \quad q_{\circ} = \frac{P.a.\cot\alpha}{M_T} \tag{12}$$

The longitudinal shear stress on a section making an angle $\theta$ with the axis is,

$$f_s = \frac{V\,Q}{t\,I} \qquad \text{and} \qquad q_1 = f_s.t = \frac{V\,Q}{t\,I}\,t = \frac{V\,Q}{I}$$

But $V = 2P$

$I = \frac{a^2 t}{6}(a+3b)$ about an axis through the plane of anti-symmetry at A and B,

Q is the moment about A of the area between the section and point A, and is equal to X.A.

| For, | A | X | Q |
|---|---|---|---|
| $0 < \theta < \varphi_1$ | $\frac{1}{2}(b+a-b.\tan\theta)t$ | $\frac{(2ab + a^2 - b^2\tan^2\theta)}{4(b+a-b\tan\theta)}$ | $\frac{1}{8}(2ab + a^2 - b^2\tan^2\theta)$ |
| $\varphi_1 < \theta < \pi-\varphi_1$ | $(\frac{a}{2}\cot\theta)t$ | $\frac{a}{2}$ | $(\frac{a^2}{4}t\cot\theta)$ |
| $\pi-\varphi_1 < \theta < \pi$ | $-\frac{1}{2}(+b+a+b\tan\theta)t$ | $\frac{(2ab + a^2 - b^2\tan^2\theta)}{4(b+a+b\tan\theta)}$ | $-\frac{1}{8}(2ab + a^2 - b^2\tan^2\theta)$ |

And the total shear force $q = q_0 + q_1$

$$0 < \theta < \varphi_1 \qquad q = P\left\{\frac{a\cot\alpha}{M_T} + \frac{3(2ab + a^2 - b^2\tan^2\theta)}{2a^2(a+3b)}\right\} \tag{13a}$$

$$\varphi_1 < \theta < \pi-\varphi_1 \qquad q = P\left\{\frac{a\cot\alpha}{M_T} + \frac{3\cot\theta}{(a+3b)}\right\} \tag{13b}$$

$$\pi-\varphi_1 < \theta < \pi \qquad q = P\left\{\frac{a\cot\alpha}{M_T} - \frac{3(2ab + a^2 - b^2\tan^2\theta)}{2a^2(a+3b)}\right\} \tag{13c}$$

For the anti-symmetrically loaded section, the following formula has been derived:[1]

$$\int_0^\pi \frac{M(R\sin\varphi)\,ds}{EI} = 0 \tag{14}$$

Eq. (14) expresses the moment relation for the radial component of the deflection at B, taken with respect to the tangent to the elastic curve at A.

For,

$$0 < \varphi < \varphi_1 \qquad M_1 = S_A\frac{b}{2}\tan\varphi \tag{15a}$$

$$\varphi_1 < \varphi < \pi-\varphi_1 \qquad M_2 = S_A\frac{a}{2} + \int_0^{\varphi_1} q\left(\frac{b}{2} - \frac{a}{2}\cot\varphi\right)\frac{b}{2}\sec^2\theta\,d\theta \tag{15b}$$

$$\alpha < \varphi < \pi-\varphi_1 \qquad M_3 = M_2 - P\frac{a}{2}(\cot\alpha - \cot\varphi) \tag{15c}$$

$$\pi-\varphi_1 < \varphi < \pi \qquad M_4 = S_A\frac{b}{2}\tan(\pi-\varphi) - P\left(\frac{a}{2}\cot\alpha + \frac{b}{2}\right) + \int_0^{\varphi_1} q\left(\frac{b^2}{2}\sec^2\theta\,d\theta\right) + \int_{\varphi_1}^{\pi-\varphi_1} q\left[\frac{a}{2} - \frac{b}{2}\tan(\pi-\varphi)\right]\frac{a}{2}\operatorname{cosec}^2\theta\,d\theta \tag{15d}$$

For the circular ring $(R\sin\varphi)$ is the distance from the section to the plane of anti-symmetry. Let Y be this distance for the rectangular ring.

For ,

$0 < \varphi < \varphi_1$ $\qquad Y = \frac{b}{2} \tan \varphi$

$\varphi_1 < \varphi < (\pi - \varphi_1)$ $\qquad Y = \frac{a}{2}$

$\pi - \varphi_1 < \varphi < \pi$ $\qquad Y = \frac{b}{2} \tan (\pi - \varphi)$

Substituting in Eq. (14) these values of Y and assuming EI constant,

$$\int_0^{\pi} M ( R \sin \varphi ) \, ds = \int_0^{\pi} M Y \, ds = 0$$

$$= \int_0^{\varphi_1} M_1 \left(\frac{b}{2} \tan \varphi\right)\left(\frac{b}{2} \sec^2 \varphi \, d\varphi\right) + \int_{\varphi_1}^{\pi-\varphi_1} M_2 \left(\frac{a}{2}\right)\left(\frac{a}{2} \sec^2 \varphi \, d\varphi\right)$$

$$- \int_{\alpha}^{\pi-\varphi_1} P \frac{a}{2} ( \cot \alpha - \cot \varphi )\left(\frac{a}{2}\right)\left(\frac{a}{2} \sec^2 \varphi \, d\varphi\right) + \int_{\pi-\varphi_1}^{\pi} M_4 \frac{b}{2} \tan (\pi - \varphi) \, \frac{b}{2} \sec^2 \varphi \, d\varphi \qquad (16)$$

Substituting Eq.(13) for the values of q in (15) and in integrating with respect to $\theta$, gives $M_2$ and $M_4$; substituting Eq.(15) in Eq. (16) and integrating with respect to $\varphi$ and simplifying

$$S_a = \frac{P \left\{ 3a^2 \cot^2 \alpha + \cot \alpha \left[ 3a ( a + 2b ) - \frac{2a^2 b ( 2a + 3b )}{M_T} - 3b^2 \right] \right\}}{4a ( a + 3b )} \qquad (17)$$

In Eq.(17), $M_T$ is a quantity depending on the lengths of the sides a and b , and can be expressed in terms of a or b if the ratio a/b is known.

The variable $\alpha$ defines the position of the loads P and can vary from $\varphi_1$ to $(\pi - \varphi_1)$ and from $(\pi + \varphi_1)$ to $(2\pi - \varphi_1)$

When the load is acting parallel with the planes of symmetry and anti-symmetry, (fig.4) the following expressions are derived ;

Symmetrical loading:

The tangential shear stress is,

$$q_1 = f_s t = \frac{V Q}{t I} t = \frac{V Q}{I}$$

where $V = 2P$

$I = \frac{b^2 t ( 3a + b )}{6}$ with respect to an axis perpendicular to the plane of symmetry.

For

| | $A_o$ | $X_o$ | Q |
|---|---|---|---|
| $0 < \theta < \varphi_1$ | $t\left(\frac{b}{2}\tan\theta\right)$ | $\frac{b}{2}$ | $t\,\frac{b^2}{4}\tan\theta$ |
| $\varphi_1 < \theta < \pi-\varphi_1$ | $\frac{t}{2}(a + b - a\cot\theta)$ | $\frac{+(2ab + b^2 - a^2\cot^2\theta)}{4(a + b - a\cot\theta)}$ | $\frac{t(2ab + b^2 - a^2\cot^2\theta)}{8}$ |
| $\pi-\varphi_1 < \theta < \pi$ | $t\left(a + b + \frac{b}{2}\tan\theta\right)$ | $\frac{-(b^2\tan\theta)}{4\left(a + b + \frac{b}{2}\tan\theta\right)}$ | $\frac{t}{4}(-b^2\tan\theta)$ |

So, for

$0 < \theta < \varphi_1$ $$q = \frac{3\tan\theta . P}{(3a + b)} \qquad (18a)$$

$\varphi_1 < \theta < \pi-\varphi_1$ $$q = \frac{3P(2ab + b^2 - a^2\cot^2\theta)}{2b^2(3a + b)} \qquad (18b)$$

$\pi-\varphi_1 < \theta < \pi$ $$q = \frac{-3P\tan\theta}{(3a + b)} \qquad (18c)$$

Also, for

$0 < \varphi < \varphi_1$ $$M_1 = M_a \qquad (19a)$$

$\alpha < \varphi < \varphi_1$ $$M_1' = M_a - P\frac{b}{2}(\tan\varphi - \tan\alpha) \qquad (19b)$$

$\varphi_1 < \varphi < \pi-\varphi_1$ $$M_2 = M_a + N_a\left(\frac{b}{2} - \frac{a}{2}\cot\varphi\right) - P\left(\frac{a}{2} - \frac{b}{2}\tan\alpha\right) - \int_0^{\varphi_1} q_1\left(\frac{b}{2} - \frac{a}{2}\cot\varphi\right)\frac{b}{2}\sec^2\theta . d \qquad (19c)$$

$\pi-\varphi_1 < \varphi < \pi$ $$M_3 = M_a + b\,N_a - \int_0^{\phi_1} q_1\, b\,\frac{b}{2}\sec^2\theta . d\theta - \int_{\phi_1}^{\pi-\phi_1} q_2(a + b\tan\varphi)\frac{a}{4}\operatorname{cosec}^2\theta\, d\theta \qquad (19d)$$

$\alpha < \phi < \pi$ $$M_3' = M_3 - P\frac{b}{2}\left[\tan\varphi - \tan\alpha\right] \qquad (19e)$$

For this symmetrical loading: from Eq. (7) and (8)

$$\int_0^{\pi} M . ds = 0 \qquad \text{and} \qquad \int_0^{\pi} M(R\cos\varphi)\, ds = 0$$

$$\int_0^{\pi} M\, ds = 0 = \int_0^{\varphi_1} M_1\left(\frac{b}{2}\sec^2\varphi\, d\phi\right) - \int_{\alpha}^{\phi_1} P\frac{b}{2}(\tan\phi - \tan\alpha)\frac{b}{2}\sec^2\phi\, d\phi + \int_{\phi_1}^{\pi-\phi_1} M_2\left(\frac{a}{2}\operatorname{cosec}^2\phi\, d\phi\right)$$
$$+ \int_{\pi-\phi_1}^{\pi} M_3\left(\frac{b}{2}\sec^2\phi\, d\phi\right) - \int_{\alpha}^{\pi} P\frac{b}{2}(\tan\phi - \tan\alpha)\frac{b}{2}\sec^2\phi . d\phi = 0 \qquad (20)$$

$$\int_0^{\pi} M . (R\cos\phi)\, ds = 0 = \int^{\phi_1} M_1\frac{b}{2}\left(\frac{b}{2}\sec^2\phi\, d\phi\right) - \int_{\alpha}^{\phi_1} P\frac{b}{2}(\tan\phi - \tan\alpha)\frac{b}{2}\left(\frac{b}{2}\sec^2\phi . d\phi\right)$$
$$+ \int_{\varphi_1}^{\pi-\varphi_1} M_2\left(\frac{a}{2}\cot\phi\right)\left(\frac{a}{2}\operatorname{cosec}^2\phi\, d\phi\right) + \int_{\pi-\varphi_1}^{\pi} M_3\left(-\frac{b}{2}\frac{b}{2}\sec^2\phi . d\phi\right)$$
$$- \int_{\alpha}^{\pi} P\frac{b}{2}(\tan\phi - \tan\alpha)\left(-\frac{b}{2}\right)\left(\frac{b}{2}\sec^2\phi\, d\phi\right) = 0 \qquad (21)$$

Substituting Eq.(18a,b) in Eq.(19c,d) and integrating with respect to $\theta$ gives $M_2$ and $M_3$.

Substituting Eq19(a,b,c,d,e) in (20) and in (21) and integrating with respect to $\phi$ gives the following relations:

$$M_a = \frac{P\left\{b^2(3a+b)\tan^2\alpha - 2b(3a^2+5ab+b^2)\tan\alpha + 2a(3a^2+5ab+b^2)\right\}}{4(a+b)(3a+b)} \tag{22}$$

$$N_a = \frac{P(6ab\tan\alpha - 3a^2)}{4b(3a+b)} \tag{23}$$

For the anti-symmetrical loading; (fig.4c)

The twisting moment is:

$$2P\frac{b}{2}\tan\alpha = M_T q_o \qquad \text{or} \qquad q_o = \frac{P\,b\tan\alpha}{M_T} \tag{24}$$

where $M_T$ has the same definition as in Eq.(11)

For

$$0 < \phi < \phi_1 \qquad M_1 = S_a \cdot \frac{b}{2}\tan\phi \tag{25a}$$

$$\alpha < \phi < \phi_1 \qquad M_1' = M_1 - P\frac{b}{2}(\tan\phi - \tan\alpha) \tag{25b}$$

$$\phi_1 < \phi < \pi-\phi_1 \quad M_2 = S_a \cdot \frac{a}{2} - P\left(\frac{a}{2} - \frac{b}{2}\tan\alpha\right) - \int_0^{\phi_1} q_o\left(\frac{b}{2} - \frac{a}{2}\cot\phi\right)\frac{b}{2}\sec^2\theta\,d\theta \tag{25c}$$

$$\pi-\phi_1 < \phi < \pi \quad M_3 = S_a \cdot \frac{b}{2}\tan(\pi-\phi) - \int_0^{\phi_1} q_o b\left(\frac{b}{2}\sec^2\theta\,d\theta\right) - \int_{\phi_1}^{\pi-\phi_1} q_o\left(\frac{a}{2} + \frac{b}{2}\tan\phi\right)\frac{a}{2}\operatorname{cosec}^2\theta\,d\theta \tag{25d}$$

$$\alpha < \phi < \pi \qquad M_3' = M_3 - P\frac{b}{2}(\tan\phi - \tan\alpha) \tag{25e}$$

From Eq.(14),

$$\int_0^{\pi} M(R\sin\phi)\,ds = \int_0^{\pi} M\,Y\,ds = 0$$

$$\int_0^{\pi} M.Y.\,ds = \int_0^{\phi_1} M_1\left(\frac{b}{2}\tan\phi\right)\left(\frac{b}{2}\sec^2\phi\,d\phi\right) - P\frac{b}{2}(\tan\phi - \tan\alpha)\left(\frac{b}{2}\tan\phi\right)\left(\frac{b}{2}\sec^2\phi\,d\phi\right)$$
$$+ \int_{\phi_1}^{\pi-\phi_1} M_2\left(\frac{a}{2}\right)\left(\frac{a}{2}\operatorname{cosec}^2\phi\,d\phi\right) + \int_{\pi-\phi_1}^{\pi} M_3\left(-\frac{b}{2}\tan\phi\right)\left(\frac{b}{2}\sec^2\phi\,d\phi\right)$$
$$- \int_{\alpha}^{\pi} P\frac{b}{2}(\tan\phi - \tan\alpha)\left(-\frac{b}{2}\sec\phi\,d\phi\right) = 0 \tag{26}$$

Substituting Eq.(24) in Eq.(25) and integrating with respect to $\theta$ then substituting in Eq.(26), and integrating with respect to $\phi$ gives:

$$S_a = \frac{P\left\{\tan\alpha\left[\frac{2ab^2}{M_T}(a + 3b) - 3b(a + 4b)\right] + 2a(a + 6b)\right\}}{4a(a + 3b)} \tag{27}$$

## Results.

Coefficients for bending moment, normal and shearing forces have been tabulated in the appendix. From these, curves have been drawn , for various positions of the loads , and also for various values of a/b , in the case of the rectangular rings. For these rings , b/a range from 1/5 to 1, so that it is possible to interpolate between these curves for any value of b/a.

## Discussion.

The curves for the circular rings are symmetrical about the horizontal diameter. Stresses at all points can be determined, but it is necessary to resolve the external forces into components normal and parallel to the horizontal diameter.It is to be noted that the concentrated load for the original loading was 2P. So that when using the curves , the coefficients should be multiplied by half this load or P, in order to obtain the stresses at any point.

For the rectangular rings, the curves are not symmetrical about the horizontal diameter. When the load is perpendicular to the plane of symmetry, the results for $\alpha$ varying from $\phi_1$ to $\frac{\pi}{2}$ should equal to those for $\alpha$ varying from $(2\pi - \phi_1)$ to $\frac{3\pi}{2}$ . But that is not the case. The variable $\alpha$ is expressed in these formulas in the form of a tangent or a cotangent, and these two trigonometric functions are positive in the first and third quadrants, and negative in the second and in the fourth quadrants. For this reason , the curves give the same results when $\alpha$ is the first or third quadrant, or in the second or fourth quadrant. All the formulas for rectangular rings have been checked , by using X and Y as variables instead of $\theta$ and $\phi$ . It is possible that the assumptions made for the shear distribution in the case of rectangular rings should be different from those made for a circular ring, due to the discontinuity in a rectangle.

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# APPENDIX.

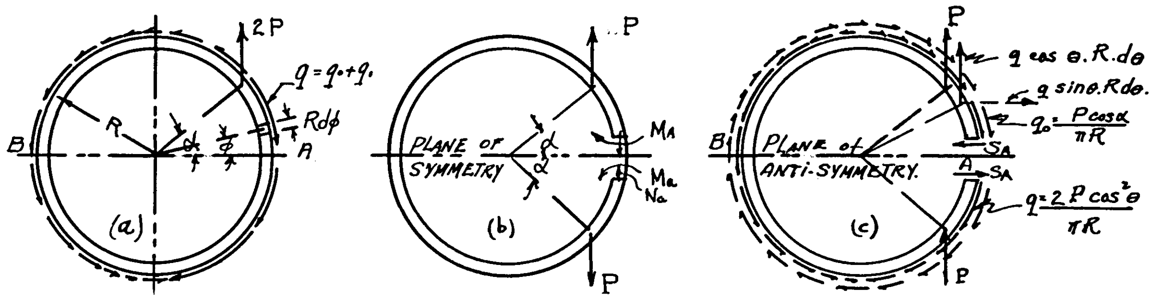



FIG. 1.

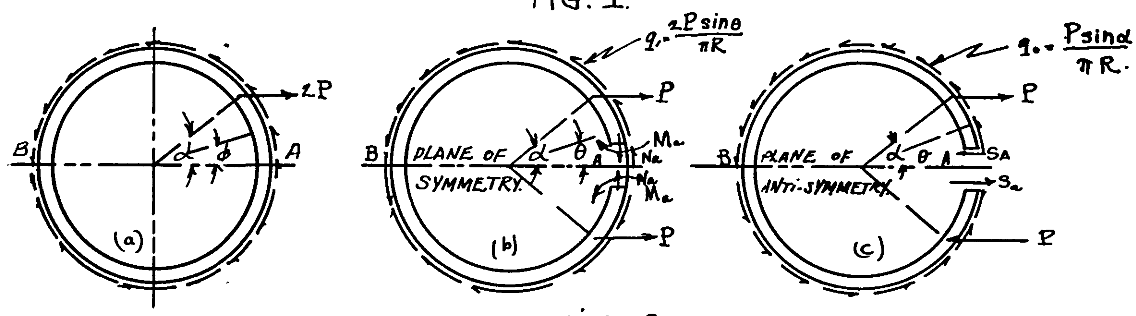



FIG. 2

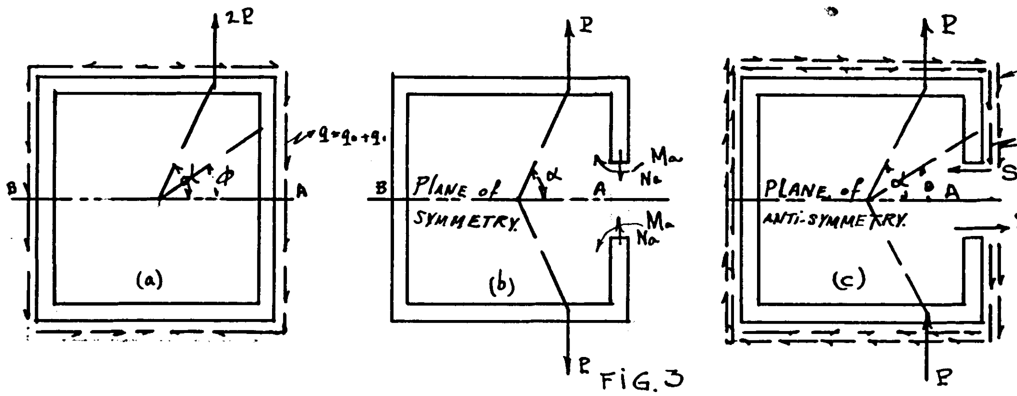



FIG. 3

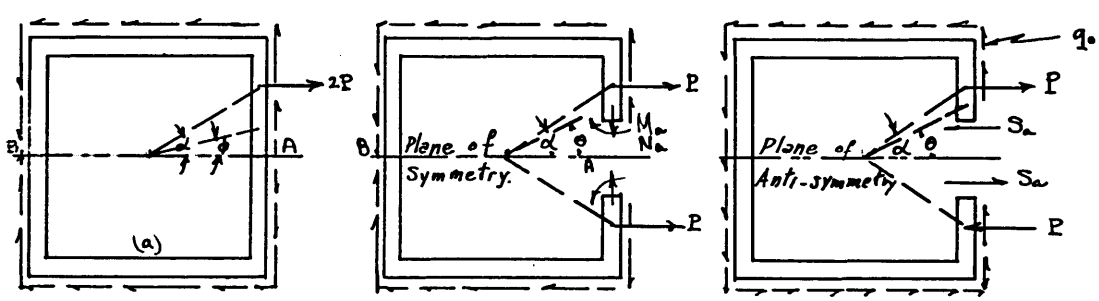



FIG. 4.

## SYMBOLS

$2P$ = the applied concentrated load.

$R$ = the radius of the centroidal axis of the ring.

$M_a$ = the bending moment at point A ( clockwise moments are positive)

$N_a$ = the normal force at point A

$S_a$ = the shearing force at point A

$\alpha$ = the angle between the radius at point of application of the load and the horizontal diameter.

$\phi$ = a variable angle used in determining the moment at any section

$\theta$ = a variable angle used in determining the shear at any section

$q_o$ = the torsional force in pounds per inch of circumference or perimeter

$q_1$ = the variable tangential force in pounds per inch

$q$ = the total shear force in pounds per inch $= q_o + q_1$

$C_{M_\alpha}$ = a moment coefficient $= \frac{M_\alpha}{PR/\pi}$ for the circular ring

$= \frac{M_\alpha}{P.a}$ or $\frac{M_\alpha}{P.b}$ for the rectangular ring

$C_{N_\alpha}$ = the normal force coefficient $= \frac{N_\alpha}{P/\pi}$ for the circular ring

$= \frac{N_\alpha}{P}$ for the rectangular ring

$C_{S_\alpha}$ = a shear force coefficient $= \frac{S_\alpha}{P/\pi}$ for the circular ring

$= \frac{S_\alpha}{P}$ for the rectangular ring

$\phi_1 = \tan^{-1} \frac{a}{b}$

RESULTS

## Circular rings .

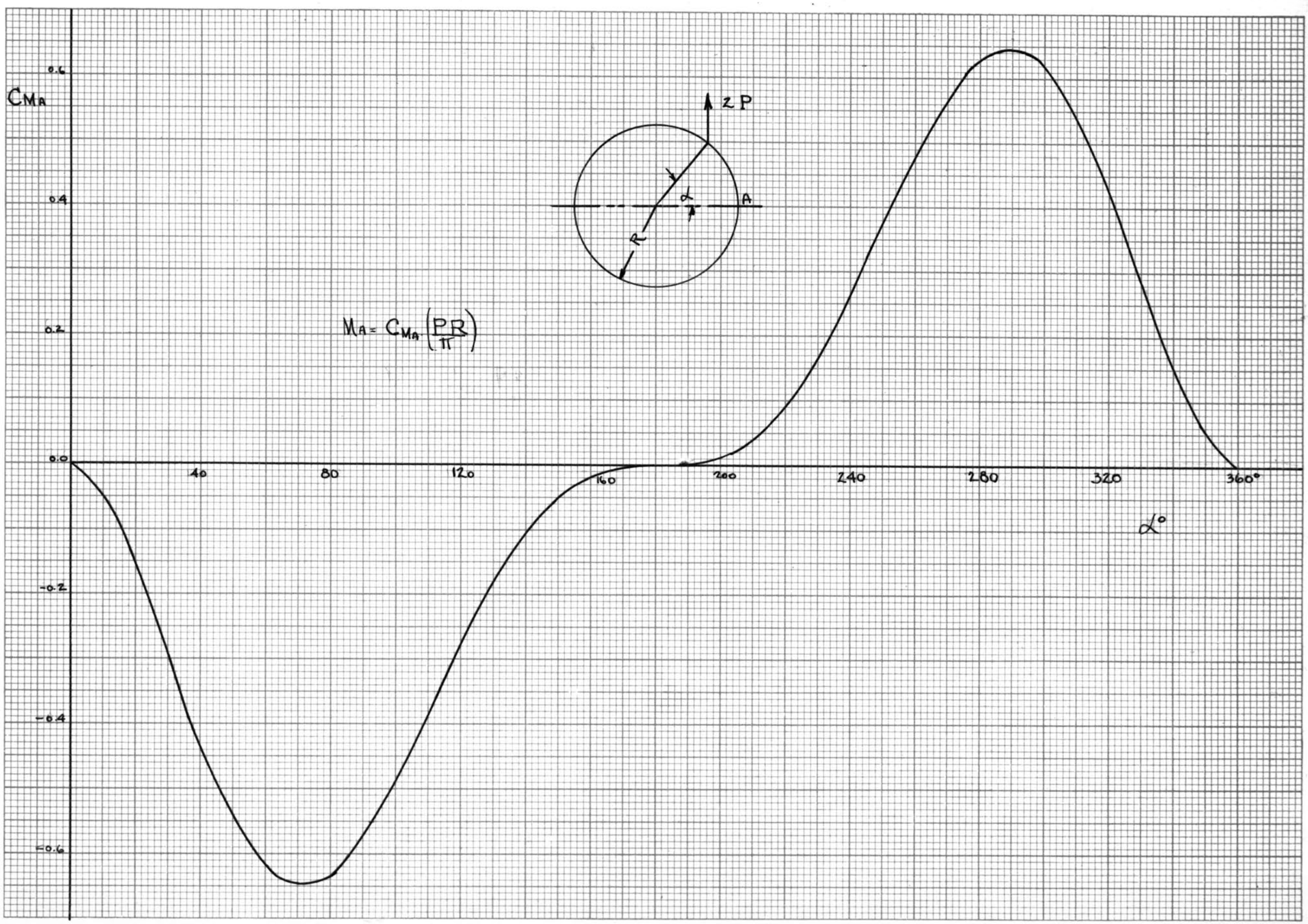
| | Loads perpendicular to diameter. | | | Loads parallel to diameter. | | |
|---|---|---|---|---|---|---|
| $\alpha$ | $C_{M\alpha}$ | $C_{N\alpha}$ | $C_{S\alpha}$ | $C_{M\alpha}$ | $C_{N\alpha}$ | $C_{S\alpha}$ |
| 0 | 0.000 | 3.141 | 1.500 | 1.500 | 1.500 | 3.141 |
| 10 | -0.042 | 3.138 | 1.455 | 1.002 | 1.470 | 2.622 |
| 20 | -0.148 | 3.114 | 1.320 | 0.604 | 1.383 | 2.129 |
| 30 | -0.281 | 3.071 | 1.120 | 0.307 | 1.250 | 1.685 |
| 40 | -0.420 | 2.936 | 0.856 | 0.108 | 1.088 | 1.308 |
| 50 | -0.536 | 2.759 | 0.560 | -0.009 | 0.912 | 1.011 |
| 60 | -0.614 | 2.527 | 0.250 | -0.063 | 0.750 | 0.795 |
| 70 | -0.642 | 2.241 | -0.040 | -0.077 | 0.620 | 0.659 |
| 80 | -0.628 | 1.916 | -0.297 | -0.074 | 0.530 | 0.590 |
| 90 | -0.571 | 1.570 | -0.500 | -0.071 | 0.500 | 0.570 |
| 100 | -0.481 | 1.226 | -0.643 | -0.074 | 0.530 | 0.583 |
| 110 | -0.378 | 0.901 | -0.725 | -0.112 | 0.620 | 0.604 |
| 120 | -0.271 | 0.614 | -0.750 | -0.157 | 0.750 | 0.614 |
| 130 | -0.176 | 0.380 | -0.730 | -0.225 | 0.912 | 0.599 |
| 140 | -0.098 | 0.209 | -0.678 | -0.301 | 1.088 | 0.548 |
| 150 | -0.044 | 0.091 | -0.616 | -0.378 | 1.250 | 0.457 |
| 160 | -0.015 | 0.029 | -0.557 | -0.442 | 1.383 | 0.329 |
| 170 | -0.003 | 0.004 | -0.516 | -0.484 | 1.470 | 0.172 |
| 180 | 0.000 | 0.000 | -0.500 | -0.500 | 1.500 | 0.000 |
| 190 | 0.004 | -0.004 | -0.516 | -0.484 | 1.470 | -0.172 |
| 200 | 0.003 | -0.029 | -0.557 | -0.442 | 1.383 | -0.329 |
| 210 | 0.044 | -0.091 | -0.616 | -0.378 | 1.250 | -0.457 |
| 220 | 0.098 | -0.209 | -0.678 | -0.301 | 1.088 | -0.548 |
| 230 | 0.176 | -0,380 | -0.730 | -0.225 | 0.912 | -0.599 |
| 240 | 0.271 | -0.614 | -0.750 | -0.157 | 0.750 | -0.614 |
| 250 | 0.378 | -0.901 | -0.725 | -0.112 | 0.620 | -0.604 |
| 260 | 0.481 | -1.226 | -0.643 | -0.074 | 0.530 | -0.583 |
| 270 | 0.571 | -1.570 | -0.500 | -0.071 | 0.500 | -0.570 |
| 280 | 0.628 | -1.916 | -0.297 | -0.074 | 0.530 | -0.590 |
| 290 | 0.642 | -2.241 | -0.040 | -0.077 | 0.620 | -0.659 |
| 300 | 0.614 | -2.527 | 0.250 | -0,063 | 0.750 | -0.795 |
| 310 | 0.536 | -2.759 | 0.560 | -0.009 | 0.912 | -1.011 |
| 320 | 0.420 | -2.936 | 0.856 | -0.108 | 1.088 | -1.308 |
| 330 | 0.281 | -3.071 | 1.120 | 0.307 | 1.250 | -1.685 |
| 340 | 0.148 | -3.114 | 1.320 | 0.604 | 1.383 | -2.129 |
| 350 | 0.042 | -3.138 | 1.455 | 1.002 | 1.470 | -2.622 |
| 360 | 0.000 | -3.141 | 1.500 | 1.500 | 1.500 | -3.141 |

## Rectangular rings.

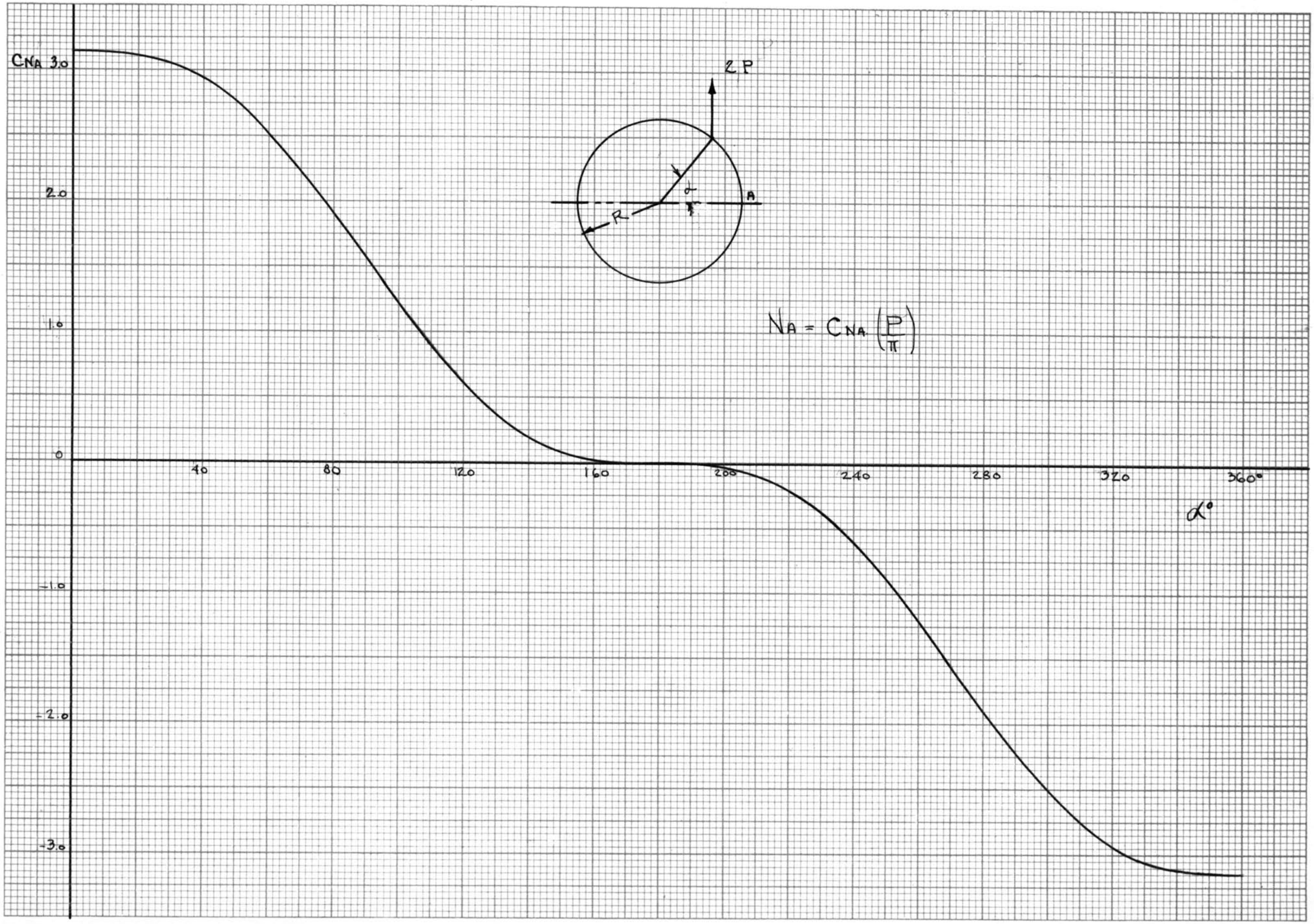
| Loads perpendicular to horizontal axis | | | | | Loads parallel to hor. axis | | | | |
|---|---|---|---|---|---|---|---|---|---|
| X | b/a | $C_{M_a}$ | $C_{N_a}$ | $C_{S_a}$ | X | b/a | $C_{M_a}$ | $C_{N_a}$ | $C_{S_a}$ |
| 0.0 b | 1 | 0.0000 | 1.000 | 0.291 | 0.0 a | 1 | 1.250 | -0.562 | 1.595 |
| 0.1 b | 1 | -0.0314 | 0.917 | 0.165 | 0.1 a | 1 | 1.094 | -0.487 | 1.453 |
| 0.2 b | 1 | -0.0520 | 0.824 | 0.054 | 0.2 a | 1 | 0.946 | -0.412 | 1.305 |
| 0.3 b | 1 | -0.0630 | 0.721 | -0.041 | 0.3 a | 1 | 0.808 | -0.337 | 1.162 |
| 0.4 b | 1 | -0.0660 | 0.612 | -0.122 | 0.4 a | 1 | 0.681 | -0.262 | 1.020 |
| 0.5 b | 1 | -0.0625 | 0.500 | -0.187 | 0.5 a | 1 | 0.562 | -0.187 | 0.875 |
| 0.6 b | 1 | -0.0540 | 0.388 | -0.238 | 0.6 a | 1 | 0.455 | -0.112 | 0.737 |
| 0.7 b | 1 | -0.0420 | 0.279 | -0.274 | 0.7 a | 1 | 0.358 | -0.037 | 0.587 |
| 0.8 b | 1 | -0.0280 | 0.176 | -0.295 | 0.8 a | 1 | 0.270 | 0.037 | 0.442 |
| 0.9 b | 1 | -0.0135 | 0.082 | -0.300 | 0.9 a | 1 | 0.193 | 0.112 | 0.300 |
| 1.0 b | 1 | 0.0000 | 0.000 | -0.291 | 1.0 a | 1 | 0.125 | 0.187 | 0.156 |
| 0.0 b | .8 | 0.0000 | 1.000 | 0.235 | 0.0 a | .8 | 1.250 | -0.742 | 1.565 |
| 0.1 b | .8 | -0.0274 | 0.833 | 0.137 | 0.1 a | .8 | 1.093 | -0.642 | 1.420 |
| 0.2 b | .8 | -0.0457 | 0.820 | 0.051 | 0.2 a | .8 | 0.942 | -0.542 | 1.280 |
| 0.3 b | .8 | -0.0554 | 0.719 | -0.024 | 0.3 a | .8 | 0.804 | -0.445 | 1.135 |
| 0.4 b | .8 | -0.0582 | 0.611 | -0.088 | 0.4 a | .8 | 0.677 | -0.345 | 0.996 |
| 0.5 b | .8 | -0.0554 | 0.500 | -0.141 | 0.5 a | .8 | 0.558 | -0.247 | 0.853 |
| 0.6 b | .8 | -0.0482 | 0.389 | -0.183 | 0.6 a | .8 | 0.453 | -0.148 | 0.711 |
| 0.7 b | .8 | -0.0378 | 0.291 | -0.215 | 0.7 a | .8 | 0.358 | -0.049 | 0.568 |
| 0.8 b | .8 | -0.0254 | 0.180 | -0.231 | 0.8 a | .8 | 0.244 | 0.049 | 0.427 |
| 0.9 b | .8 | -0.0123 | 0.167 | -0.239 | 0.9 a | .8 | 0.200 | 0.148 | 0.284 |
| 1.0 b | .8 | 0.0000 | 0.000 | -0.235 | 1.0 a | .8 | 0.138 | 0.247 | 0.142 |
| 0.0 b | .6 | -0.0000 | 1.000 | 0.184 | 0.0 a | .6 | 1.260 | -1.040 | 1.528 |
| 0.1 b | .6 | -0.0378 | 0.915 | 0.113 | 0.1 a | .6 | 1.094 | -0.902 | 1.384 |
| 0.2 b | .6 | -0.0463 | 0.816 | -0.049 | 0.2 a | .6 | 0.939 | -0.766 | 1.242 |
| 0.3 b | .6 | -0.0492 | 0.713 | -0.007 | 0.3 a | .6 | 0.797 | -0.625 | 1.104 |
| 0.4 b | .6 | -0.0468 | 0.608 | -0.056 | 0.4 a | .6 | 0.670 | -0.486 | 0.961 |
| 0.5 b | .6 | -0.0411 | 0.500 | -0.096 | 0.5 a | .6 | 0.552 | -0.347 | 0.821 |
| 0.6 b | .6 | -0.0324 | 0.392 | -0.129 | 0.6 a | .6 | 0.448 | -0.208 | 0.682 |
| 0.7 b | .6 | -0.0220 | 0.286 | -0.155 | 0.7 a | .6 | 0.357 | -0.069 | 0.540 |
| 0.8 b | .6 | -0.0109 | 0.184 | -0.172 | 0.8 a | .6 | 0.277 | 0.069 | 0.402 |
| 0.9 b | .6 | -0.0103 | 0.090 | -0.182 | 0.9 a | .6 | 0.210 | 0.208 | 0.258 |
| 1.0 b | .6 | -0.0000 | 0.000 | -0.184 | 1.0 a | .6 | 0.156 | 0.347 | 0.119 |

# Rectangular rings.

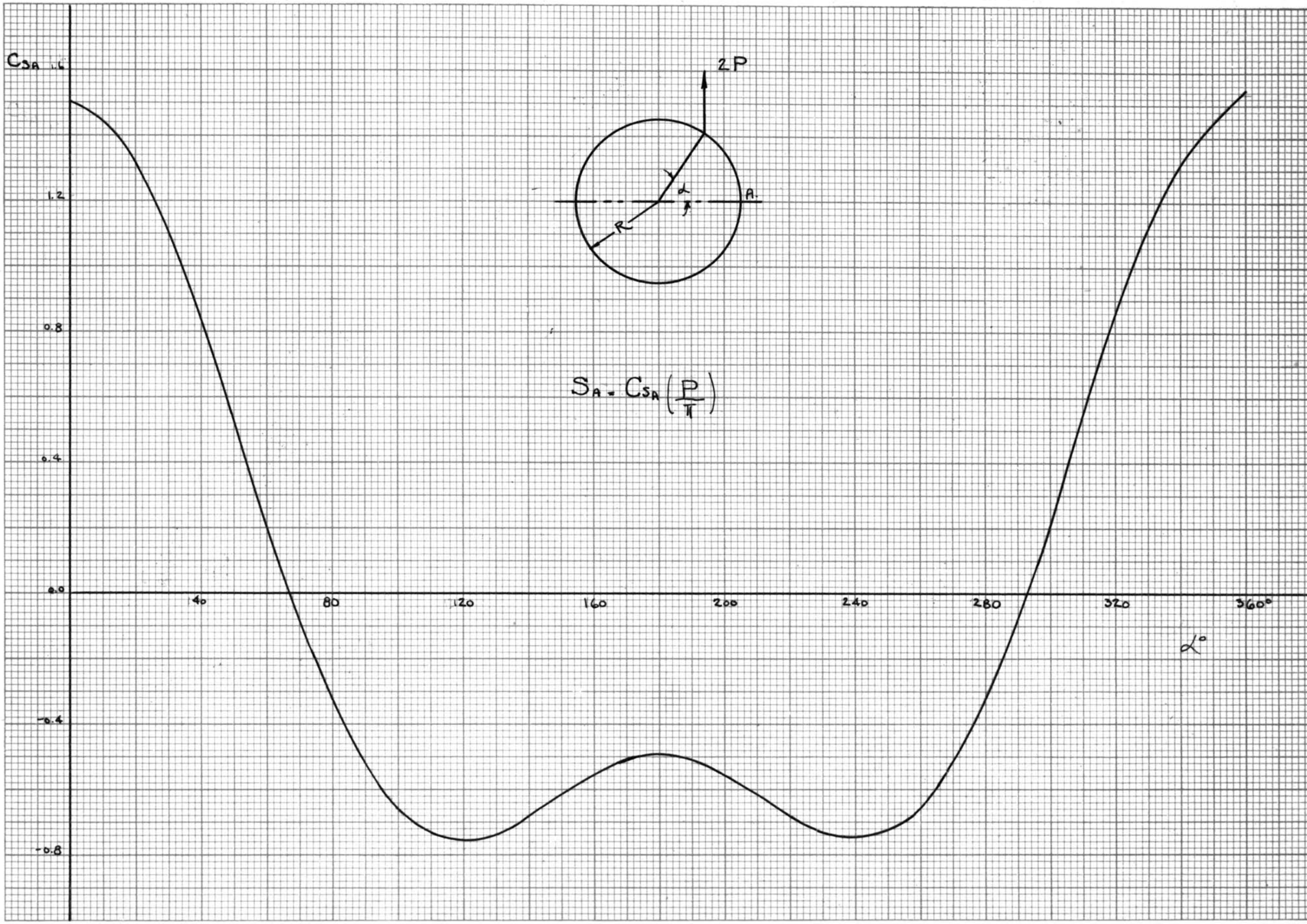
| Loads prependicular to horizontal axis | | | | | Loads parallel to horizontal axis | | | | |
|---|---|---|---|---|---|---|---|---|---|
| X | b/a | $C_{M_a}$ | $C_{N_a}$ | $C_{S_a}$ | X | b/a | $C_{M_a}$ | $C_{N_a}$ | $C_{S_a}$ |
| 0.0 b | .4 | 0.0000 | 1.000 | 0.132 | 0.0 a | .4 | 1.260 | -1.653 | 1.470 |
| 0.1 b | .4 | -0.0169 | 0.909 | 0.091 | 0.1 a | .4 | 1.088 | -1.433 | 1.330 |
| 0.2 b | .4 | -0.0284 | 0.810 | 0.048 | 0.2 a | .4 | 0.932 | -1.213 | 1.190 |
| 0.3 b | .4 | -0.0349 | 0.672 | 0.009 | 0.3 a | .4 | 0.788 | -0.994 | 1.050 |
| 0.4 b | .4 | -0.0372 | 0.570 | -0.025 | 0.4 a | .4 | 0.659 | -0.772 | 0.912 |
| 0.5 b | .4 | -0.0357 | 0.500 | -0.055 | 0.5 a | .4 | 0.541 | -0.552 | 0.773 |
| 0.6 b | .4 | -0.0314 | 0.357 | -0.080 | 0.6 a | .4 | 0.442 | -0.331 | 0.632 |
| 0.7 b | .4 | -0.0250 | 0.290 | -0.101 | 0.7 a | .4 | 0.354 | -0.110 | 0.494 |
| 0.8 b | .4 | -0.0171 | 0.189 | -0.128 | 0.8 a | .4 | 0.281 | 0.110 | 0.356 |
| 0.9 b | .4 | 0.0085 | 0.091 | -0.130 | 0.9 a | .4 | 0.223 | 0.331 | 0.214 |
| 1.0 b | .4 | 0.0000 | 0.000 | -0.132 | 1.0 a | .4 | 0.178 | 0.552 | 0.074 |
| 0.0 b | .2 | 0.0000 | 1.000 | 0.0882 | 0.0 a | .2 | 1.260 | -3.515 | 1.400 |
| 0.1 b | .2 | -0.0314 | 0.904 | 0.0638 | 0.1 a | .2 | 1.080 | -3.046 | 1.257 |
| 0.2 b | .2 | -0.0520 | 0.806 | 0.041 | 0.2 a | .2 | 0.918 | -2.578 | 1.053 |
| 0.3 b | .2 | -0.0630 | 0.705 | 0.038 | 0.3 a | .2 | 0.769 | -2.110 | 0.973 |
| 0.4 b | .2 | -0.0660 | 0.603 | 0.018 | 0.4 a | .2 | 0.641 | -1.640 | 0.830 |
| 0.5 b | .2 | -0.0625 | 0.500 | -0.018 | 0.5 a | .2 | 0.527 | -1.171 | 0.688 |
| 0.6 b | .2 | -0.0540 | 0.397 | -0.036 | 0.6 a | .2 | 0.430 | -0.703 | 0.547 |
| 0.7 b | .2 | -0.0420 | 0.294 | -0.051 | 0.7 a | .2 | 0.348 | -0.234 | 0.403 |
| 0.8 b | .2 | -0.0280 | 0.194 | -0.065 | 0.8 a | .2 | 0.285 | 0.234 | 0.261 |
| 0.9 b | .2 | -0.0135 | 0.095 | -0.077 | 0.9 a | .2 | 0.238 | 0.703 | 0.186 |
| 1.0 b | .2 | 0.0000 | 0.000 | -0.088 | 1.0 a | .2 | 0.208 | 1.171 | -0.024 |



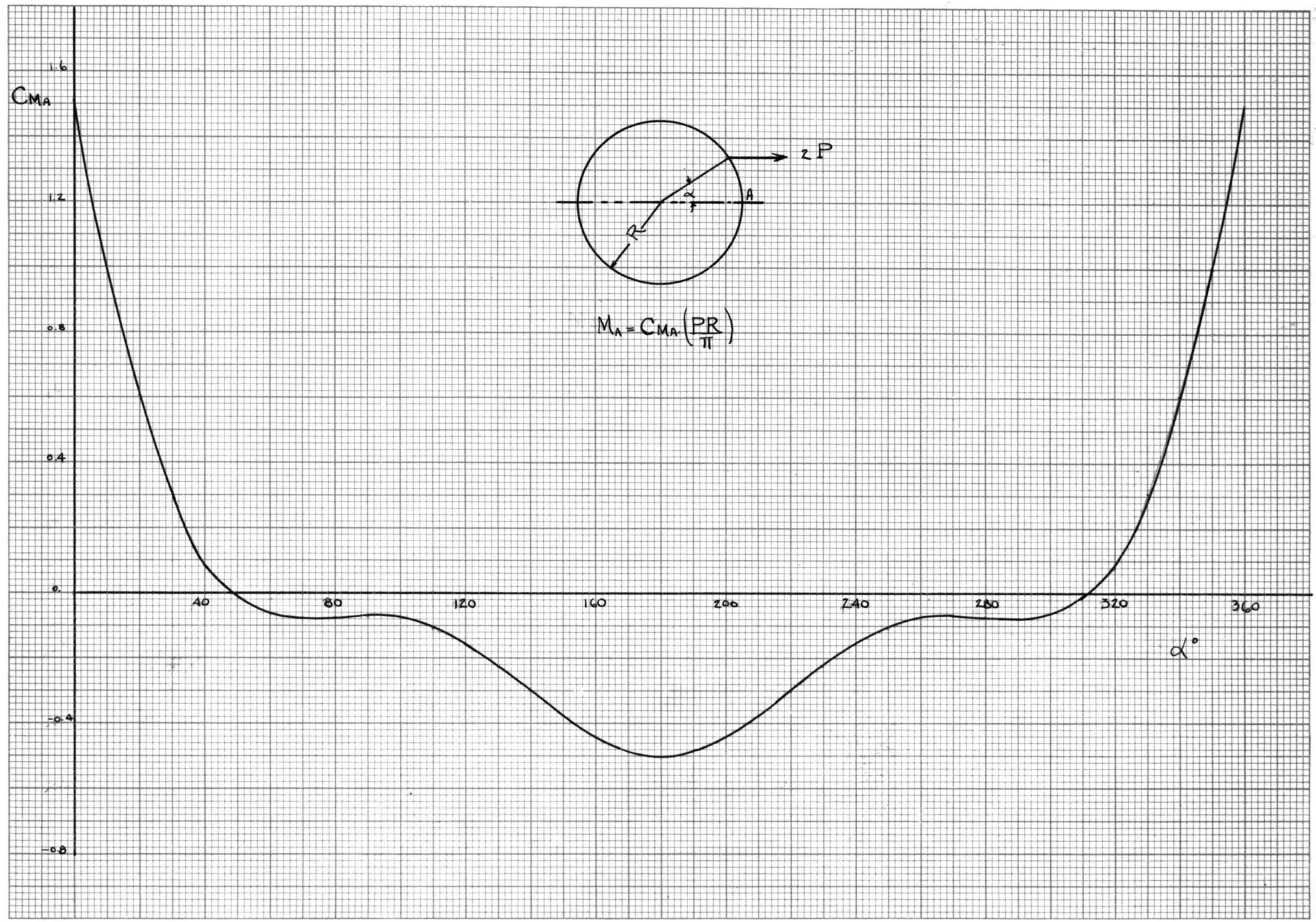
CMA
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
40
80
120
160
200
240
280
320
360°
α°
2P
A
α
R
MA = CMA (PR/π)



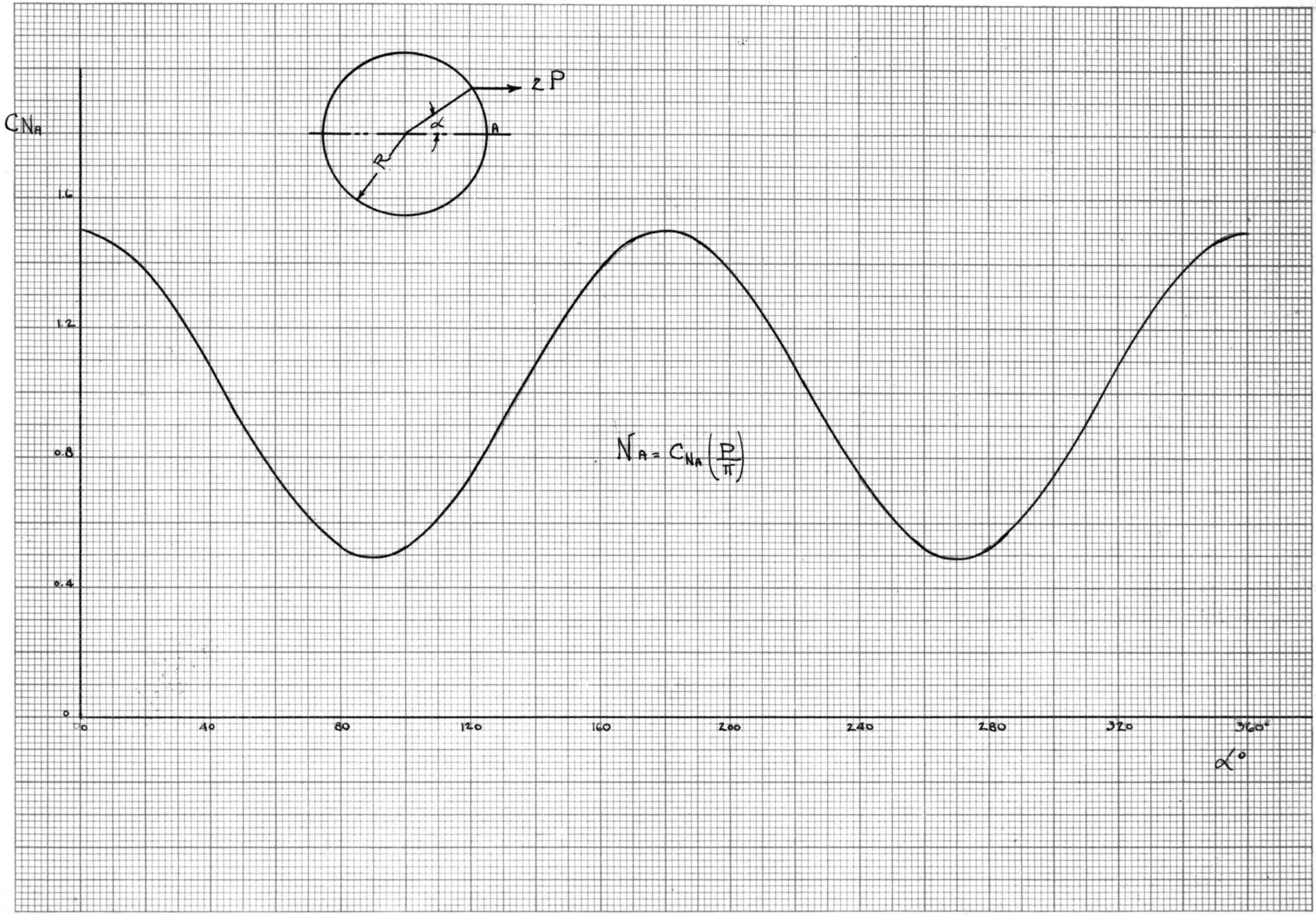
CNA 3.0
2.0
1.0
0
-1.0
-2.0
-3.0
40
80
120
160
200
240
280
320
360°
α°
2P
α
A
R
NA = CNA (P/π)



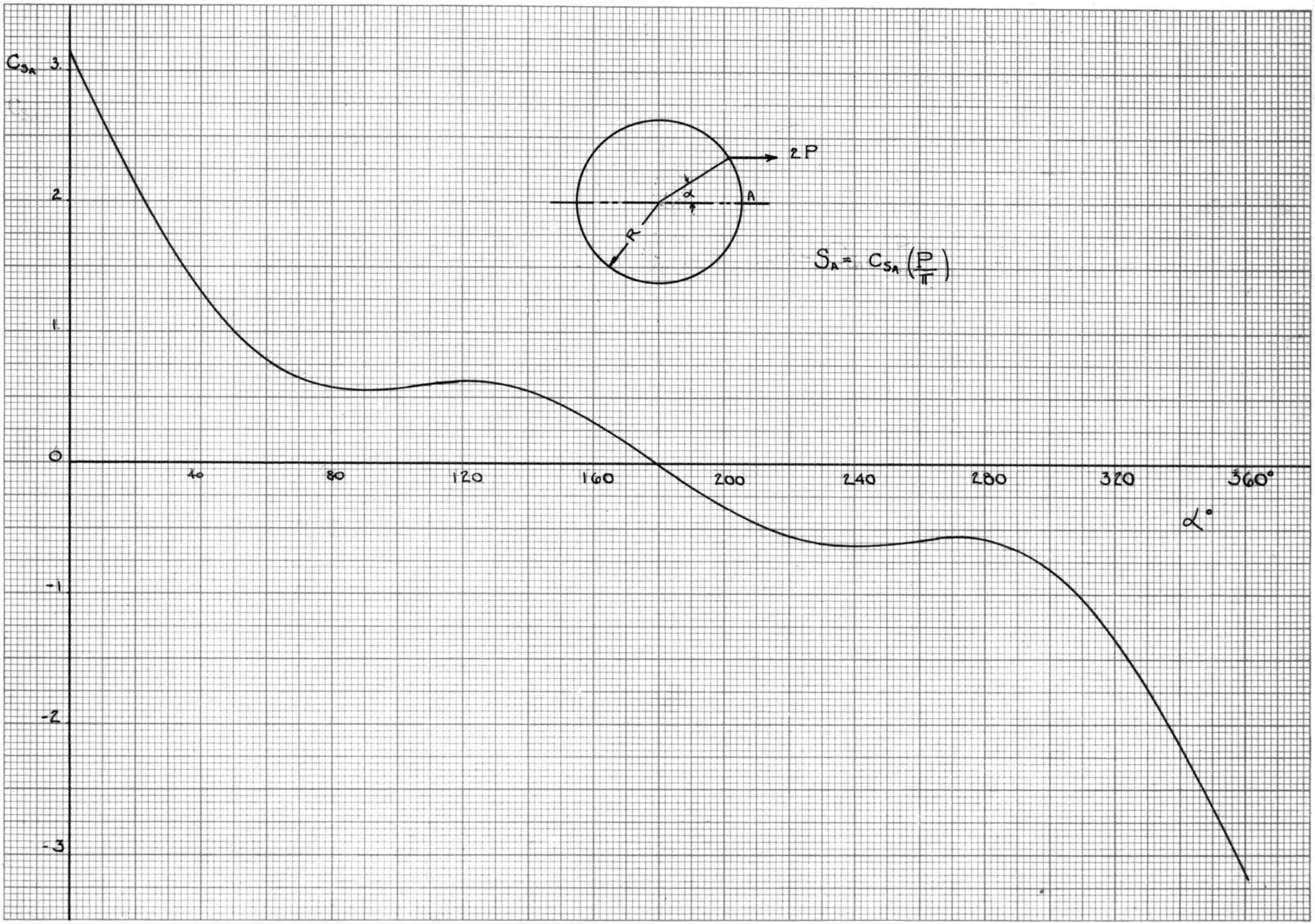
$C_{SA}$
1.6
1.2
0.8
0.4
0.0
-0.4
-0.8
40
80
120
160
200
240
280
320
360°
α°
2P
α
A.
R
$S_A = C_{SA}\left(\frac{P}{\pi}\right)$



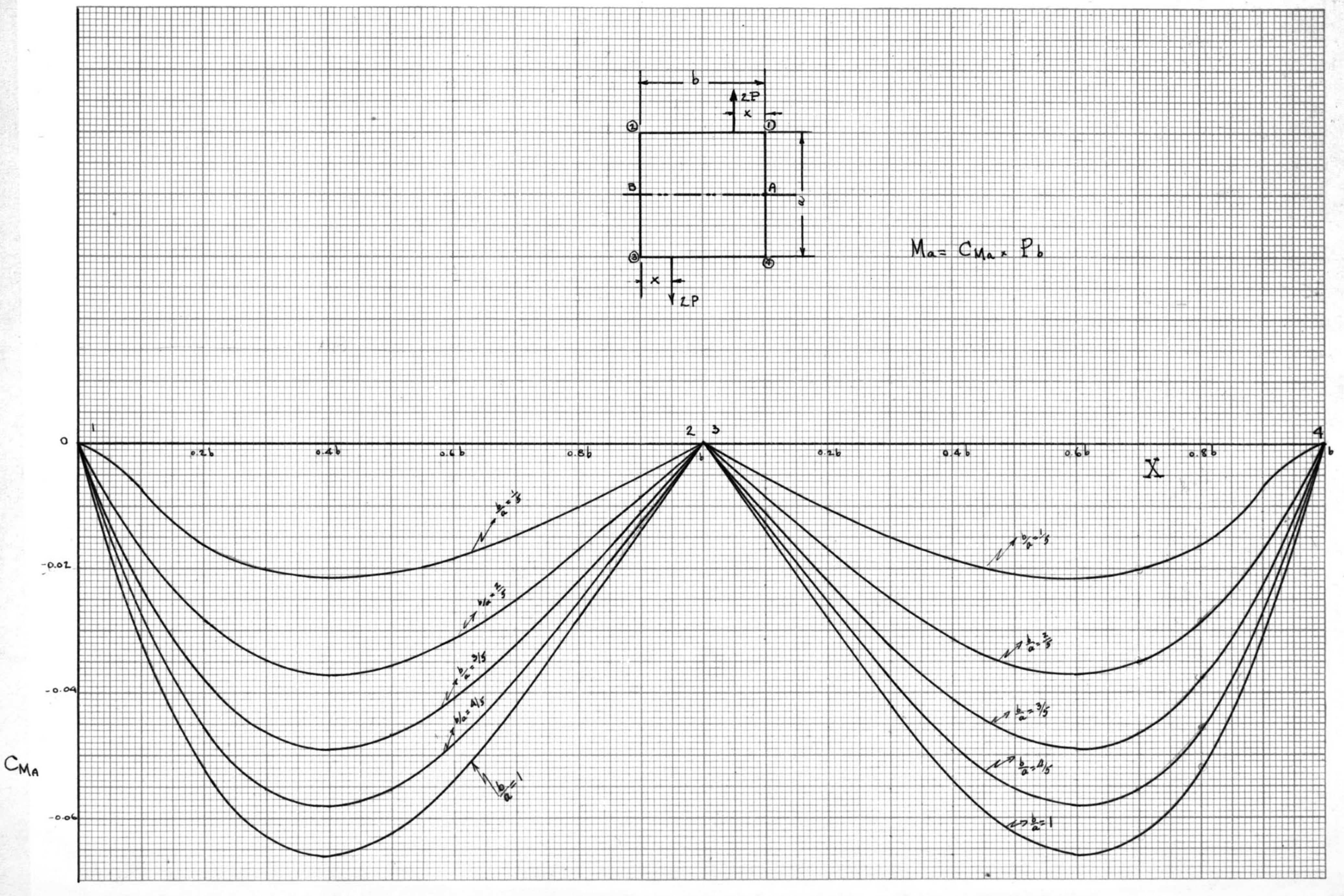
1.6
$C_{M_A}$
1.2
0.8
0.4
0.
-0.4
-0.8
40
80
120
160
200
240
280
320
360
$\alpha°$
2P
A
α
R
$M_A = C_{M_A}\left(\frac{PR}{\pi}\right)$



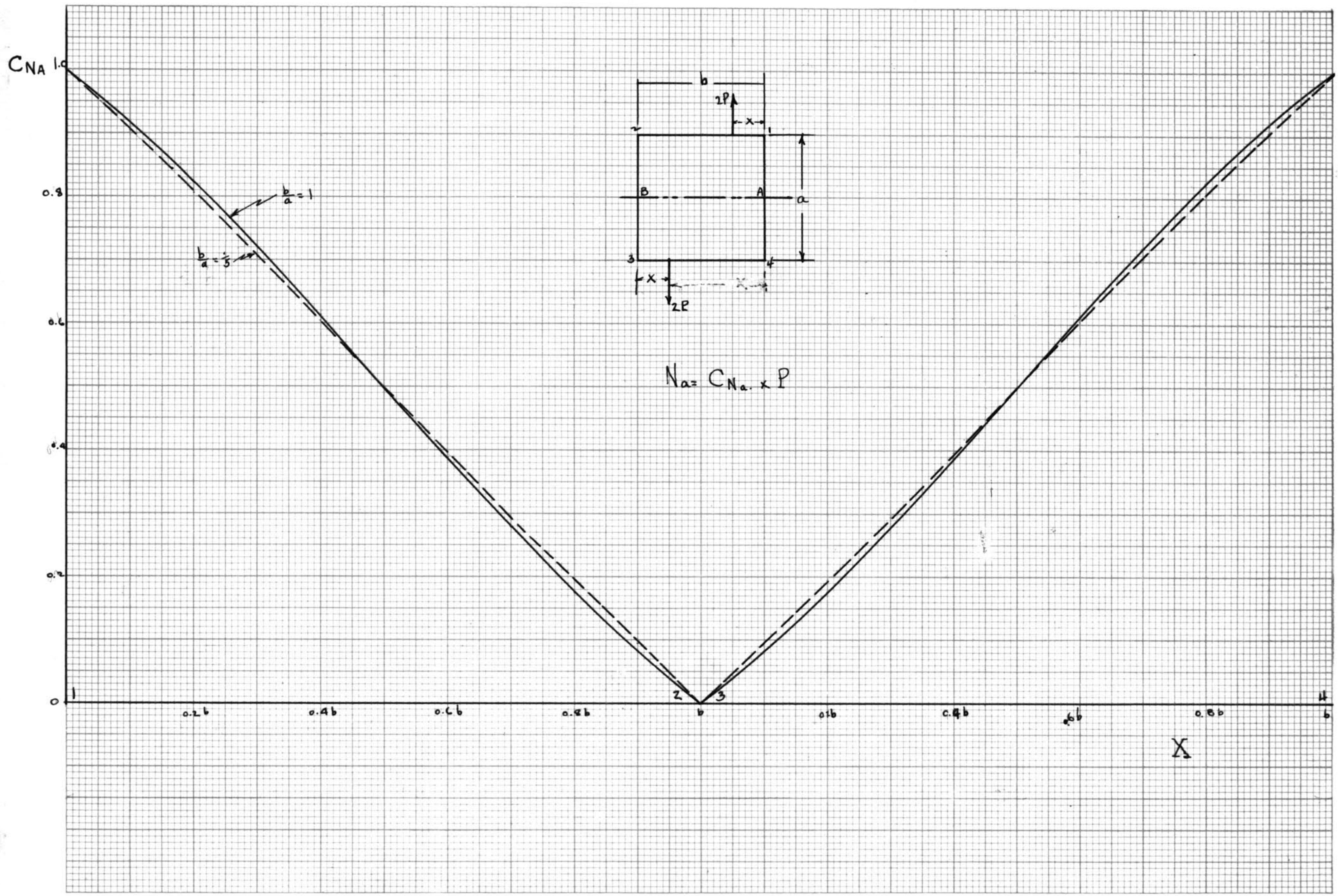
2P
α
A
R
$C_{N_A}$
1.6
1.2
0.8
0.4
0
0
40
80
120
160
200
240
280
320
360°
α°
$N_A = C_{N_A}\left(\frac{P}{\pi}\right)$



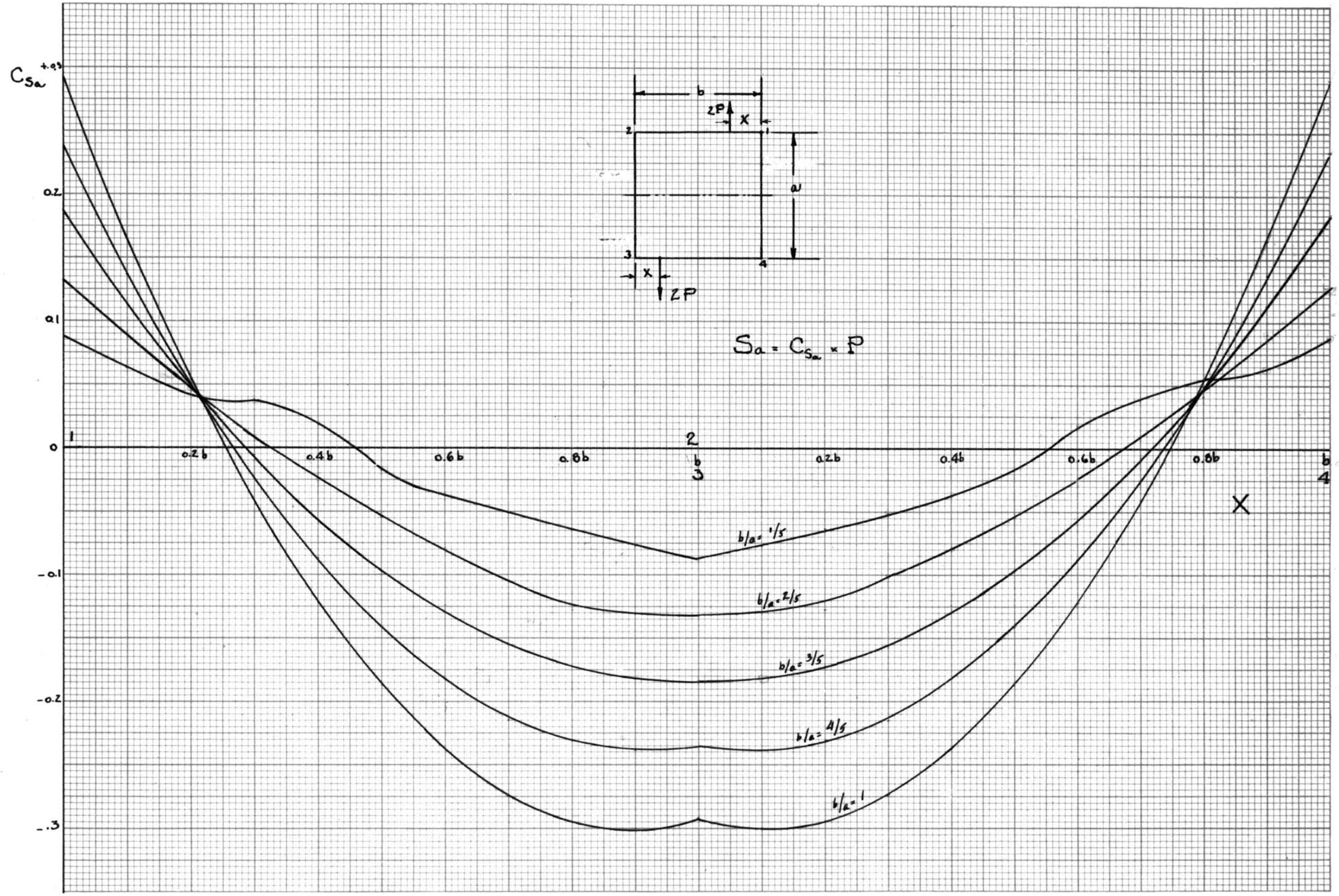
$C_{S_A}$
3.
2
1.
0
-1
-2
-3
40
80
120
160
200
240
280
320
360°
α°
2P
α
A
R
$S_A = C_{S_A}\left(\frac{P}{\pi}\right)$



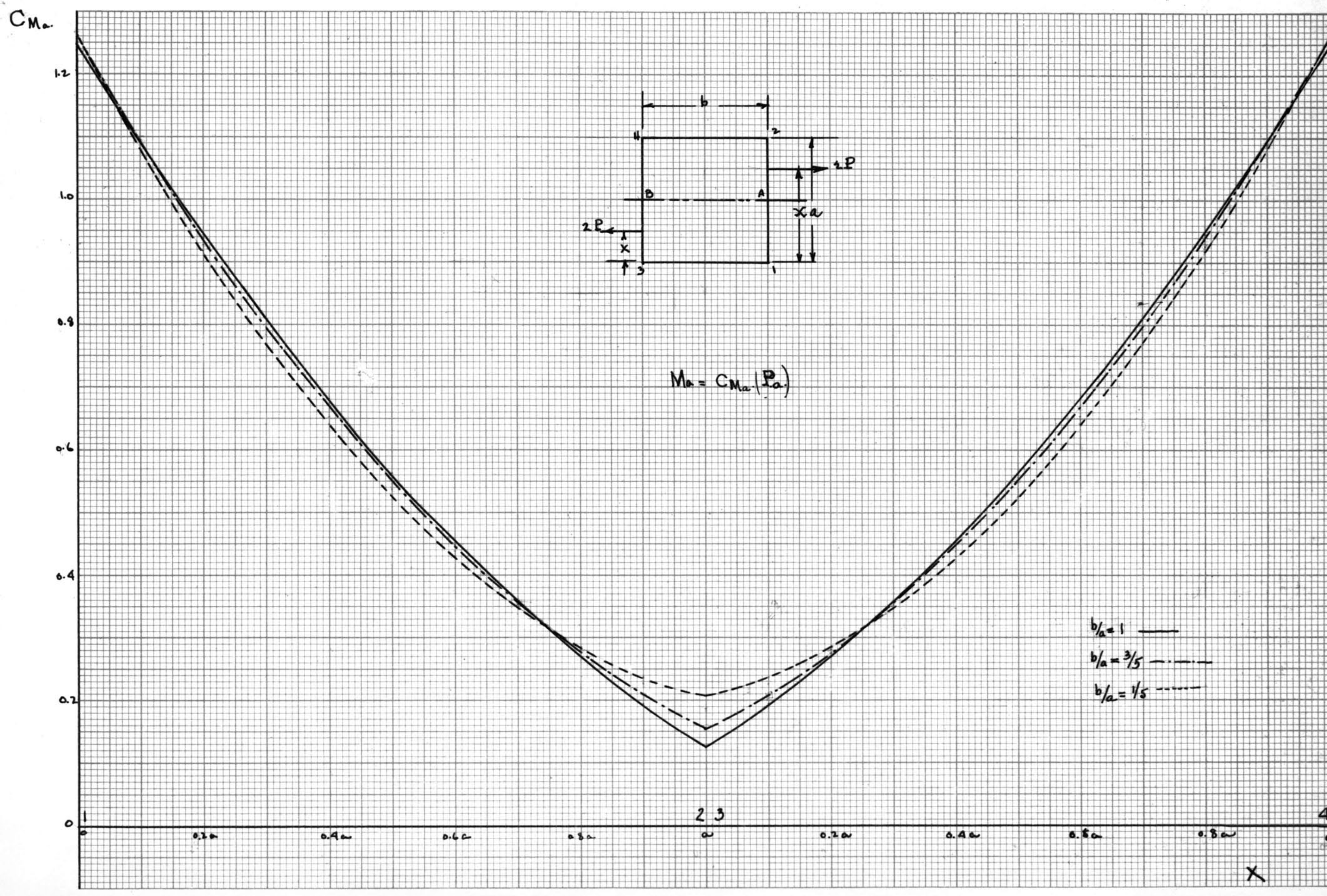
b
2P
x
1
2
A
B
a
3
4
x
2P
Ma = CMa × Pb
0
1
2 3
4
0.2b
0.4b
0.6b
0.8b
b
X
-0.02
-0.04
-0.06
CMA
b/a = 1/5
b/a = 2/5
b/a = 3/5
b/a = 4/5
b/a = 1



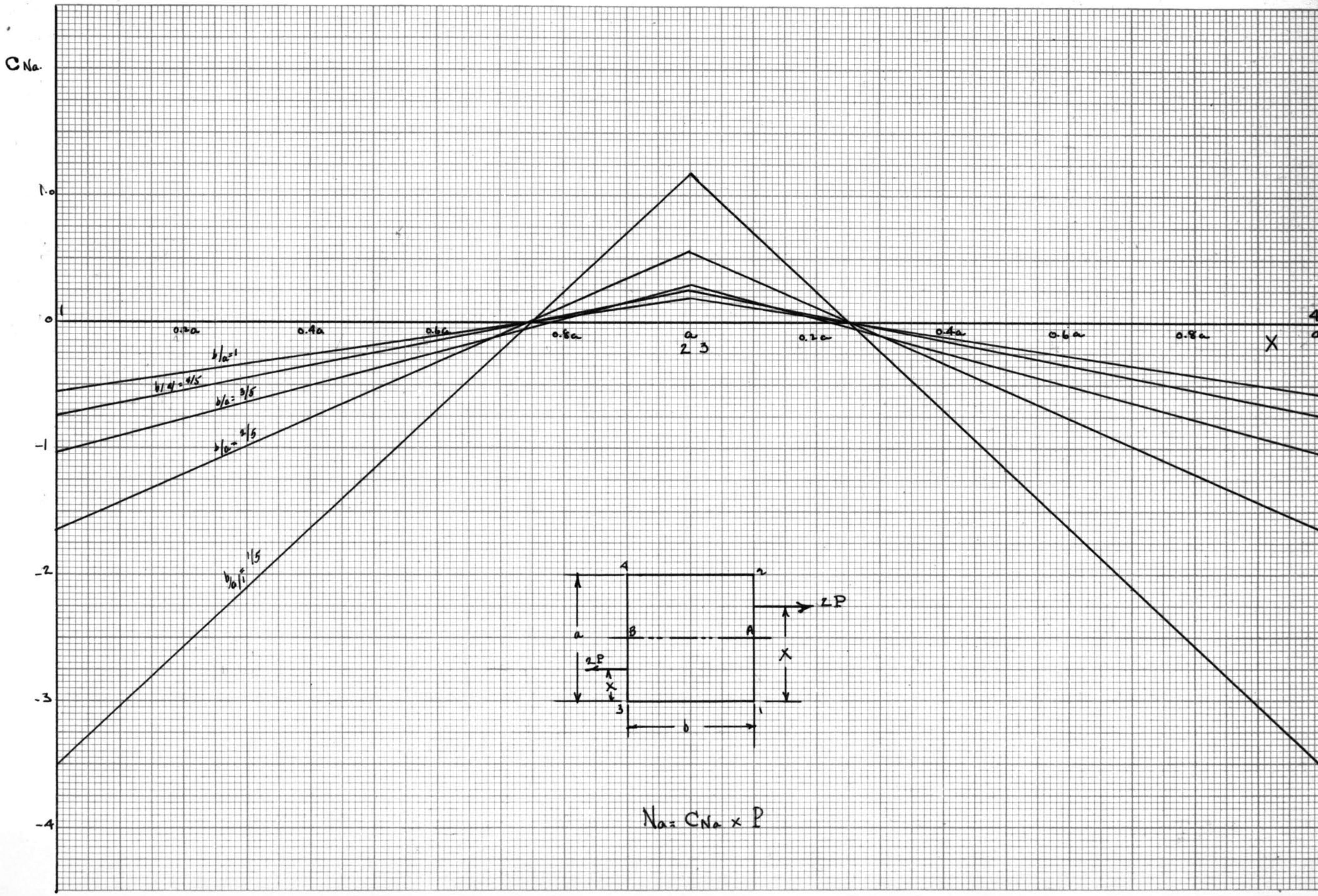
$C_{NA}$
1.0
0.8
0.6
0.4
0.2
0
b/a = 1
b/a = 1/3
b
2P
x
a
B
A
2P
$N_a = C_{Na} \times P$
0.2b
0.4b
0.6b
0.8b
b
0.8b
0.6b
0.4b
0.2b
b
X



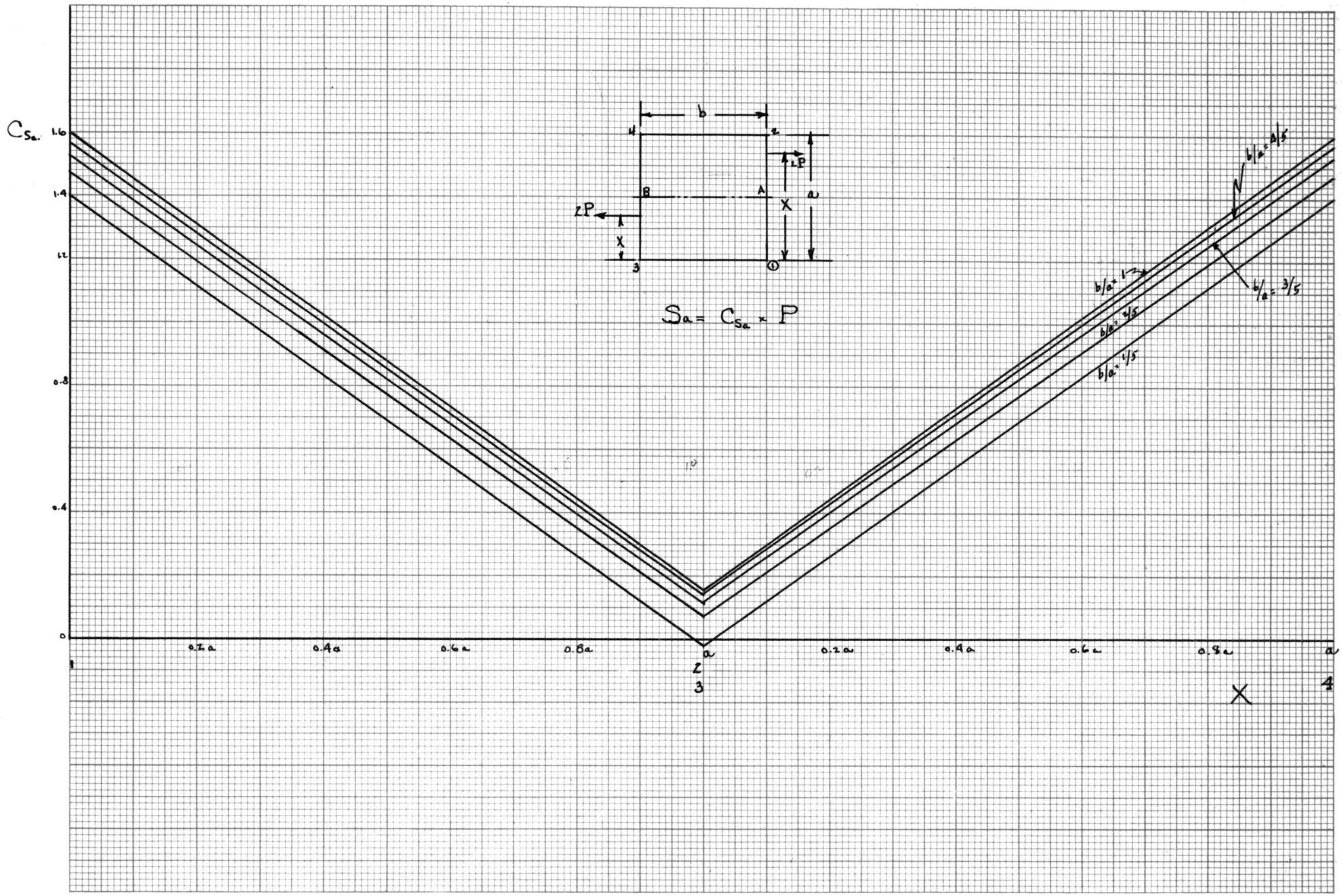
$C_{S_a}$
+0.3
0.2
0.1
0
-0.1
-0.2
-.3
1
0.2b
0.4b
0.6b
0.8b
2
b
3
0.2b
0.4b
0.6b
0.8b
b
4
X
b
2P
X
2
1
a
3
4
X
2P
$S_a = C_{S_a} \times P$
b/a = 1/5
b/a = 2/5
b/a = 3/5
b/a = 4/5
b/a = 1



$C_{Ma}$
1.2
1.0
0.8
0.6
0.4
0.2
0
b
1
2
3
4
B
A
2P
x
xa
$M_a = C_{Ma} (Pa)$
b/a = 1
b/a = 3/5
b/a = 1/5
0
0.2a
0.4a
0.6a
0.8a
a
X



CNa.
1.0
0
-1
-2
-3
-4
0.2a
0.4a
0.6a
0.8a
a
2 3
0.2a
0.4a
0.6a
0.8a
a
X
b/a=1
b/a=4/5
b/a=3/5
b/a=2/5
b/a=1/5
2P
a
B
A
X
b
Na = CNa × P



$C_{Sa}$
1.6
1.4
1.2
0.8
0.4
0
0.2a
0.4a
0.6a
0.8a
a
2
3
X
4
b
a
2P
B
A
X
$S_a = C_{Sa} \times P$
b/a = 4/5
b/a = 1
b/a = 3/5
b/a = 2/5
b/a = 1/5